KYBERNETIKA — VOLUME 54 (2018), NUMBER 3, PAGES 413–426

X-SIMPLICITY OF INTERVAL MAX-MIN MATRICES

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A matrix A is said to have $\boldsymbol{X}$-simple image eigenspace if any eigenvector x belonging to the interval $\boldsymbol{X} = \{x\colon \underline{x} \leq x \leq \overline{x}\}$ containing a constant vector is the unique solution of the system $A \otimes y = x$ in $\boldsymbol{X}$. The main result of this paper is an extension of $\boldsymbol{X}$-simplicity to interval max-min matrix $\boldsymbol{A} = \{A\colon \underline{A} \leq A \leq \overline{A}\}$ distinguishing two possibilities, that at least one matrix or all matrices from a given interval have $\boldsymbol{X}$-simple image eigenspace. $\boldsymbol{X}$-simplicity of interval matrices in max-min algebra are studied and equivalent conditions for interval matrices which have $\boldsymbol{X}$-simple image eigenspace are presented. The characterized property is related to and motivated by the general development of tropical linear algebra and interval analysis, as well as the notions of simple image set and weak robustness (or weak stability) that have been studied in max-min and max-plus algebras.

Keywords: max-min algebra, interval, eigenspace, simple image set

Classification: 15A80, 15A18, 08A72

1. INTRODUCTION

Max-min algebra (the addition and the multiplication are formally replaced by operations of maximum and minimum) can be used in a range of practical problems related to scheduling, optimization, modeling of fuzzy discrete dynamic systems, graph theory, knowledge engineering, cluster analysis, fuzzy systems and also related to describing diagnosis of technical devices [30] or medical diagnosis [26]. There are several monographs [8, 10, 11, 12, 13] and collections of papers [15, 16] on fuzzy algebra and its applications. Let us also mention some connections between idempotent algebra and fuzzy sets theory [5, 6].

In the max-min algebra, sometimes also called the “fuzzy algebra” [8, 17, 20, 21], the arithmetical operations $a \oplus b := \max(a, b)$ and $a \otimes b := \min(a, b)$ are defined over a linearly ordered set. As usual, the two arithmetical operations are naturally extended to matrices and vectors.

The development of linear algebra over idempotent semirings was historically motivated by multi-machine interaction processes. In these processes we have n machines which work in stages, and in the algebraic model of their interactive work, entry $x_i^{(k)}$ of a vector $x^{(k)} \in \mathbb{B}(n)$ where $i \in \{1, \ldots, n\}$ and $\mathbb{B}$ is an idempotent semiring, represents the state of machine i after some stage k, and the entry a_{ij} of a matrix $A \in \mathbb{B}(n, n)$,

DOI: 10.14736/kyb-2018-3-0413

where $i, j \in \{1, \dots, n\}$, encodes the influence of the work of machine j in the previous stage on the work of machine i in the current stage. For simplicity, the process is assumed to be homogeneous, like in the discrete time Markov chains, so that A does not change from stage to stage. Summing up all the influence effects multiplied by the results of previous stages, we have $x_i^{(k+1)} = \bigoplus_j a_{ij} \otimes x_j^{(k)}$. In the case of $\oplus = \max$ this "summation" is often interpreted as waiting till all the processes are finished and all the necessary influence constraints are satisfied.

The orbit x, $A \otimes x, \dots, A^k \otimes x$, where $A^k = A \otimes \dots \otimes A$, represents the evolution of such a process. Regarding the orbits, one wishes to know the set of starting vectors from which a given objective can be achieved. One of the most natural objectives in tropical algebra, where the ultimate periodicity of the orbits often occurs, is to arrive at an eigenvector. The set of starting vectors from which one reaches an eigenvector of A after a finite number of stages, is called attraction set of A (see [2, 9, 22, 27]). In general, attraction set contains the set of all eigenvectors, but it can be also as big as the whole space. This leads us, in turn, to another question: in which case is attraction set precisely the same as the set of all eigenvectors? Matrices with this property are called weakly robust [23] or weakly stable [3]. Therefore, by eigenvectors of A we shall mean the fixed points of A (satisfying $A \otimes x = x$).

In terms of the systems $A \otimes x = b$, weak robustness (with eigenvectors understood as fixed points) is equivalent to the following condition: every eigenvector y belongs to the *simple image set* of A, that is, for every eigenvector y, the system $A \otimes x = y$ has unique solution $x = y$ [21].

In the present paper, we consider an interval version of this condition. Namely, we describe matrices $A \in \boldsymbol{A} = [\underline{A}, \overline{A}] := \{A \in \mathbb{B}(n, n);\ \underline{A} \leq A \leq \overline{A}\}$ such that for any eigenvector y belonging to an interval $\boldsymbol{X} = [\underline{x}, \overline{x}] := \{x \in \mathbb{B}(n);\ \underline{x} \leq x \leq \overline{x}\}$ containing a constant vector the system has a unique solution $x = y$ in $\boldsymbol{X}$. This is what we mean by saying that "A has $\boldsymbol{X}$-simple image eigenspace". Notice that if $\boldsymbol{X}$ does not contain a constant vector the problem is difficult since it is not possible to guarantee a unique solution of the system $A \otimes x = y$. We will distinguish two possibilities, that at least one matrix or all matrices from a given interval have $\boldsymbol{X}$-simple image eigenspace. It was shown that A has $\boldsymbol{X}$-simple image eigenspace if and only if it satisfies a nontrivial combinatorial criterion, which makes use of threshold digraphs and to which we refer as "$\boldsymbol{X}$-conformity" (see Definition 3.4).

Let us now give more details on the organization of the paper and on the results obtained there. The next section will be occupied by some definitions and notation of the max-min algebra, leading to the discussion of weak $\boldsymbol{X}$-robustness and $\boldsymbol{X}$-simple image eigenvectors. Sections 3, 4 and 5 are devoted to the main results of the paper which characterize interval matrices with universal and possible $\boldsymbol{X}$-simple image eigenspaces.

