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QMLE OF PERIODIC BILINEAR MODELS AND OF PARMA MODELS WITH PERIODIC BILINEAR INNOVATIONS

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This paper develops an asymptotic inference theory for bilinear (BL) time series models with periodic coefficients (PBL for short). For this purpose, we establish firstly a necessary and sufficient conditions for such models to have a unique stationary and ergodic solutions (in periodic sense). Secondly, we examine the consistency and the asymptotic normality of the quasi-maximum likelihood estimator ($QMLE$) under very mild moment condition for the innovation errors. As a result, it is shown that whenever the model is strictly stationary, the moment of some positive order of PBL model exists and is finite, under which the strong consistency and asymptotic normality of $QMLE$ for PBL are proved. Moreover, we consider also the periodic $ARMA$ ($PARMA$) models with PBL innovations and we prove the consistency and the asymptotic normality of its $QMLE$.

Keywords: periodic bilinear model, periodic $ARMA$ model, strict and second-order periodic stationarity, strong consistency, asymptotic normality

Classification: 2M10, 62M15

1. INTRODUCTION

Periodically varying parameters can arise in modelling nonstationary time series having significant periodic behavior in mean, variance and in covariance structures, namely in economic, hydrological and meteorological ones. Data of this type are frequently analyzed using a $s-$periodic autoregressive moving average ($PARMA_s$) models (interested readers are advised to see Gardner et al. [21] for references dealing with $PARMA_s$ models). However, many real time series encountered in practice exhibit not only nonstationary behavior, but also certain phenomena commonly observed by the practitioners such as, limit cycles, self-excitation, asymmetric distributions, leptokurtosis and sudden jumping that cannot be adequately modeled by linear models and hence the resort to some non linear models becomes inevitable. Among others, the most prominent discrete-time model for modelling the non-Gaussian and nonstationary time series is certainly the bilinear models with time-varying coefficients which have been attracting a great deal of interest in the recent statistical literature. This class of models is an extension of $PARMA_s$ models by adding one or more interaction components between the observed and the innovations processes. It becomes an appealing tool for investigating both non Gaussianty and distinct "seasonal" patterns appearing for instance in

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finance, macroeconomics, econometrics, etc..., and continue to gain a growing interest and a relevant attention of researchers. In other words, a discrete-time process $(X_t)_{t\in\mathbb{Z}}$, $\mathbb{Z}=\{0,\pm1,\pm2,\ldots.\}$ defined on some probability space $(\Omega,\Im,P)$, is called periodic bilinear model denoted by $PBL_s(p,q,P,Q)$ if it satisfies the following nonlinear stochastic difference equation

$$X_n = a_0\left(n\right) + \sum_{i=1}^{p} a_i(n)X_{n-i} + \sum_{j=0}^{q} b_j(n)e_{n-j} + \sum_{j=1}^{Q}\sum_{i=1}^{P} c_{ij}(n)X_{n-i}e_{n-j}. \tag{1.1}$$

In (1.1), $(e_n)_{n\in\mathbb{Z}}$ is a sequence of independent and identically distributed ($i.i.d.$) random variables defined on the same probability space $(\Omega,\mathcal{A},P)$, with $E\left\{e_n\right\}=0$ and $E\left\{\log^+|e_n|\right\}<+\infty$ where $\log^+ x=\max(\log x,0)$, $x>0$, and e_k is independent of X_n for $k>n$, the coefficients $a_i(n)$, $b_j(n)$ and $c_{ij}(n)$ are periodic functions with period s. Therefore, by setting $n=st+v$, $X_{st+v}=X_t\left(v\right)$ and $e_{st+v}=e_t\left(v\right)$, Model (1.1) may be equivalently written in periodic version as

$$X_t\left(v\right) = a_0(v) + \sum_{i=1}^{p} a_i(v)X_t(v-i) + \sum_{j=0}^{q} b_j(v)e_t(v-j) + \sum_{j=1}^{Q}\sum_{i=1}^{P} c_{ij}(v)X_t(v-i)e_t(v-j). \tag{1.2}$$

In (1.2) the notation $X_t\left(v\right)$ refers to X_t during the $v-$th "season" or regime $v\in\{1,\ldots,s\}$ of cycle t and hence the functions $a_i(v)$, $b_j(v)$ and $c_{ij}(v)$ may be interpreted as the coefficients of model corresponding to the $v-$th regime. For the convenience, $X_t\left(v\right)=X_{t-1}\left(v+s\right)$, $e_t\left(v\right)=e_{t-1}\left(v+s\right)$ if $v<0$. The non-periodic notations $\{X_n\}$, $\{e_n\}$, $\{a_i(n)\}$, $\{b_i(n)\}$, $\{c_{ij}(n)\}$, etc. will be used interchangeably with the periodic notations $\{X_t\left(v\right)\}$, $\{e_t\left(v\right)\}$, $\{a_i(v)\}$, $\{b_i(v)\},\{c_{ij}(v)\}$ etc. whenever emphasis on seasonality is not paramount. It is worth noting that when $s>1$, the process is globally nonstationary, but is stationary within each period.

Remark 1.1. Recently, many empirical works have showed after fitting some data to bilinear models, that the $(G)\,ARCH$ effect in residual process is significant (see Pan et al. [30] for more general settings) and hence the $i.i.d$ hypothesis on $(e_n)_{n\in\mathbb{Z}}$ may be relaxed for an heteroscedastic process.

The model (1.1) has been studied by Bibi et al. {[5, 7, 8, 9, 10, 11]}, more precisely, with respect to the probabilistic properties, Bibi [5], beside the motivations proposed to introduce the class of bilinear models with time-dependent coefficients, he also studied the $\mathbb{L}_2-$structure and some empirical properties of such models. In particular, with periodic coefficients, Bibi and Lessak [9] have established some sufficient conditions for the existence of causal, periodically correlated (PC) and $\beta-$mixing of the solution process for some specifications of (1.2) (see the subsection 2.2 for a definition of PC processes). More general, necessary and sufficient conditions for the existence of PC solution process of (1.2) are given by Bibi and Lescheb [10]. From statistical point of view of such models, Bibi and Oyet [7] have studied a subclass of time-dependent coefficients (not necessarily periodic) and established the consistency and the normality asymptotic (CAN) of the conditional least square estimator. The asymptotic properties of Yule-Walker type estimator have been established by Bibi and Aknouche [8] for some restrictive periodic bilinear models. Recently, Bibi and Ghezal [11] have studied the CAN properties of the generalized method of moments (GMM) estimator for the model (1.2).