Let us conclude with a brief overview of the works on max-min algebra to which this paper is related. The concepts of robustness in max-min algebra were introduced and studied in [21]. Following that work, some equivalent conditions and efficient algorithms were presented in [17, 23]. In particular, see [23] for some polynomial procedures checking the weak robustness (weak stability) in max-min algebra.

2. PRELIMINARIES

2.1. Max-min algebra and associated digraphs

Let us denote the set of all natural numbers by $\mathbb{N}$. Let $(\mathbb{B}, \leq)$ be a bounded linearly ordered set with the least element in $\mathbb{B}$ denoted by O and the greatest one by I. Notice that $\mathbb{B}$ can be not only infinite but also discrete or even finite.

A max-min algebra is a set $\mathbb{B}$ equipped with two binary operations $\oplus = \max$ and $\otimes = \min$, called addition and multiplication, such that $(\mathbb{B}, \oplus)$ is a commutative monoid with identity element O, $(\mathbb{B}, \otimes)$ is a monoid with identity element I, multiplication left and right distributes over addition and multiplication by O annihilates $\mathbb{B}$.

We will use the notations N for the set of natural numbers not exceeding n, i. e., $N = \{1, 2, \ldots, n\}$. The set of $n \times n$ matrices over $\mathbb{B}$ is denoted by $\mathbb{B}(n, n)$, and the set of $n \times 1$ vectors over $\mathbb{B}$ is denoted by $\mathbb{B}(n)$. If each entry of a matrix $A \in \mathbb{B}(n, n)$ (a vector $x \in \mathbb{B}(n)$) is equal to O we shall denote it as $A = O$ $(x = O)$.

Let $x = (x_1, \ldots, x_n) \in \mathbb{B}(n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{B}(n)$ be vectors. We write $x \leq y$ $(x < y)$ if $x_i \leq y_i$ $(x_i < y_i)$ holds for each $i \in N$.

For a matrix $A \in \mathbb{B}(n, n)$ the symbol $G(A) = (N, E)$ stands for a complete, arc-weighted digraph associated with A, i. e., the node set of $G(A)$ is N, and the weight (capacity) of any arc (i, j) is $a_{ij} \geq O$. For given $h \in \mathbb{B}$, the *threshold digraph* $G(A, h)$ is the digraph with the node set N and with the arc set $E = \{(i, j);\ i, j \in N,\ a_{ij} \geq h\}$. A path in the digraph $G(A) = (N, E)$ is a sequence of nodes $p = (i_1, \ldots, i_{k+1})$ such that $(i_j, i_{j+1}) \in E$ for $j = 1, \ldots, k$. The number k is the length of the path p and is denoted by $l(p)$. If $i_1 = i_{k+1}$, then p is called a cycle and it is called an elementary cycle if moreover $i_j \neq i_m$ for $j, m = 1, \ldots, k$.

2.2. Orbits, eigenvectors and weak robustness

For $A \in \mathbb{B}(n, n)$ and $x \in \mathbb{B}(n)$, the orbit $O(A, x)$ of $x = x^{(0)}$ generated by A is the sequence $x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots$, where $x^{(r)} = A^r \otimes x^{(0)}$ for each $r \in \mathbb{N}$.

The operations $\max, \min$ are idempotent, so no new numbers are created in the process of generating of an orbit. Therefore any orbit contains only a finite number of different vectors. It follows that any orbit starts repeating itself after some time, in other words, it is ultimately periodic. The same holds for the power sequence $(A^k; k \in \mathbb{N})$.

We are interested in the case when the ultimate period is 1, or in other words, when the orbit is ultimately stable. Note that in this case the ultimate vector of the orbit necessarily satisfies $A \otimes x = x$. This is the main reason why in this paper by eigenvectors we mean fixed points. (Also observe that if x is not a fixed point but a more general eigenvector satisfying $A \otimes x = \lambda \otimes x$, then $A \otimes x$ is already a fixed point due to the idempotency of multiplication.)

For a given matrix $A \in \mathbb{B}(n, n)$, the number $\lambda \in \mathbb{B}$ and the n-tuple $x \in \mathbb{B}(n)$ are the so-called *eigenvalue* of A and *eigenvector* of A, respectively, if $A \otimes x = \lambda \otimes x$.

The *eigenspace* $V(A, \lambda)$ is defined as the set of all eigenvectors of A with associated eigenvalue λ, i. e., $V(A, \lambda) = \{x \in \mathbb{B}(n);\ A \otimes x = \lambda \otimes x\}$.

In case $\lambda = I$ let us denote $V(A, I)$ by abbreviation $V(A)$.

Formally we can define the attraction set $\mathrm{attr}(A)$ as follows

$$\mathrm{attr}(A) = \{x \in \mathbb{B}(n);\ O(A, x) \cap V(A) \neq \emptyset\}.$$

The present paper investigates the following kind of matrices.

Definition 2.1. Let $A \in \mathbb{B}(n, n)$ be a matrix. Then A is called weakly robust (or weakly stable), if $\mathrm{attr}(A) = V(A)$.

Observe that in general $V(A) \subseteq \mathrm{attr}(A) \subseteq \mathbb{B}^n$. The matrices for which $\mathrm{attr}(A) = \mathbb{B}^n$ are called (strongly) robust or (strongly) stable, as opposed to weakly robust (weakly stable). The following fact, which holds in max-min algebra and max-plus algebra alike, is one of the main motivations for our paper.

Theorem 2.2. (Plavka and Szabó [21], Butkovič et al. [3]) Let $A \in \mathbb{B}(n, n)$ be a matrix. Then A is weakly robust if and only if $(\forall x \in \mathbb{B}(n))[A \otimes x \in V(A) \Rightarrow x \in V(A)]$.

Let us conclude this section with recalling some information on 1) the greatest eigenvector and 2) constant eigenvectors in max-min algebra.

Let $A = (a_{ij}) \in \mathbb{B}(n, n)$, $\lambda \in \mathbb{B}$ be given and define the greatest eigenvector $x^{\oplus}(A, \lambda)$ corresponding to a matrix A and λ as

$$x^{\oplus}(A, \lambda) = \bigoplus_{x \in V(A, \lambda)} x.$$

In case $\lambda = I$ let us denote $x^{\oplus}(A, I)$ by abbreviation $x^{\oplus}(A)$.