In this paper, we propose the conditional quasi-maximum likelihood method for estimating the unknown parameters of $PBL_s\,(p,q,P,Q)$ and we prove its CAN properties. Moreover, the proposal method is also applied for $PARMA_s$ model with some periodic BL innovation. Note here that the $PBL_s\,(p,q,P,Q)$ encompasses many commonly used models in the literature, indeed,

(i) Standard $BL\,(p,q,P,Q)$ models: These models are obtained by assuming constant the functions $a_i(.)$, $b_j(.)$ and $c_{ij}(.)$ in (1.1) (e. g., Subba Rao and Gabr [32] and the references therein), i. e.,

$$X_n = a_0 + \sum_{i=1}^{p} a_i X_{n-i} + \sum_{j=0}^{q} b_j e_{n-j} + \sum_{j=1}^{Q}\sum_{i=1}^{P} c_{ij} X_{n-i} e_{n-j}.$$

(ii) Periodic $ARMA$ models ($PARMA_s$): These models are obtained by setting $c_{ij}(.) = 0$ for all i and j in (1.1) (e. g., Francq et al. [18] for recent references on $PARMA_s$ models), i. e.,

$$X_n = a_0\,(n) + \sum_{i=1}^{p} a_i(n) X_{n-i} + \sum_{j=0}^{q} b_j(n) e_{n-j}.$$

(iii) Some classes of periodic $GARCH\,(p,q)$ ($PGARCH_s$): (see Bibi and Lessak [9] and Kristensen [26] for the building of $GARCH$ models as special case of BL), i. e. $X_n = \sqrt{h_n} e_n$ where the volatility process $(h_n)_n$ satisfy

$$h_n = a_0\,(n) + \sum_{i=1}^{p} a_i(n) h_{n-i} + \sum_{i=1}^{p} c_{ii}(n) h_{n-i} e_{n-i}^2,$$

in which the sequences $(a_i(n), 0 \le i \le p)_{n\in\mathbb{Z}}$, $(c_{ii}(n), 1 \le i \le p)_{n\in\mathbb{Z}}$ are positive with $a_0\,(n) > 0$ for all n.

Since, it is difficult to handle in (1.1) the product terms, like $X_n e_{n-k}$, $k > 0$, so in the sequel, we shall restrict ourselves on the so-called superdiagonal models noted $SPBL_s(p,q,p,q)$ in which $c_{ij}(n) = 0$ in (1.1) for all $i < j$ and all $n \in \mathbb{Z}$, i. e.,

$$X_n = a_0\,(n) + \sum_{i=1}^{p} a_i(n) X_{n-i} + \sum_{j=0}^{q} b_j(n) e_{n-j} + \sum_{j=1}^{q}\sum_{i=j}^{p} c_{ij}(n) X_{n-i} e_{n-j}. \tag{1.3}$$

The main purpose of this paper is to investigate some probabilistic and statistical properties of equation (1.3). Recalling here that a process $(Y_t)_{t\in\mathbb{Z}}$, defined on some probability space $(\Omega, \mathcal{A}, P)$ is said to be strictly periodically stationary (SPS) (with period $s > 0$) if the distribution of $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})'$ is the same as that of $(Y_{t_1+sh}, Y_{t_2+sh}, \ldots, Y_{t_n+sh})'$ for all $n \in \mathbb{N}^*$ and h, $t_1, t_2, \ldots, t_n \in \mathbb{Z}$. Furthermore, it is called *periodically ergodic*

(PE) if for all Borel set B and all integer m, $\frac{1}{n}\sum_{t=1}^{n}\mathbb{I}_B\left(Y_{v+st}, Y_{v+1+st}, \ldots, Y_{v+m+st}\right)$ converges almost surely $(a.s)$ to $P\left((Y_v, Y_{v+1}, \ldots, Y_{v+m}) \in B\right)$ as $n \to \infty$ for all $1 \leq v \leq s$, where $\mathbb{I}_B(.)$ denotes the indicator function of the set B (cf., Boyles and Gardner [14] for further discussion). As for the stationary case (see for instance Billingsley [12] Theorem 36.4), periodic ergodicity is closed under certain transformations. In particular if $(\varepsilon_t)_{t\in\mathbb{Z}}$ is SPS and PE and if $(Y_t)_{t\in\mathbb{Z}}$ is defined by $Y_t = f(\ldots, \varepsilon_t, \varepsilon_{t+1}, \ldots)$ where f is some measurable function mapping from $\mathbb{R}^\infty$ to $\mathbb{R}$, then $(Y_t)_{t\in\mathbb{Z}}$ is also SPS and PE. Moreover, if $(Y_t)_{t\in\mathbb{Z}}$ is SPS and PE and if f is a measurable function from $\mathbb{R}^\infty$ to $\mathbb{R}$ such that $E\{f(\ldots, Y_{t-1}, Y_t, Y_{t+1}, \ldots)\} < \infty$ then $\frac{1}{n}\sum_{t=1}^{n} f\left(\ldots, Y_{v+s(t-1)}, Y_{v+st}, Y_{v+s(t+1)}, \ldots\right)$ converges $a.s.$ to $E\left\{f\left(\ldots, Y_{v+s(t-1)}, Y_{v+st}, Y_{v+s(t+1)}, \ldots\right)\right\}$ as $n \to \infty$ for all $1 \leq v \leq s$. Before we proceed, we need to introduce some notations:

1.1. Algebraic notation

Throughout the paper, the following notations are used.

- $I_{(n)}$ is the $n \times n$ identity matrix, $O_{(k,l)}$ denotes the matrix of order $k \times l$ whose entries are zeros, for simplicity we set $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$.
- The spectral radius of square matrix M is noted $\rho(M)$.
- Let $\|.\|$ denote any induced matrix norm on the set of $m \times n$ and $m \times 1$ matrices. For any $\gamma \in]0, 1]$, we set $|M|^\gamma := (|m_{ij}|^\gamma)$, then it is easy to see that the operator $|.|^\gamma$ is submultiplicative, i.e., $|M_1M_2|^\gamma \leq |M_1|^\gamma |M_2|^\gamma$, $|M\underline{X}|^\gamma \leq |M|^\gamma |\underline{X}|^\gamma$ for any appropriate vector $\underline{X}$ and $\left|\sum_i M_i\right|^\gamma \leq \sum_i |M_i|^\gamma$ where the inequality $M \leq N$ denotes the elementwise relation, i.e., $m_{ij} \leq n_{ij}$ for all i and j.
- The symbol $\otimes$ is the usual Kronecker product of matrices and $M^{\otimes r} = M \otimes M \otimes \ldots \otimes M$ $r-$times, $Vec(M)$ is the vector obtained from a matrix $M := (m_{ij})$ by setting down the column of M underneath each other, " $\rightsquigarrow$ ", "$p\lim$" and "$a.s.$" respectively means convergence in law, in probability and almost surely.

The remaining sections are organized as follows. In the next section, we first give necessary and sufficient conditions ensuring the existence of SPS, causal and PE solution of (1.3) and other probabilistic properties such as existence of moments for some finite order. The conditions are shown to reduce to the usual conditions that we find often in literature of time series and especially for the models cited above. Section 3 deals with the strong consistency and the asymptotic normality properties of conditional $QMLE$ for $SPBL_s$ while in Section 4, we investigate those of $PARMA_s$ models with periodic bilinear innovations. Section 5, concludes the article.

2. MARKOVIAN REPRESENTATION AND ITS PROBABILISTIC PROPERTIES

With constant coefficients (i.e., when $s = 1$), the Markovian representation and its properties were discussed in [31]. This representation can easily be extended to the class of $SPBL_s(p,q,p,q)$ models as follows $X_n = \underline{H}'\underline{X}_{n-1} + a_0(n) + b_0(n)e_n$ and

$$\underline{X}_n = \Gamma_n(e_n)\underline{X}_{n-1} + \underline{\eta}_n(e_n), \tag{2.2}$$

in which $\Gamma_n(e_n) = \Gamma_0(n) + e_n\Gamma_1(n)$, $\underline{\eta}_n(e_n) = \underline{b}_0(n) + \underline{b}_1(n)e_n + \underline{b}_2(n)e_n^2$ where

$$\Gamma_0(n) = \begin{pmatrix} J & A_{(0)}(n) \\ A_{(1)}(n) & A_{(2)}(n) \end{pmatrix}_{r\times r}, \Gamma_1(n) = \begin{pmatrix} O_{(p,p)} & O_{(p,q)} \\ C_{(1)}(n) & C_{(2)}(n) \end{pmatrix}_{r\times r}.$$

The matrices J, $\left(A_{(i)}(n)\right)_{0\leq i\leq 2}$ and $\left(C_{(j)}(n)\right)_{1\leq j\leq 2}$ and the vectors $\underline{H}$, $\underline{b}_0(n), \underline{b}_1(n)$ and $\underline{b}_2(n)$ are periodic in n with period s, its explicit forms can be found in Bibi [5]. Equation (2.2) is the same as defining periodic random coefficient autoregressions ($PRCAR$) (see Aknouche and Guerbyenne [2]) except that the matrix $\Gamma_n(e_n)$ is not independent of the vector $\underline{\eta}_n(e_n)$ as it is required in these models, whereas $\Gamma_n(e_n)$ and $\underline{\eta}_m(e_m)$ are independent for all $m \neq n$. Noting here that, because of $s-$periodic time-varying coefficients in equation (2.2), the solution process (when existing) are not strictly stationary nor ergodic unless $s = 1$. To remedy this problem, the idea is to introduce the concept of SPS and PE solutions already discussed above. For this purpose, iterate (2.2) $s-$times to get the following equation $\underline{X}_{(n+1)s} = \Gamma(\underline{e}_n)\underline{X}_{ns} + \underline{\eta}(\underline{e}_n)$ where $\Gamma(\underline{e}_n) = \left\{\prod_{i=0}^{s-1} \Gamma_{s-i}\left(e_{(n+1)s-i}\right)\right\}$ and $\underline{\eta}(\underline{e}_n) = \sum_{j=1}^{s}\left\{\prod_{i=0}^{s-j-1}\Gamma_{s-i}\left(e_{(n+1)s-i}\right)\right\}\underline{\eta}_j(e_{ns+j})$ with $\underline{e}_{n+1} = \left(e_{(n+1)s}, \ldots .e_{sn+1}\right)$. Now, let $\underline{X}(n) = \underline{X}_{ns}$ and rewrite the above equation as

$$\underline{X}(n) = \Gamma(\underline{e}_n)\underline{X}(n-1) + \underline{\eta}(\underline{e}_n). \tag{2.3}$$

It is worth noting however that $\Gamma(\underline{e}_n)$ is independent of $\underline{X}(k)$ for any $k < n$ and $\left(\left(\Gamma(\underline{e}_n), \underline{\eta}(\underline{e}_n)\right)\right)_{n\in\mathbb{Z}}$ is an *i.i.d.* process and hence a formal solution for (2.3) can be given by

$$\underline{X}(n) = \underline{\eta}(\underline{e}_n) + \sum_{k\geq 1}\left\{\prod_{j=0}^{k-1}\Gamma(\underline{e}_{n-j})\right\}\underline{\eta}(\underline{e}_{n-k}). \tag{2.4}$$

Therefore, (1.3) has a unique, causal, SPS and PE solution given by $(\underline{H}'\underline{X}(n-1)+a_0(n) + b_0(n)e_n)_{n\in\mathbb{Z}}$, if and only if (2.4) has a unique, causal, strictly stationary and ergodic solution.

2.1. Strict periodic stationarity

Processes $(\underline{X}(n))_{n\in\mathbb{Z}}$ similar to (2.3) has been examined by Brandt [15] (in scalar case) and by Bougerol and Picard [13], who have showed that the unique solution for (2.3) should be given by the series (2.4) if and only if

$$\gamma(\Gamma) < 0, \tag{2.5}$$

where $\gamma(\Gamma)$ is the top$-$Lyapunov exponent associated with the strictly stationary and ergodic sequence of random matrices $(\Gamma(\underline{e}_n))_{n\in\mathbb{Z}}$ defined by

$$\gamma(\Gamma) := \inf_{n\geq 1}\frac{1}{n}E\left\{\log\left\|\prod_{i=0}^{n-1}\Gamma(\underline{e}_{n-i})\right\|\right\} \overset{a.s}{=} \lim_{n\to\infty}\frac{1}{n}\log\left\|\prod_{i=0}^{n-1}\Gamma(\underline{e}_{n-i})\right\|.$$

Note that the right-hand member in above equality can be justified using Kingman's subadditive ergodic theorem and the existence of $\gamma(\Gamma)$ is guaranteed however by the fact that $E\left\{\log^{+}\|\Gamma(\underline{e}_n)\|\right\}$ and $E\left\{\log^{+}\left\|\underline{\eta}(\underline{e}_n)\right\|\right\}$ are finite. The following theorem due to Bougerol and Picard [13], gives us the main result for stochastic difference equation (2.3).

Theorem 2.1. Consider the $SPBL_s(p,q,p,q)$ model (1.3) with Markovian representation (2.2). Then, if (2.3) has a strictly stationary solution, then (2.5) hold true. Conversely, if (2.5) hold true, then for any $n \in \mathbb{Z}$, the series (2.4) converges absolutely $a.s.$ and constitute the unique, causal, strictly stationary and ergodic solution for (2.3) and hence (1.3) has a unique, causal, SPS and PE solution (given by $\left(\underline{H}'\underline{X}(n-1)+a_0(n)+b_0(n)e_n\right)_{n\in\mathbb{Z}}$).

The properties of solution process (2.4) is given in the following theorem.

Theorem 2.2. Consider the $SPBL_s(p,q,p,q)$ model (1.3) with Markovian representation (2.2). Under the condition (2.5), we have

1. Equation (2.2) has an unique, causal, SPS and PE solution given by the series (2.4).
2. The dual (multivariate) process $\left(\underline{X}'_{st+1},\underline{X}'_{st+2},\ldots,\underline{X}'_{st+s}\right)'$ is strictly stationary and ergodic.

P r o o f. The proof follows essentially the same arguments as in Bibi and Lescheb [10]. □

Though the condition (2.5) could be used as a test for the SPS, it is of little use in practice since this condition involves the limit of products of infinitely many random matrices. However, some simple sufficient conditions ensuring the negativity of $\gamma(\Gamma)$ can be given.

Theorem 2.3. Let $|\Gamma| = E\{|\Gamma(\underline{e}_n)|\}$ and $\left|\underline{\eta}\right| = E\left\{\left|\underline{\eta}(\underline{e}_n)\right|\right\}$, and consider the model (1.3) with Markovian representation (2.2), then $\gamma(\Gamma) < 0$, if one of the following conditions holds true.