Notice that the greatest eigenvector $x^{\oplus}(A, \lambda)$ exists for every matrix A and each $\lambda \in \mathbb{B}$ whereby its entries are given by the efficient formula, see [29]. Moreover, in [4] it was shown that entries of the greatest eigenvector $x^{\oplus}(A)$ can be computed by the following $O(n^2 \log n)$ iterative procedure. Let us denote $x^1_i(A) = \bigoplus_{j \in N} a_{ij}$ for each $i \in N$ and $x^{k+1}(A) = A \otimes x^k(A)$ for all $k \in \{1, 2, \dots\}$. Then $x^{k+1}(A) \leq x^k(A)$ and $x^{\oplus}(A) = x^n(A)$.

Next, denote

$$c(A) = \bigotimes_{i \in N} \bigoplus_{j \in N} a_{ij}, \quad c^*(A) = (c(A), \dots, c(A))^T \in \mathbb{B}(n).$$

It can be checked that $A \otimes c^*(A) = c^*(A)$, so $c^*(A)$ is a constant eigenvector of A. As $x^{\oplus}(A)$ is the greatest eigenvector of A, we have $c^*(A) \leq x^{\oplus}(A)$.

2.3. Weak $\boldsymbol{X}$-robustness and $\boldsymbol{X}$-simplicity

In this subsection we consider an interval extension of weak robustness and its connection to $\boldsymbol{X}$-simplicity. We remind that throughout the paper,

$$\boldsymbol{X} = [\underline{x}, \overline{x}] = \{x \colon \underline{x} \leq x \leq \overline{x}, \},\ \text{where } \underline{x}, \overline{x} \in \mathbb{B}(n).$$

Consider the following interval extension of weak $\boldsymbol{X}$-robustness.

Definition 2.3. $A \in \mathbb{B}(n,n)$ is called weakly $\boldsymbol{X}$-robust if $\operatorname{attr}(A) \cap \boldsymbol{X} \subseteq V(A)$.

The notion of $\boldsymbol{X}$-simplicity is related to the concept of simple image set [1]: by definition, this is the set of vectors b such that the system $A \otimes x = b$ has a unique solution, which is usually denoted by $|S(A,b)| = 1$ ($S(A,b)$ standing for the solution set of $A \otimes x = b$). If the only solution of the system $A \otimes x = b$ is $x = b$, then b is called a *simple image eigenvector.*

If $\boldsymbol{X} \subseteq \mathbb{B}$ then the notion of weak robustness can be described in terms of simple image eigenvectors:

Proposition 2.4. (Plavka and Sergeev [24]) Let $A \in \mathbb{B}(n,n)$. The following are equivalent:

(i) A is weakly robust;

(ii) $(\forall x \in V(A))[|S(A,x)| = 1]$.

This motivates us to consider an interval version of simple image eigenvectors.

Definition 2.5. Let $A = (a_{ij}) \in \mathbb{B}(n,n)$. Then

(i) An eigenvector $x \in V(A) \cap \boldsymbol{X}$ is called an $\boldsymbol{X}$-simple image eigenvector if x is the unique solution of the equation $A \otimes y = x$ in interval $\boldsymbol{X}$.

(ii) Matrix A is said to have $\boldsymbol{X}$-simple image eigenspace if any $x \in V(A) \cap \boldsymbol{X}$ is an $\boldsymbol{X}$-simple image eigenvector.

Definition 2.6. Let $A \in \mathbb{B}(n,n)$ and $\boldsymbol{X} = [\underline{x}, \overline{x}] \subseteq \mathbb{B}(n)$ be given. We say that $\boldsymbol{X}$ is invariant under A if $x \in \boldsymbol{X}$ implies $A \otimes x \in \boldsymbol{X}$.

Theorem 2.7. (Plavka and Sergeev [24]) Let $A \in \mathbb{B}(n,n)$ be a matrix and $\boldsymbol{X} = [\underline{x}, \overline{x}] \subseteq \mathbb{B}(n)$ be an interval vector.

(i) If A is weakly $\boldsymbol{X}$-robust then A has $\boldsymbol{X}$-simple image eigenspace.

(ii) If A has $\boldsymbol{X}$-simple image eigenspace and if $\boldsymbol{X}$ is invariant under A then A is weakly $\boldsymbol{X}$-robust.

Proposition 2.8. $\boldsymbol{X}$ is invariant under A if and only if $A \otimes \underline{x} \geq \underline{x}$ and $A \otimes \overline{x} \leq \overline{x}$.

Thus the $\boldsymbol{X}$-simplicity is a necessary condition for weak $\boldsymbol{X}$-robustness. It is also sufficient if the interval $\boldsymbol{X}$ is invariant under A, i. e., $\underline{x} \leq A \otimes \underline{x}$ and $A \otimes \overline{x} \leq \overline{x}$.

3. INTERVAL $\boldsymbol{X}$-SIMPLICITY

The purpose of this section is to define the condition for matrix A which will ensure that each eigenvector $x \in V(A) \cap \boldsymbol{X}$ is an $\boldsymbol{X}$-simple image eigenvector and to deal with matrices with interval elements. Sufficient and necessary conditions for an interval matrix which have $\boldsymbol{X}$-simple eigenspace will be proved. In addition we introduce a polynomial algorithm to check the $\boldsymbol{X}$-simplicity of interval max-min matrices.

Definition 3.1. Let $A \in \mathbb{B}(n,n)$ be a matrix. A is called a *generalized level α-permutation matrix* (abbr. level α-permutation) if all entries greater than or equal to α of A lie on disjoint elementary cycles covering all the nodes. In other words, the threshold digraph $G(A,\alpha)$ is a set of disjoint elementary cycle containing all nodes.

Remark 3.2. If $A = (a_{ij}) \in \mathbb{B}(n,n)$ is a level $c(A)$-permutation matrix, $(i_1,\dots,i_n)$ is a permutation of N such that $a_{i_j i_{j+1}} \geq c(A)$ and

$$(i_1,\dots,i_n) = (i_1^1,\dots,i_{s_1}^1)\dots(i_1^k,\dots,i_{s_k}^k)$$

($c_u = (i_1^u,\dots,i_{s_u}^u)$ is an elementary cycle in digraph $G(A,c(A))$, $u = 1,\dots,k$), then $x_v^{\oplus}(A) = \min_{(k,l)\in c_u} a_{kl}$ for all $v \in c_u$ (see [23]).