1. $E\left\{\log\left\|\prod_{i=0}^{n-1}\Gamma(\underline{e}_{n-i})\right\|\right\} < 0$ for some $n \geq 1$,
2. $\rho(|\Gamma|) < 1$.

P r o o f. Because the top−Lyapunov exponent is independent of the norm, by choosing an absolute norm, i. e., a norm $\|.\|$ such that $\|.\| \leq \||.|\|$, then from the definition of $\gamma(\Gamma)$ and according to Kesten and Spitzer [25] we have almost surely $\lim\limits_{n\to\infty}\frac{1}{n}\log\left\|\prod_{i=0}^{n-1}\Gamma(\underline{e}_{n-i})\right\| \leq \log\rho(|\Gamma|)$. On the other hand, by Jensen's inequality we get almost surely

$$\gamma(\Gamma) \leq \frac{1}{n}E\left\{\log\left\|\prod_{i=0}^{n-1}\Gamma(\underline{e}_{n-i})\right\|\right\} \leq \frac{1}{n}\log E\left\{\left\|\prod_{i=0}^{n-1}\left|\Gamma(\underline{e}_{n-i})\right|\right\|\right\} \leq \log\rho(|\Gamma|),$$

so the result follows. □

Corollary 2.1. If $\gamma(\Gamma) < 0$ then there is a $\delta \in]0,1]$ such that $E\left\{\left\|\underline{X}(n)\right\|^{\delta}\right\} < +\infty$ and hence $E\left\{|X_n|^{\delta}\right\} < +\infty$, $\forall n \in \mathbb{Z}$.

Proof. See for instance Aknouche and Bibi [3]. □

Example 2.1. For $PBL_s(1,1,1,1)$ model, we obtain $\Gamma(\underline{e}_n) := \prod_{v=0}^{s-1}(a_1(s-v) + c_{11}(s-v)e_{n-1}(s-v-1))$. Hence $PBL_s(1,1,1,1)$ admits a causal, SPS and PE solution if and only if $\gamma(\Gamma) := \sum_{i=0}^{s-1} E\{\log(|a_1(i) + c_{11}(i)e_0|\} < 0$. It is worth noting that the existence of "explosive regimes" (i. e., $E\{\log(|a_1(i) + c_1(i)e_0|\} > 0$) does not preclude the existence of SPS solution.

Example 2.2. For the $PARMA_s$ model, the necessary and sufficient condition reduces to $\rho\left(\prod_{v=0}^{s-1} A_{(2)}(v)\right) < 1$ where $A_{(2)}(n)$ is the companion matrix associated with $PAR_s(p)$ part in $PARMA_s$ model.

Remark 2.1. The condition (2.5) provides a certain stability of model (1.3). However when $\gamma(\Gamma) \geq 0$, the model (1.3) is said to be unstable and hence does not admit a SPS solution. As an example, consider $PBL_s(0,1,1,1)$ then it is not difficult to verify that $\gamma(\Gamma) \geq 0$ if and only if $\prod_{v=0}^{s-1} |c_{11}(v)| \geq \exp(-sE\{\log|e_0|\})$. Since, if $(e_t)_{t\in\mathbb{Z}}$ follow $\mathcal{N}(0,1)$ then $E\{\log|e_0|\} = \frac{1}{2}(\log(2) + \dfrac{\Gamma'(0.5)}{\Gamma(0.5)})$ where $\Gamma(.)$ and $\Gamma'(.)$ are the Gamma function and its first derivative respectively. Hence, $\exp(-sE\{\log|e_0|\}) \approx \exp(0.1048s)$.

Remark 2.2. For the $PBL_s(1,1,1,1)$ model and if e_t admits a density, then the distribution of the polynomial $P_v(e_0) = c_{11}(v)e_0 + a_1(v)$ can be expressed in terms of the CDF of e_t and hence $\sum_{v=0}^{s-1} E\{\log|P_v(e_0)|\}$ follows.

When $p > 1$, the top-Lyapunov exponent criterion $\gamma(\Gamma)$ is defined as a product of infinitely many random matrices, so, one encounters fundamental difficulties in determination of its asymptotic distribution explicitly. However, a potential method to verify the negativity of $\gamma(\Gamma)$ is via Monte-Carlo simulation using Equation (2.4) (see for instance He et al. [24] and the references therein). On the other hand, in statistical applications, we often suggest conditions ensuring the existence of some moments for the stable process under investigation, but this suggestion cannot be achieved by the top-Lyapunov exponent criterion. Therefore, we need to search for conditions ensuring the existence of moments for the SPS solutions.

2.2. Second-order periodic stationarity

The problem of finding conditions ensuring the existence of solution of (1.3) having moments up to second-order were addressed initially by Gladyshev [22] and continues to receive more attention in literature under the so-called periodically correlated (PC) processes (e. g. [21] and the references therein). Formally a second-order process $(X_n)_{n\in\mathbb{Z}}$ is said to be PC with periodic s, if for any integers n and n', $E\{X_{n+s}\} = E\{X_n\}$ and $Cov(X_{n+s}, X_{n'+s}) = Cov(X_n, X_{n'})$. In other word, the mean function $\mu_n = E\{X_n\}$ and the covariance functions $\gamma_n(h) = Cov(X_n, X_{n+h})$, $h \in \mathbb{Z}$ are both periodic in n with s period, so when $s = 1$, a PC process is equivalent to second-order stationary process. In this subsection, we give the necessary and sufficient conditions for the existence of causal, SPS and PE solution to Equation (1.3). For this purpose and for convenience, we shall consider the centered version of the state vector (2.2), i. e., $\widetilde{\underline{X}}_n = \Gamma_n(e_n)\widetilde{\underline{X}}_{n-1} + \widetilde{\underline{\eta}}_n(e_n)$ in which $\widetilde{\underline{X}}_n = \underline{X}_n - \underline{\mu}_n$ where $\underline{\mu}_n = E\{\underline{X}_n\}$ and $\widetilde{\underline{\eta}}_n(e_n) = e_n\left(\underline{b}_1(n) + \Gamma_1(n)\underline{\mu}_{n-1}\right) + \underline{b}_2(n)\left(e_n^2 - \sigma^2\right)$ so a similar version of (2.3) is now

$$\widetilde{\underline{X}}(n) = \Gamma(\underline{e}_n)\widetilde{\underline{X}}(n-1) + \widetilde{\underline{\eta}}(\underline{e}_n), \tag{2.6}$$

in which the process $\left(\widetilde{\underline{\eta}}(\underline{e}_n)\right)_{n\in\mathbb{Z}}$ is centered and is orthogonal to $\widetilde{\underline{X}}(k)$ for any $k < n-s$. However, the process $(X_t)_{t\in\mathbb{Z}}$ has a PC and PE solution if and only if (2.6) has a second-order and ergodic solution.

Theorem 2.4. Suppose that $E\{e_n^4\} < +\infty$. Then there exists a PC process $(X_t)_{t\in\mathbb{Z}}$ generated by the equation (1.3) with state-space representation (2.6) if

$$\lambda_{(2)} = \rho\left(\Gamma^{(2)}\right) < 1, \tag{2.7}$$

where $\Gamma^{(2)} := E\left\{\Gamma^{\otimes 2}(\underline{e}_n)\right\} = \prod_{v=0}^{s-1}\left\{\Gamma_0^{\otimes 2}(s-v) + \sigma^2\Gamma_1^{\otimes 2}(s-v)\right\}$. The covariance matrix $\Sigma_{\underline{X}} = E\left\{\widetilde{\underline{X}}(n)\widetilde{\underline{X}}'(n)\right\}$ of $\left(\widetilde{\underline{X}}(n)\right)_{n\in\mathbb{Z}}$ is then given by

$$Vec\left(\Sigma_{\underline{X}}\right) = \left(I_{(r)} - \Gamma^{(2)}\right)^{-1}\underline{\Sigma}_{\underline{\eta}}^{\otimes 2},$$

where $\underline{\Sigma}_{\underline{\eta}}^{\otimes 2} = E\left\{\widetilde{\underline{\eta}}^{\otimes 2}(\underline{e}_n)\right\}$, moreover, its solution process is causal, unique, strictly stationary and ergodic.

Conversely, a necessary condition for existence of a PC process solution to (1.3) is that there exists a covariance matrix $\Sigma_{\underline{X}}$ associated with state-space representation (2.6) solution of the equation

$$\left(I_{(r)} - \Gamma^{(2)}\right)Vec\left(\Sigma_{\underline{X}}\right) = \underline{\Sigma}_{\underline{\eta}}^{\otimes 2}.$$

Proof. The proof follows essentially the same arguments as in Bibi and Lessak [9]. □

Remark 2.3. It is not difficult to see that if $E\{e_n^4\} < +\infty$ and $E\left\{\left\|\Gamma(\underline{e}_{n_0})\right\|^2\right\} < 1$ for some n_0, then $(\underline{X}_n)_{n\in\mathbb{Z}}$ is square integrable.

Example 2.3. The following table summarizes the condition (2.7) for some particular cases.

Specification	Condition (2.7)	Particular case: $p = q = 1$
Standard BL	$\rho\left(\Gamma_0^{\otimes 2}(1) + \sigma^2\Gamma_1^{\otimes 2}(1)\right) < 1$	$a^2(1) + \sigma^2 c_{11}^2(1) < 1$
$PARMA_s$	$\rho\left(\prod_{v=0}^{s-1} A_{(2)}^{\otimes 2}(v)\right) < 1$	$\prod_{v=0}^{s-1} \lvert a(v)\rvert < 1$
$PGARCH_s(p,p)^{(a)}$	$\rho\left(\prod_{v=0}^{s-1} \Xi(v)\right) < 1$	$\prod_{v=0}^{s-1} \left(a(v) + \sigma^2 c_{11}(v)\right) < 1$

$$^{(a)}\Xi(v) = \begin{pmatrix} M_0(v) & \frac{1}{\sigma^2}(\widetilde{M}_1(v) - J) \\ \widetilde{M}_0(v) & \widetilde{M}_1(v) \end{pmatrix} \text{ where } M_0(v) = \begin{pmatrix} a_1(v), \ldots, a_p(v) \\ I_{(p-1)} \quad \underline{O}_{(p-1)} \end{pmatrix}$$

$$\widetilde{M}_0(v) = \begin{pmatrix} \sigma^2 a_1(v), \ldots, \sigma^2 a_p(v) \\ O_{(p-1)} \end{pmatrix}, \widetilde{M}_1(v) = \begin{pmatrix} \sigma^2 c_{11}(v), \ldots, \sigma^2 c_{pp}(v) \\ I_{(p-1)} \quad \underline{O}_{(p-1)} \end{pmatrix}$$

Tab. 1. Conditions ensuring $E\left\{X_t^2\right\} < +\infty$ for certain specifications.

Remark 2.4. Noting here that some extensions of PC processes which can account for more complex cyclical phenomena have been proposed (see Bibi and Francq [6] for further discussion). In particular almost periodically correlated (APC) processes were introduced by Gladyshev [22] and have been discussed by several authors. A discrete−time process is said to be APC if its covariance function is an almost periodic sequence in the sense of Bohr, i. e., for each n and every $\epsilon > 0$, the set of $\epsilon-$almost periods of the function $h \to \gamma_n(h) = Cov(X_n, X_{n+h})$ defined as the integer number τ_ϵ such that $|\gamma_n(h+\tau_\epsilon) - \gamma_n(h)| < \epsilon$ for every $h \in \mathbb{Z}$ is relatively dense in $\mathbb{Z}$. Similarly, a continuous−time process $(X_n)_{n\in\mathbb{R}}$ is said to be APC, if its covariance function is an almost periodic function on $\mathbb{R}$ in the sense of Bohr. Example of APC processes are obtained from contemporaneous aggregation of independent periodic process with incommensurate periods, for instance $X_n = e_n \sin(n) + \eta_n \sin(\pi n)$, $n \in \mathbb{R}$ where $(e_n)_{n\in\mathbb{R}}$ and $(\eta_n)_{n\in\mathbb{R}}$ are independent stationary processes.

3. QUASI-MAXIMUM LIKELIHOOD ESTIMATION FOR $SPBL_S(P, 0, P, 1)$

The estimating of general bilinear model (1.1) is quite challenging even when $s = 1$. However, in the literature many ideas have been proposed for estimating the unknown parameters of some restrictive stationary and ergodic bilinear models. The most frequently used methods are the (generalized) method of moments $(G)(MM)$ and the (conditional) least squares $(C)(LS)$ method. The asymptotic properties of the $(G)MM$ and $(C)LS$ estimates have been also discussed under certain restrictions (see for example, Liu [28], Grahn [23], Wittwer [34] and among others). Recently, Bibi et al. [8, 11] have developed certain procedures for estimating some special bilinear processes with periodically time-varying parameters. In this section, we focus on the estimation of the parameter governing the following model

$$X_t = a_1(t) + \sum_{i=2}^{p} a_i(t)X_{t-i} + e_t + \sum_{i=2}^{p} \phi_i(t)X_{t-i}e_{t-1}, \tag{3.1}$$

in which, the order p and the period s are assumed to be known and fixed. The $d = s(2p-1)+1-$unknown parameter gathered in $\underline{\theta} := \left(\underline{a}', \underline{\phi}'\right)' = (\theta_1, \ldots, \theta_d)'$ and its true value denoted by $\underline{\theta}^0$ lying in some parameter space $\Theta \subset \mathbb{R}^d$ where the vectors coordinate projections $\underline{a} := \left(\underline{a}_1', \ldots, \underline{a}_p'\right)'$ and $\underline{\phi} := \left(\underline{\phi}_2', \ldots, \underline{\phi}_p', \sigma^2\right)'$, with $\underline{a}_i = (a_i(1), \ldots a_i(s))', 1 \leq i \leq p$ and $\underline{\phi}_j = (\phi_j(1), \ldots \phi_j(s))'$, $2 \leq j \leq p$. The state-space representation of (3.1) may be obtained as $X_t = \underline{H}'\underline{X}_t$ where