Lemma 3.3. (Plavka and Sergeev [24]) Let $A = (a_{ij}) \in \mathbb{B}(n,n)$ be a matrix, $\boldsymbol{X} = [\underline{x},\overline{x}] \in \mathbb{B}(n)$ be an interval vector and $\underline{x} < c^*(A) \leq \overline{x}$. If A has $\boldsymbol{X}$-simple image eigenspace then A is level $c(A)$-permutation.

Definition 3.4. Let $\boldsymbol{X} = [\underline{x},\overline{x}] \subseteq \mathbb{B}(n)$ be an interval vector such that $\underline{x} < c^*(A) \leq \overline{x}$ and $A = (a_{ij}) \in \mathbb{B}(n,n)$ be a level $c(A)$-permutation matrix, $(i_1,\dots,i_n)$ be a permutation of N such that $a_{i_j i_{j+1}} \geq c(A)$ and

$$(i_1,\dots,i_n) = (i_1^1,\dots,i_{s_1}^1)\dots(i_1^k,\dots,i_{s_k}^k)$$

($c_u = (i_1^u,\dots,i_{s_u}^u)$ is an elementary cycle in digraph $G(A,c(A))$, $u = 1,\dots,k$). Then vectors $e_{\underline{x}}(A) = (e_1(A),\dots,e_n(A))^T$ and $f_{\overline{x}}(A) = (f_1(A),\dots,f_n(A))^T$ are called $\underline{x}$-vector of A and $\overline{x}$-vector of A if

$$e_i(A)(=e_i) = \max_{v\in c_u} \underline{x}_v \text{ and } f_i(A)(=f_i) = \min_{v\in c_u} \overline{x}_v \otimes x_v^{\oplus}(A), \tag{1}$$

respectively, for $i \in c_u$, $u \in \{1,\dots,k\}$ and matrix A is called $\boldsymbol{X}$-conforming if

(i) $\underline{x}_{i_{j+1}} < e_{i_{j+1}} \Rightarrow a_{i_j k} < e_{i_j}$ for $k \neq i_{j+1},\ k \in N$

(ii) $\underline{x}_{i_{j+1}} = e_{i_{j+1}} \Rightarrow a_{i_j k} \leq e_{i_j}$ for $k \neq i_{j+1},\ k \in N$

(iii) $a_{i_j i_{j+1}} = \min\limits_{(k,l)\in c_u} a_{kl} = x_{i_{j+1}}^{\oplus}(A) = f_{i_{j+1}} \Rightarrow \overline{x}_{i_{j+1}} \leq x_{i_{j+1}}^{\oplus}(A)$.

Notice that $e_{i_j} = e_{i_{j+1}}$ and $f_{i_j} = f_{i_{j+1}}$ by definition of $e_{\underline{x}}(A)$ and $f_{\overline{x}}(A)$ (nodes i_j, i_{j+1} are lying in the same cycle c_u). Notation $(k,l) \in c_u$ means that the edge (k,l) is lying in c_u (see example 3.4 in [24]).

Theorem 3.5. (Plavka and Sergeev [24]) Let $A = (a_{ij}) \in \mathbb{B}(n,n)$ be a matrix, $\boldsymbol{X} = [\underline{x},\overline{x}] \in \mathbb{B}(n)$ be an interval vector and $\underline{x} < c^*(A) \leq \overline{x}$. Then A has $\boldsymbol{X}$-simple image eigenspace if and only if A is an $\boldsymbol{X}$-conforming matrix.

Similarly to [7, 14, 18, 19], we define an interval matrix $\boldsymbol{A}$.

Definition 3.6. Let $\underline{A}, \overline{A} \in B(n,n)$. An interval matrix $\boldsymbol{A}$ with bounds $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$ is defined as follows

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \left\{ A \in B(n,n);\ \underline{A} \le A \le \overline{A} \right\}.$$

Investigating $\boldsymbol{X}$-simplicity for an interval matrix $\boldsymbol{A}$ following questions can arise. Does $\boldsymbol{A}$ have $\boldsymbol{X}$-simple image eigenspace for some $A \in \boldsymbol{A}$ or for all $A \in \boldsymbol{A}$?

Definition 3.7. Let $\boldsymbol{A}$ be an interval matrix. $\boldsymbol{A}$ has

(i) possible $\boldsymbol{X}$-simple image eigenspace if there exists a matrix $A \in \boldsymbol{A}$ such that A has $\boldsymbol{X}$-simple image eigenspace,

(ii) universal $\boldsymbol{X}$-simple image eigenspace if each matrix $A \in \boldsymbol{A}$ has $\boldsymbol{X}$-simple image eigenspace.

Definition 3.8. Let $\boldsymbol{X} = [\underline{x}, \overline{x}] \subseteq \mathbb{B}(n)$ be an interval vector such that $\underline{x} < c^*(\underline{A}) \le c^*(\overline{A}) \le \overline{x}$. The matrix $\boldsymbol{A}$ is called

(i) possibly $\boldsymbol{X}$-conforming if there is a matrix $A \in \boldsymbol{A}$ which is $\boldsymbol{X}$-conforming.

(ii) universally $\boldsymbol{X}$-conforming if each matrix $A \in \boldsymbol{A}$ is $\boldsymbol{X}$-conforming.

4. POSSIBLE $\boldsymbol{X}$-SIMPLICITY

Denote P_n the set of all permutations on N and suppose that $\pi = (i_1, \dots, i_n) \in P_n$. Define the matrix $\overline{A}^{\pi} = (\overline{a}^{\pi}_{kl})$ as follows

$$\overline{a}^{\pi}_{kl} = \begin{cases} \overline{a}_{kl}, & \text{if } \exists j : (k,l) = (i_j, i_{j+1}) \\ \underline{a}_{kl}, & \text{otherwise.} \end{cases}$$

Lemma 4.1. Let $\boldsymbol{A}$ be given, $A = (a_{ij})$ be a level $c(A)$-permutation matrix and $(i_1, \dots, i_n)$ be a permutation π of N such that $a_{i_j i_{j+1}} \ge c(A)$. Then $\overline{A}^{\pi}$ is level $c(\overline{A}^{\pi})$-permutation.