$$\underline{X}_t = (A_0(t) + e_{t-1}A_1(t))\,\underline{X}_{t-1} + \underline{H}e_t + \underline{a}_1(t), \tag{3.2}$$

in which $\underline{H} = (1, 0, \ldots, 0)'$, $\underline{a}_1(t) = (a_1(t), 0, \ldots, 0)'$ and $\underline{X}_t = (X_t, X_{t-1}, \ldots, X_{t-p+1})'$ are vectors in $\mathbb{R}^p$, the periodic matrices $A_0(t)$ and $A_1(t)$ are however easily determined. Let $\mu_t(\underline{a})$ (resp. $h_t(\underline{\phi})$) be the conditional expectation (resp. variance) of X_t given $\Im_{t-2}$ where $\Im_t = \sigma(X_s, s \leq t)$, then $\mu_t(\underline{a}) = E\{X_t|\Im_{t-2}\} = a_1(t) + \sum_{i=2}^{p} a_i(t)X_{t-i}$ and $h_t(\underline{\phi}) = Var\{X_t|\Im_{t-2}\} = \sigma^2\left(1 + \alpha_t^2(\underline{\phi})\right)$ with $\alpha_t(\underline{\phi}) = \sum_{i=2}^{p} \phi_i(t)X_{t-i}$ and hence, the model (3.1) is conditionally heteroscedastic (but not a $GARCH$ model). Let $\{X_1, \ldots, X_n, n = sN\}$ be a realization from the unique, causal and SPS solution of (3.1) initialized from $\{X_0, \ldots, X_{1-p}, e_0\}$ which may be chosen as $X_0 = \ldots = X_{1-p} = e_0 = X_1$ or

$$X_0 = \ldots = X_{1-p} = e_0 = 0. \tag{3.3}$$

A Gaussian quasi-likelihood function for estimating $\underline{\theta}^0$ based on observed data $(X_t, 1-p \leq t \leq n)$ is given by

$$\widetilde{L}_n(\underline{\theta}) = \left\{\prod_{t=1}^{n} \frac{1}{\left(2\pi\widetilde{h}_t(\underline{\phi})\right)^{\frac{1}{2}}}\right\} \exp\left\{-\sum_{t=1}^{n} \frac{\left(X_t - \widetilde{\mu}_t(\underline{a})\right)^2}{2\widetilde{h}_t(\underline{\phi})}\right\}, \tag{3.4}$$

in which $\widetilde{\mu}_t(\underline{a})$ and $\widetilde{h}_t(\underline{\phi})$ are constructed under the initial values (3.3). A quasi-maximum likelihood estimator ($QMLE$) of $\underline{\theta}^0$ is defined as any measurable solution $\widehat{\underline{\theta}}_n$ of

$$\widehat{\underline{\theta}}_n = Arg\max_{\underline{\theta}\in\Theta} \widetilde{L}_n(\underline{\theta}) = Arg\min_{\underline{\theta}\in\Theta}(-\widetilde{L}_n(\underline{\theta})),$$

where (ignoring the constants) $\widetilde{L}_{sN}(\underline{\theta}) = -\frac{1}{sN}\sum_{t=1}^{N}\sum_{v=0}^{s-1}\widetilde{l}_{st+v}(\underline{\theta})$ with $\widetilde{l}_t(\underline{\theta}) = \frac{\left(X_t - \widetilde{\mu}_t(\underline{a})\right)^2}{\widetilde{h}_t(\underline{\phi})} + \log\widetilde{h}_t(\underline{\phi})$. Note that $\widehat{\underline{\theta}}_n$ would provide the exact conditional maximum likelihood. Due to the strong dependency of $\widetilde{\mu}_t(\underline{a})$ and $\widetilde{h}_t(\underline{\phi})$ on initial values, $\left(\widetilde{l}_t(\underline{\theta})\right)_{t\geq 1}$ is not a SPS nor PE process, and hence, it will be convenient to approximate it by its SPS and PE version $(l_t(\underline{\theta}))_{t\geq 1}$, so we work with an approximate version $L_n(\underline{\theta})$ of the likelihood (3.4), i. e.,

$$L_{sN}(\underline{\theta}) = -\frac{1}{sN}\sum_{t=1}^{N}\sum_{v=0}^{s-1} l_{st+v}(\underline{\theta}) \text{ with } l_t(\underline{\theta}) = \frac{\left(X_t - \mu_t(\underline{a})\right)^2}{h_t(\underline{\phi})} + \log h_t(\underline{\phi}).$$

Remark 3.1. It is worth noting that the choice of initial values have importance from a practical point of view and doesn't affect the asymptotic properties of $QMLE$.

Remark 3.2. Azrek and Mélard [2] (resp. Bibi and Oyet [7]) established asymptotic results for $ARMA$ (resp. bilinear) models with time-dependent coefficients using the penalty function $L_n(\underline{\theta})$. Recently Ling et al. [27] have used the same approach for estimating special time-invariant bilinear model.

Remark 3.3. Aknouche and Bibi [3] established asymptotic results for periodic $GARCH$ model, in which the process $\left(\frac{\partial l_t}{\partial \underline{\theta}}\left(\underline{\theta}^0\right), \Im_t\right)_{t\geq 1}$ is required to form a martingale difference (MD) sequence, however, in our case $\left(\frac{\partial l_t}{\partial \underline{\theta}}\left(\underline{\theta}^0\right), \Im_t\right)_{t\geq 1}$ is not a MD sequence.

Remark 3.4. Florian [17] established asymptotic properties for general periodic autoregressive, conditionally heteroscedastic models. When applied to the model (3.1), her consistency result coincides with our.

In what follows, we will give conditions ensuring the strong consistency of $\widehat{\underline{\theta}}_n$ and its asymptotic normality. Our approach has benefited from the papers by Bibi and Oyet [7] for some restrictive time-dependent bilinear models, Azrak and Melard for Time-dependent $ARMA$ models, Francq et al. [18], Basawa and Lund [4] and Florian [17] for periodic $ARMA$ models, Aknouche and Bibi [3] for the periodic $GARCH$ model, Ling et al. [27] for some standard simple bilinear models, Ngatchou-Wandji [29], Chatterjee and Das [16] for general conditionally heteroscedastic time series.

3.1. Strong consistency of $QMLE$

To study the strong consistency of $\hat{\underline{\theta}}_n$, we consider the following regularity assumptions.

A.1 $\underline{\theta}^0 \in \Theta$ and Θ is a compact subset of $\mathbb{R}^d$ and $\theta_j^0(v)\theta_j(v) > 0$ for all $v \in \{1, \ldots, s\}, j = 1, \ldots, d$.

A.2 $E\left\{|e_0|^{2\delta}\right\} < \infty$ for some $0 < \delta \leq 1$ and $\rho(\Gamma_\delta) < 1$ where

$$\Gamma_\delta = \prod_{v=0}^{s-1} E\left\{|A_0(v) + A_1(v)e_0|^\delta\right\}.$$

As usual in nonlinear models, the compactness of Θ in **A.1,** is assumed in order that several results from real analysis maybe used. The second assumption is imposed for the identification purpose. **A.2,** is more practical than $\gamma(L) < 0$ and ensure that the process $(X_t, t \in \mathbb{Z})$ defined by (3.1) admits a SPS and PE solution with some finite moments (see corollary 2.1).