Proof. Suppose that $A = (a_{ij})$ is a level $c(A)$-permutation matrix and $(i_1, \dots, i_n)$ is a permutation π of N such that $a_{i_j i_{j+1}} \ge c(A)$. Then for entires of $\overline{A}^{\pi}$ we have $\overline{a}^{\pi}_{kl} = \overline{a}_{kl} \ge a_{kl}$ if there is $j : (k,l) = (i_j, i_{j+1})$ and $\overline{a}^{\pi}_{kl} = \underline{a}_{kl} \le a_{kl}$ if such $j : (k,l) = (i_j, i_{j+1})$ does not exist. Hence we get $c(\overline{A}^{\pi}) \ge c(A)$ and the assertion follows. □

Theorem 4.2. Let $\boldsymbol{A}$, $\boldsymbol{X}$ be given and $\underline{x} < c^*(\underline{A}) \le c^*(\overline{A}) \le \overline{x}$. Then $\boldsymbol{A}$ is possibly $\boldsymbol{X}$-conforming if and only if there is $\pi \in P_n$ such that $\overline{A}^{\pi}$ is $\boldsymbol{X}$-conforming.

Proof. Suppose that $\boldsymbol{A}$ is possibly $\boldsymbol{X}$-conforming, i.e., there is $A \in \boldsymbol{A}$ which is $\boldsymbol{X}$-conforming, i.e., $A = (a_{ij})$ is a level $c(A)$-permutation matrix, $(i_1, \dots, i_n)$ is a permutation π of N such that $a_{i_j i_{j+1}} \geq c(A)$ and

$$(i_1, \dots, i_n) = (i_1^1, \dots, i_{s_1}^1) \dots (i_1^k, \dots, i_{s_k}^k)$$

($c_u = (i_1^u, \dots, i_{s_u}^u)$ is an elementary cycle in digraph $G(A, c(A))$, $u = 1, \dots, k$). By Lemma 4.1 matrix $\overline{A}^\pi$ is level $c(\overline{A}^\pi)$-permutation. In particular, $A \in \boldsymbol{A}$ is $\boldsymbol{X}$-conforming and by Definition 3.4 we obtain

(i) $\underline{x}_{i_{j+1}} < e_{i_{j+1}} \Rightarrow a_{i_j k} < e_{i_j}$ for $k \neq i_{j+1},\ k \in N$

(ii) $\underline{x}_{i_{j+1}} = e_{i_{j+1}} \Rightarrow a_{i_j k} \leq e_{i_j}$ for $k \neq i_{j+1},\ k \in N$

and hence

(i) $\underline{x}_{i_{j+1}} < e_{i_{j+1}} \Rightarrow \overline{a}^\pi_{i_j k} = \underline{a}_{i_j k} \leq a_{i_j k} < e_{i_j}$ for $k \neq i_{j+1},\ k \in N$

(ii) $\underline{x}_{i_{j+1}} = e_{i_{j+1}} \Rightarrow \overline{a}^\pi_{i_j k} = \underline{a}_{i_j k} \leq a_{i_j k} \leq e_{i_j}$ for $k \neq i_{j+1},\ k \in N$.

To prove the condition (iii) of Definition 3.4 suppose that

$$\overline{a}^\pi_{i_j i_{j+1}} = \min_{(k,l) \in c_u} \overline{a}^\pi_{kl} = x^\oplus_{i_{j+1}}(\overline{A}^\pi) = f_{i_{j+1}}(\overline{A}^\pi)$$

and for the sake of a contrary let $\overline{x}_{i_{j+1}} > x^\oplus_{i_{j+1}}(\overline{A}^\pi)$.

We shall consider four cases.

Case 1. $a_{i_j i_{j+1}} = \min\limits_{(k,l) \in c_u} a_{kl} = x^\oplus_{i_{j+1}}(A) = f_{i_{j+1}}(A) \Rightarrow \overline{x}_{i_{j+1}} \leq x^\oplus_{i_{j+1}}(A)$.
By Remark 3.2 ($x^\oplus_v(A) = \min\limits_{(k,l) \in c_u} a_{kl}$ for all $v \in c_u$) it is easy to see that

$$x^\oplus_{i_{j+1}}(A) = \min_{(k,l) \in c_u} a_{kl} \leq \min_{(k,l) \in c_u} \overline{a}_{kl} = x^\oplus_{i_{j+1}}(\overline{A}^\pi) \tag{2}$$

and the following inequalities contradict the assertion

$$\overline{x}_{i_{j+1}} \leq x^\oplus_{i_{j+1}}(A) \leq x^\oplus_{i_{j+1}}(\overline{A}^\pi) < \overline{x}_{i_{j+1}}.$$

Case 2. $a_{i_j i_{j+1}} = \min\limits_{(k,l) \in c_u} a_{kl} = x^\oplus_{i_{j+1}}(A) > f_{i_{j+1}}(A)$.
In this case we have that

$$x^\oplus_{i_{j+1}}(\overline{A}^\pi) \geq x^\oplus_{i_{j+1}}(A) > f_{i_{j+1}}(A) \Rightarrow (\exists v \in c_u)[f_{i_{j+1}}(A) = \overline{x}_v]. \tag{3}$$

According to the definition of $f_{i_{j+1}}(A)$ (see (1) and (3)) we get

$$\overline{x}_v = f_{i_{j+1}}(A) = \min_{r \in c_u} \overline{x}_r \otimes x^\oplus_r(A) = \min_{r \in c_u} \overline{x}_r = \min_{r \in c_u} \overline{x}_r \otimes x^\oplus_r(\overline{A}^\pi) = f_{i_{j+1}}(\overline{A}^\pi). \tag{4}$$

Thus, we obtain

$$\overline{x}_v = f_{i_{j+1}}(A) = f_{i_{j+1}}(\overline{A}^{\pi}) < x^{\oplus}_{i_{j+1}}(A) \leq x^{\oplus}_{i_{j+1}}(\overline{A}^{\pi})$$

which contradicts the assumption $f_{i_{j+1}}(\overline{A}^{\pi}) = x^{\oplus}_{i_{j+1}}(\overline{A}^{\pi})$.

Case 3. $a_{i_t i_{t+1}} = \min\limits_{(k,l)\in c_u} a_{kl} = x^{\oplus}_{i_{t+1}}(A) = f_{i_{t+1}}(A) \Rightarrow \overline{x}_{i_{t+1}} \leq x^{\oplus}_{i_{t+1}}(A)$, $t \neq j$.
Then we have $a_{i_t i_{t+1}} = \overline{x}_{i_{t+1}}$ and $a_{i_t i_{t+1}} < \overline{a}^{\pi}_{i_j i_{j+1}} = x^{\oplus}_{i_{j+1}}(\overline{A}^{\pi})$ (if $a_{i_t i_{t+1}} = \overline{a}^{\pi}_{i_j i_{j+1}}$, then $a_{i_j i_{j+1}} \geq a_{i_t i_{t+1}} = \overline{a}_{i_j i_{j+1}}$ and it implies $a_{i_j i_{j+1}} = a_{i_t i_{t+1}}$, using Case 1 we get a contradiction). Further by (4) we obtain the following:

$$a_{i_t i_{t+1}} = \overline{x}_{i_{t+1}} = f_{i_{t+1}}(A) = f_{i_{t+1}}(\overline{A}^{\pi}) = f_{i_{j+1}}(\overline{A}^{\pi}) < x^{\oplus}_{i_{j+1}}(\overline{A}^{\pi})$$

which contradicts the assumption $f_{i_{j+1}}(\overline{A}^{\pi}) = x^{\oplus}_{i_{j+1}}(\overline{A}^{\pi})$.

Case 4. $a_{i_t i_{t+1}} = \min\limits_{(k,l)\in c_u} a_{kl} = x^{\oplus}_{i_{t+1}}(A) > f_{i_{t+1}}(A)$, $t \neq j$.
In this case we have that

$$x^{\oplus}_{i_{t+1}}(A) > f_{i_{t+1}}(A) \Rightarrow (\exists v \in c_u)[f_{i_{t+1}}(A) = \overline{x}_v] \tag{5}$$

and hence we get

$$\overline{x}_v = f_{i_{t+1}}(A) = f_{i_{t+1}}(\overline{A}^{\pi}) < x^{\oplus}_{i_{t+1}}(A) \leq x^{\oplus}_{i_{t+1}}(\overline{A}^{\pi}) = x^{\oplus}_{i_{j+1}}(\overline{A}^{\pi})$$

which contradicts the assumption $f_{i_{j+1}}(\overline{A}^{\pi}) = x^{\oplus}_{i_{j+1}}(\overline{A}^{\pi})$. □

Notice that Theorem 4.2 implies that the computational complexity of a procedure based on checking all permutations can be exponentially large. We are able neither to suggest polynomial algorithm nor to prove NP-hardness of the possible $\boldsymbol{X}$-conformity.

5. UNIVERSAL $\boldsymbol{X}$-SIMPLICITY

Definition 5.1. An interval matrix $\boldsymbol{A}$ is called level $c(\boldsymbol{A})$-permutation if

$$\underline{a}_{i_j i_{j+1}} \geq c(\underline{A}) \text{ for } (i_j, i_{j+1}) \in c_u \ \wedge \ \overline{a}_{i_j i_{j+1}} < c(\underline{A}) \text{ for } (i_j, i_{j+1}) \notin c_u,$$

where

$$(i_1, \dots, i_n) = (i^1_1, \dots, i^1_{s_1}) \dots (i^k_1, \dots, i^k_{s_k})$$

is a permutation of N such that $c_u = (i^u_1, \dots, i^u_{s_u})$ is an elementary cycle in $G(\underline{A}, c(\underline{A}))$, $u = 1, \dots, k$.

Lemma 5.2. Let $\boldsymbol{A}$ be an interval matrix. Then $\boldsymbol{A}$ is level $c(\boldsymbol{A})$-permutation if and only if each $A \in \boldsymbol{A}$ is level $c(A)$-permutation.

P r o o f . Suppose that each $A \in \boldsymbol{A}$ is level $c(A)$-permutation. Then the matrix $\underline{A} \in \boldsymbol{A}$ is level $c(\underline{A})$-permutation as well and the condition $\underline{a}_{i_j i_{j+1}} \geq c(\underline{A})$ for $(i_j, i_{j+1}) \in c_u$ of the last definition trivially follows.
For the sake of a contradiction assume that there exists $(s,v) \notin \bigcup_{u=1}^{k} c_u$ such that $\overline{a}_{s,v} \geq c(\underline{A})$. We shall show that the matrix $\tilde{A} = (\tilde{a}_{kl})$ is not level $c(\tilde{A})$-permutation, where

$$\tilde{a}_{kl} = \begin{cases} \overline{a}_{kl}, & \text{if } (k,l) = (s,v) \\ \underline{a}_{kl}, & \text{otherwise.} \end{cases} \tag{6}$$

We shall consider two cases.

Case 1: If $c(\underline{A}) = c(\tilde{A})$ then it trivially follows that $\tilde{A}$ is not level $c(\tilde{A})$-permutation.

Case 2. If $c(\underline{A}) < c(\tilde{A})$, then $\underline{a}_{s,v} < c(\underline{A}) < \overline{a}_{s,v}$ (the first inequality follows from the fact that the matrix $\underline{A} \in \boldsymbol{A}$ is level $c(\underline{A})$-permutation and the second one from the assumption $c(\underline{A}) < c(\tilde{A})$). Now we shall construct the next auxiliary matrix $\hat{A} = (\hat{a}_{kl})$, where

$$\hat{a}_{kl} = \begin{cases} c(\underline{A}), & \text{if } (k,l) = (s,v) \\ \underline{a}_{kl}, & \text{otherwise.} \end{cases} \tag{7}$$

It is clearly to see that $c(\underline{A}) = c(\hat{A})$ and $\hat{A}$ is not level $c(\hat{A})$-permutation. This is a contradiction.