First we show the following technical lemmas.

Lemma 3.1. Under **A.1, A.2** and with initial values (3.3), almost surely as $t \to +\infty$

$$\mathbf{1}.\sup_{\underline{\theta}\in\Theta} \left|\mu_t\left(\underline{a}\right) - \tilde{\mu}_t\left(\underline{a}\right)\right| = o(1),$$

$$\mathbf{2}.\sup_{\underline{\theta}\in\Theta} \left|h_t\left(\underline{\phi}\right) - \widetilde{h}_t\left(\underline{\phi}\right)\right| = o(1),$$

$$\mathbf{3}.\sup_{\underline{\theta}\in\Theta} \left|L_n\left(\underline{\theta}\right) - \widetilde{L}_n\left(\underline{\theta}\right)\right| = o(1).$$

Proof. The first and second assertion follows upon the observation that under initial values (3.3)

$$\mu_t\left(\underline{a}\right) - \tilde{\mu}_t\left(\underline{a}\right) = \underline{H}' A_0(t)\left(\underline{X}_{t-1} - \widetilde{\underline{X}}_{t-1}\right) \text{ and } \alpha_t - \widetilde{\alpha}_t = \underline{H}' A_1(t)\left(\underline{X}_{t-1} - \widetilde{\underline{X}}_{t-1}\right),$$

where $\widetilde{\underline{X}}_t$ is the solution process of (3.2) initializing the state-space vector $\underline{X}_t$ at $\underline{O}_{(p)}$. Therefore, under **A.2,** almost surely $\left\|\underline{X}_t - \widetilde{\underline{X}}_t\right\| \leq K e^{t\gamma(L)}$, for some positive constant K. Now

$$\begin{aligned}
&\left|\widetilde{L}_n\left(\underline{\theta}\right) - L_n\left(\underline{\theta}\right)\right| \\
&\leq \frac{1}{n}\sum_{t=1}^{n}\left\{\frac{1}{\tilde{h}_t\left(\underline{\phi}\right) h_t\left(\underline{\phi}\right)}\left(\left|\tilde{h}_t\left(\underline{\phi}\right) - h_t\left(\underline{\phi}\right)\right|\left(X_t - \widetilde{\mu}_t\left(\underline{a}\right)\right)^2 + \right.\right. \\
&\left.\left. + \widetilde{h}_t\left(\underline{\phi}\right)\left|\widetilde{\mu}_t\left(\underline{a}\right) - \mu_t\left(\underline{a}\right)\right|\left|2\left(X_t - \widetilde{\mu}_t\left(\underline{a}\right)\right) + \left(\widetilde{\mu}_t\left(\underline{a}\right) - \mu_t\left(\underline{a}\right)\right)\right|\right) + \log\left(\frac{h_t\left(\underline{\phi}\right)}{\widetilde{h}_t\left(\underline{\phi}\right)}\right)\right\}.
\end{aligned}$$

Using the inequality $\log\left(\frac{y}{x}\right) \leq \frac{|y-x|}{\min(x,y)}$ for any positive x and y, we obtain

$$\begin{aligned}
\sup_{\underline{\theta}\in\Theta}\left|\widetilde{L}_n\left(\underline{\theta}\right) - L_n\left(\underline{\theta}\right)\right| \leq \frac{1}{n\sigma^2}\sum_{t=1}^{n}\sup_{\underline{\theta}\in\Theta}\left(\left|\tilde{h}_t\left(\underline{\phi}\right) - h_t\left(\underline{\phi}\right)\right|\left(\frac{1}{\sigma^2}\left(X_t - \mu_t\left(\underline{a}\right)\right)^2 + 1\right)\right. \\
\left. + \widetilde{h}_t\left(\underline{\phi}\right)\left|\widetilde{\mu}_t\left(\underline{a}\right) - \mu_t\left(\underline{a}\right)\right|\left|2\left(X_t - \widetilde{\mu}_t\left(\underline{a}\right)\right) + \left(\widetilde{\mu}_t\left(\underline{a}\right) - \mu_t\left(\underline{a}\right)\right)\right|\right).
\end{aligned}$$

By assertions **1**, **2** of lemma 3.1, Assumption **A.2** and Borel Cantelli lemma, the result follow. □

Lemma 3.2. Under **A.1** and **A.2**, if almost surely $\mu_t\left(\underline{a}\right) = \mu_t\left(\underline{a}^0\right)$ and $h_t\left(\underline{\phi}\right) = h_t\left(\underline{\phi}^0\right)$ then $\underline{\theta} = \underline{\theta}^0$ for any $\underline{\theta} \in \Theta$.

Proof. Straightforward and hence omitted. □

Lemma 3.3. Under **A.2**, $\sum_{v=1}^{s} E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta} l_v\left(\underline{\theta}\right)\right\} < \infty$ and $\sum_{v=1}^{s} E_{\underline{\theta}^0}\left\{l_v\left(\underline{\theta}\right)\right\}$ achieves its unique minimum at $\underline{\theta} = \underline{\theta}^0$.

P r o o f. First by corollary 2.1, we have for some $\delta \in]0,1]$,

$$\frac{1}{\delta}\sum_{v=1}^{s} E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\log h^{\delta}_{st+v}\left(\underline{\phi}\right)\right\} \leq \frac{1}{\delta}\sum_{v=1}^{s}\sup_{\underline{\theta}\in\Theta}\log E_{\underline{\theta}^0}\left\{h^{\delta}_{st+v}\left(\underline{\phi}\right)\right\} < \infty,$$

and second

$$\begin{aligned}
&\sum_{v=1}^{s} E_{\underline{\theta}^0}\sup_{\underline{\theta}\in\Theta}\left\{\frac{\left(X_{st+v}-\mu_{st+v}\left(\underline{a}\right)\right)^2}{h_{st+v}\left(\underline{\phi}\right)}\right\}\\
&\leq \frac{1}{\sigma^2}\sum_{v=1}^{s}\left(E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\frac{e^2_{st+v}}{1+\alpha^2_{st+v}\left(\underline{\phi}\right)}\right\} + E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\frac{\left(a_1(v)-a_1^0(v)\right)^2}{1+\alpha^2_{st+v}\left(\underline{\phi}\right)}\right\}\right.\\
&\left.+E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\frac{e^2_{st+v-1}\alpha^2_{st+v}\left(\underline{\phi}^0\right)}{1+\alpha^2_{st+v}\left(\underline{\phi}\right)}\right\} + E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\frac{\left(\sum_{j=2}^{p}\left(a_j(v)-a_j^0(v)\right)X_{st+v-j}\right)^2}{1+\alpha^2_{st+v}\left(\underline{\phi}\right)}\right\}\right).
\end{aligned}$$