The inverse implication trivially holds true from the following inequalities

$$a_{i_j i_{j+1}} \geq \underline{a}_{i_j i_{j+1}} \geq c(\underline{A}) \text{ for } (i_j, i_{j+1}) \in c_u$$

and

$$a_{i_j i_{j+1}} \leq \overline{a}_{i_j i_{j+1}} < c(\underline{A}) \text{ for } (i_j, i_{j+1}) \notin c_u.$$

□

Lemma 5.3. Let $\boldsymbol{A}$ be an interval matrix, $\boldsymbol{X} = [\underline{x}, \overline{x}] \subseteq \mathbb{B}(n)$ be an interval vector and $\underline{x} < c^*(\underline{A}) \leq c^*(\overline{A}) \leq \overline{x}$. If $\boldsymbol{A}$ has universal $\boldsymbol{X}$-simple image eigenspace then $\boldsymbol{A}$ is level $c(\boldsymbol{A})$-permutation.

P r o o f . Let $\boldsymbol{A}$ have universal $\boldsymbol{X}$-simple image eigenspace, i.e., each matrix $A \in \boldsymbol{A}$ has $\boldsymbol{X}$-simple image eigenspace. Then by Lemma 3.3 each matrix $A \in \boldsymbol{A}$ is level $c(A)$-permutation and hence by Lemma 5.2 the interval matrix $\boldsymbol{A}$ is level $c(\boldsymbol{A})$-permutation. □

Theorem 5.4. Let $\boldsymbol{A}$ be an interval matrix, $\boldsymbol{X} = [\underline{x}, \overline{x}] \subseteq \mathbb{B}(n)$ be an interval vector and $\underline{x} < c^*(\underline{A}) \leq c^*(\overline{A}) \leq \overline{x}$ and $\boldsymbol{A}$ be a level $c(\boldsymbol{A})$-permutation matrix. Then $\boldsymbol{A}$ has universal $\boldsymbol{X}$-simple image eigenspace if and only if $\boldsymbol{A}$ is universally $\boldsymbol{X}$-conforming.

P r o o f . The proof follows from the next equivalences: $\boldsymbol{A}$ is universally $\boldsymbol{X}$-conforming $\Leftrightarrow$ each matrix $A \in \boldsymbol{A}$ is $\boldsymbol{X}$-conforming $\Leftrightarrow$ each matrix $A \in \boldsymbol{A}$ has $\boldsymbol{X}$-simple image eigenspace $\Leftrightarrow$ $\boldsymbol{A}$ has universal $\boldsymbol{X}$-simple image eigenspace. □

Theorem 5.5. Let $\boldsymbol{X} = [\underline{x}, \overline{x}] \subseteq \mathbb{B}(n)$ be an interval vector such that $\underline{x} < c^*(\underline{A}) \leq c^*(\overline{A}) \leq \overline{x}$ and $\boldsymbol{A}$ be a level $c(\boldsymbol{A})$-permutation matrix. Then $\boldsymbol{A}$ is universally $\boldsymbol{X}$-conforming if and only if

(i) $\underline{x}_{i_{j+1}} < e_{i_{j+1}} \Rightarrow \overline{a}_{i_j k} < e_{i_j}$ for $k \neq i_{j+1},\ k \in N$

(ii) $\underline{x}_{i_{j+1}} = e_{i_{j+1}} \Rightarrow \overline{a}_{i_j k} \leq e_{i_j}$ for $k \neq i_{j+1},\ k \in N$

(iii) $(\forall (i_j, i_{j+1}) \in c_u)[\underline{a}_{i_j i_{j+1}} \leq \min_{(k,l)\in c_u} \overline{a}_{kl} \Rightarrow \overline{x}_{i_{j+1}} \leq \underline{a}_{i_j i_{j+1}}]$,

where $(i_1, \ldots, i_n) = (i_1^1, \ldots, i_{s_1}^1) \ldots (i_1^k, \ldots, i_{s_k}^k)$ is a permutation of N with $\underline{a}_{i_j i_{j+1}} \geq c(\underline{A})$ ($c_u = (i_1^u, \ldots, i_{s_u}^u)$ is an elementary cycle in digraph $G(\underline{A}, c(\underline{A}))$, $u = 1, \ldots, k$) and $e_{\underline{x}} = (e_1, \ldots, e_n)^T$ with $e_i = \max_{v \in c_u} \underline{x}_v$ for $i \in c_u$, $u \in \{1, \ldots, k\}$.

Proof. Suppose that $\underline{x} < c^*(\underline{A}) \leq c^*(\overline{A}) \leq \overline{x}$, $\boldsymbol{A}$ is level $c(\boldsymbol{A})$-permutation and $\boldsymbol{A}$ is universally $\boldsymbol{X}$-conforming, i. e., each matrix $A \in \boldsymbol{A}$ is $\boldsymbol{X}$-conforming. In particular, $\overline{A} \in \boldsymbol{A}$ is $\boldsymbol{X}$-conforming and by Definition 3.4 we obtain

(i) $\underline{x}_{i_{j+1}} < e_{i_{j+1}} \Rightarrow \overline{a}_{i_j k} < e_{i_j}$ for $k \neq i_{j+1},\ k \in N$

(ii) $\underline{x}_{i_{j+1}} = e_{i_{j+1}} \Rightarrow \overline{a}_{i_j k} \leq e_{i_j}$ for $k \neq i_{j+1},\ k \in N$.

To prove the condition (iii) of the theorem suppose that

$$(\exists (r, s) \in c_u)[\underline{a}_{rs} \leq \min_{(k,l)\in c_u} \overline{a}_{kl} \wedge \overline{x}_s > \underline{a}_{rs}].$$

Define an auxiliary matrix $\tilde{A} = (\tilde{a}_{ij})$ with

$$\tilde{a}_{ij} = \begin{cases} \underline{a}_{rs} \oplus \underline{a}_{ij}, & \text{if } (i,j) \in c_u \\ \underline{a}_{ij}, & \text{otherwise.} \end{cases} \tag{8}$$

For the level $c(\tilde{A})$-permutation matrix $\tilde{A}$ we can write

$$\tilde{a}_{rs} = \underline{a}_{rs} = \min_{(k,l)\in c_u} \tilde{a}_{kl} = x_s^{\oplus}(\tilde{A})(= x_r^{\oplus}(\tilde{A})) = f_s(\tilde{A}) \Rightarrow \overline{x}_s \leq x_s^{\oplus}(\tilde{A}).$$

Hence we obtain the following $\overline{x}_s \leq x_s^{\oplus}(\tilde{A}) = \underline{a}_{rs} < \overline{x}_s$. This is a contradiction.

To prove the inverse implication suppose that conditions (i), (ii), (iii) of the theorem hold true and $A \in \boldsymbol{A}$ is an arbitrary but fixed matrix. We shall prove that A fulfills the conditions of the Definition 3.4. The first and the second condition of the Definition 3.4 straightly follows from the implications

(i) $\underline{x}_{i_{j+1}} < e_{i_{j+1}} \Rightarrow a_{i_j k} \leq \overline{a}_{i_j k} < e_{i_j}$ for $k \neq i_{j+1},\ k \in N$

(ii) $\underline{x}_{i_{j+1}} = e_{i_{j+1}} \Rightarrow a_{i_j k} \leq \overline{a}_{i_j k} \leq e_{i_j}$ for $k \neq i_{j+1},\ k \in N$.

To prove the third condition of the Definition 3.4 suppose that

$$(\forall (i_j, i_{j+1}) \in c_u)[\underline{a}_{i_j i_{j+1}} \leq \min_{(k,l)\in c_u} \overline{a}_{kl} \Rightarrow \overline{x}_{i_{j+1}} \leq \underline{a}_{i_j i_{j+1}}]$$

and for the sake of a contradiction assume that

$$(\exists (i_j, i_{j+1}) \in c_u)[a_{i_j i_{j+1}} = \min_{(k,l)\in c_u} a_{kl} = x^{\oplus}_{i_{j+1}}(A) = f_{i_{j+1}}(A) \wedge \overline{x}_{i_{j+1}} > x^{\oplus}_{i_{j+1}}(A)].$$

Then we obtain the following contradiction

$$a_{i_j i_{j+1}} = x^{\oplus}_{i_{j+1}}(A) < \overline{x}_{i_{j+1}} \leq \underline{a}_{i_j i_{j+1}} \leq a_{i_j i_{j+1}}.$$

□

According to Theorem 5.5 the complexity of checking the $\boldsymbol{X}$-simplicity of eigenspace of a given interval matrix $\boldsymbol{A}$ and an interval vector $\boldsymbol{X}$ in case that $\underline{x} < c^*(\underline{A}) \leq c^*(\overline{A}) \leq \overline{x}$ consists of $O(n^2)$ arithmetic operations needed for the checking conditions (i) – (iii) and the condition of $\boldsymbol{A}$ to be a level $c(\boldsymbol{A})$-permutation matrix.

Example 5.6. Let us consider $\mathbb{B} = [0, 10]$, $\lambda = 10$ and

$$\underline{A} = \begin{pmatrix} 0 & 0 & 1 & 5 \\ 1 & 1 & 6 & 0 \\ 0 & 8 & 0 & 1 \\ 5 & 1 & 0 & 0 \end{pmatrix}, \ \overline{A} = \begin{pmatrix} 3 & 3 & 2 & 5 \\ 2 & 2 & 7 & 2 \\ 3 & 9 & 3 & 3 \\ 7 & 2 & 1 & 2 \end{pmatrix}, \ \underline{x} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 3 \end{pmatrix}, \ \overline{x} = \begin{pmatrix} 5 \\ 6 \\ 6 \\ 5 \end{pmatrix}.$$

Matrix $\underline{A}$, $\overline{A}$ are level 5-permutation with $c_1 = (i_1, i_2) = (1, 4)$, $c_2 = (i_3, i_4) = (2, 3)$ and $e_{\underline{x}} = (3, 3, 3, 3)^T$.

Now, we shall argue that $\boldsymbol{A}$ is $\boldsymbol{X}$-conforming,

$$\begin{aligned} i_1 = 1, i_2 = 4;\ \ & \underline{x}_1 < e_1 \Rightarrow \overline{a}_{4j} < e_4 \ (\forall j \neq 1), \\ i_2 = 4, i_1 = 1;\ \ & \underline{x}_4 = e_4 \Rightarrow \overline{a}_{1j} \leq e_1 \ (\forall j \neq 4), \\ i_3 = 2, i_4 = 3;\ \ & \underline{x}_2 = e_2 \Rightarrow \overline{a}_{3j} \leq e_3 \ (\forall j \neq 2), \\ i_4 = 3, i_3 = 2;\ \ & \underline{x}_3 < e_3 \Rightarrow \overline{a}_{2j} < e_2 \ (\forall j \neq 3) \end{aligned}$$

and

$$\begin{aligned} \underline{a}_{14} = 5 \leq \min_{(k,l)\in c_1} \overline{a}_{kl} = 5 \Rightarrow \overline{x}_4 = 5 \leq \underline{a}_{14} = 5, \\ \underline{a}_{41} = 5 \leq \min_{(k,l)\in c_1} \overline{a}_{kl} = 5 \Rightarrow \overline{x}_1 = 5 \leq \underline{a}_{41} = 5, \\ \underline{a}_{23} = 6 \leq \min_{(k,l)\in c_2} \overline{a}_{kl} = 7 \Rightarrow \overline{x}_3 = 6 \leq \underline{a}_{23} = 6, \\ \underline{a}_{32} = 8 \not\leq \min_{(k,l)\in c_2} \overline{a}_{kl} = 7. \end{aligned}$$

Hence matrix $\boldsymbol{A}$ is $\boldsymbol{X}$-conforming.

ACKNOWLEDGEMENTS

The authors would like to thank the referees for the invaluable remarks and suggestions that helped to increase the clarity of arguments in the paper.

(Received May 9, 2017)

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