It can be shown that each term between the above bracket on the right-hand side is finite and hence the first assertion follows. To show the second, we observe that for each $v \in \{1,\ldots,s\}$

$$\begin{aligned}
&E_{\underline{\theta}^0}\left\{\left(X_{st+v}-\mu_{st+v}\left(\underline{a}\right)\right)^2 |\Im_{st+v-2}\right\}\\
&=\left(\left(a_1(v)-a_1^0(v)\right)+\sum_{j=2}^{p}\left(a_j(v)-a_j^0(v)\right)X_{st+v-j}\right)^2 + h_{st+v}\left(\underline{\phi}^0\right),
\end{aligned}$$

so,

$$\begin{aligned}
&E_{\underline{\theta}^0}\left\{l_v\left(\underline{\theta}\right)\right\}\\
&= E_{\underline{\theta}^0}\left\{\log h_v\left(\underline{\phi}\right)\right\} + E_{\underline{\theta}^0}\left\{\frac{h_v\left(\underline{\phi}^0\right)}{h_v\left(\underline{\phi}\right)}\right\}\\
&+ E_{\underline{\theta}^0}\left\{\frac{1}{h_v\left(\underline{\phi}\right)}\left(\left(a_1(v)-a_1^0(v)\right)+\sum_{j=2}^{p}\left(a_j(v)-a_j^0(v)\right)X_{st+v-j}\right)^2\right\}\\
&= E_{\underline{\theta}^0}\left\{\frac{h_v\left(\underline{\phi}^0\right)}{h_v\left(\underline{\phi}\right)} - \log\left(\frac{h_v\left(\underline{\phi}^0\right)}{h_v\left(\underline{\phi}\right)}\right) + \log h_v\left(\underline{\phi}^0\right)\right\}\\
&+ E_{\underline{\theta}^0}\left\{\frac{1}{h_v\left(\underline{\phi}\right)}\left(\left(a_1(v)-a_1^0(v)\right)+\sum_{j=2}^{p}\left(a_j(v)-a_j^0(v)\right)X_{st+v-j}\right)^2\right\},
\end{aligned}$$

using the inequality $x - \log x \geq 1$ for all $x > 0$ with equality if and only if $x = 1$, we can see that $E_{\underline{\theta}^0}\left\{l_v\left(\underline{\theta}\right)\right\}$ reaches its minimum if and only if $\sigma^2 = \sigma_0^2$, $\phi_j(v) = \phi_j^0(v)$ and

$a_j(v) = a_j^0(v)$ for all $j = 2, \ldots, p$ and $v \in \{1, \ldots, s\}$, i.e., $E_{\underline{\theta}^0}\{l_v(\underline{\theta})\} \geq E_{\underline{\theta}^0}\{l_v(\underline{\theta}^0)\}$ solving that the criterion is minimized at $\underline{\theta}^0$. □

Lemma 3.4. For all $\underline{\theta} \neq \underline{\theta}^0$ there is a neighborhood $\mathcal{V}(\underline{\theta})$ of $\underline{\theta}$ such that almost surely

$$\lim_{n\to+\infty}\inf \inf_{\underline{\theta}^* \in \mathcal{V}(\underline{\theta})} \left(-\tilde{L}_n(\underline{\theta}^*)\right) > \sum_{v=1}^{s} E_{\underline{\theta}^0}\left\{l_v\left(\underline{\theta}^0\right)\right\}.$$

Proof. The proof follows essentially the same arguments as in Aknouche and Bibi [3]. □

We are now in a position to state the following result.

Theorem 3.1. Under Assumptions **A.1** and **A.2**, $\hat{\underline{\theta}}_n \longrightarrow \underline{\theta}^0$ *a.s.* as $n \to +\infty$.

Proof. In view of lemmas 3.1–3.4, the proof of the theorem is completed by using the compactness of Θ. First, for all neighborhood $\mathcal{V}\left(\underline{\theta}^0\right)$ of $\underline{\theta}^0$, we have

$$\lim_{n\to+\infty}\sup \inf_{\underline{\tilde{\theta}}\in\mathcal{V}(\underline{\theta}^0)} \left(-\tilde{L}_n\left(\underline{\tilde{\theta}}\right)\right) \leq \lim_{n\to+\infty} \left(-\tilde{L}_n\left(\underline{\theta}^0\right)\right) = \sum_{v=1}^{s} E_{\underline{\theta}^0}\left\{l_v\left(\underline{\theta}^0\right)\right\}. \tag{3.5}$$

The compact Θ is covered by a finite union of a neighborhood $\mathcal{V}\left(\underline{\theta}^0\right)$ of $\underline{\theta}^0$ and the set of neighborhoods $\mathcal{V}(\underline{\theta})$, $\underline{\theta} \in \Theta\backslash\mathcal{V}\left(\underline{\theta}^0\right)$ where $\mathcal{V}(\underline{\theta})$ fulfills lemma 3.4. Therefore, there exists a finite sub-covering $\mathcal{V}\left(\underline{\theta}^0\right), \mathcal{V}\left(\underline{\theta}_1\right), \ldots, \mathcal{V}\left(\underline{\theta}_j\right)$ of Θ such that $\inf_{\underline{\tilde{\theta}}\in\Theta}\left(-\tilde{L}_n\left(\underline{\theta}^*\right)\right) = \min_{i\in\{1,\ldots,j\}} \inf_{\underline{\theta}^*\in\mathcal{V}\left(\underline{\theta}_i\right)\cap\Theta}\left(-\tilde{L}_n\left(\underline{\tilde{\theta}}\right)\right)$. From (3.5) and lemma 3.4, the above equality shows that $\hat{\underline{\theta}}_n \in \mathcal{V}\left(\underline{\theta}^0\right)$ for n large enough, which completes the proof of the theorem. □

3.2. Asymptotic normality of $QMLE$

To prove the asymptotic normality of $QMLE$ it is unavoidable to explore the derivatives of $L_n(\underline{\theta})$ under the following additional assumptions

A.3 $\underline{\theta}^0 \in \overset{o}{\Theta}$ where $\overset{o}{\Theta}$ is the interior of Θ.

A.4 $\kappa_4 = E\left\{e_t^4\right\} < +\infty$.

Assumption **A.3** is necessary for the asymptotic normality of the QML estimator, it ensures also the existence of a suitable compact convex subset $\Theta_0 \subset \overset{o}{\Theta}$ on which we investigate of the differentiability and the validity of the Taylor series expansion of the penalty functions $L_n(.)$ and $\widetilde{L}_n(.)$ and their components and hence the processes $\tilde{h}_t(.)$ and $h_t(.)$. Assumption **A.4** is imposed in accordance with the Markovian representation of (3.2).

Theorem 3.2. Under Assumption **A.1** – **A.4**, $\sqrt{sN}\left(\widehat{\underline{\theta}}_{sN}-\underline{\theta}^0\right)\rightsquigarrow\mathcal{N}\left(\underline{O},\Sigma\left(\underline{\theta}^0\right)\right)$ where $\Sigma\left(\underline{\theta}^0\right):=J^{-1}\left(\underline{\theta}^0\right)\Omega\left(\underline{\theta}^0\right)J^{-1}\left(\underline{\theta}^0\right)$, the matrices $\Omega\left(\underline{\theta}^0\right)$ and $J\left(\underline{\theta}^0\right)$ are nonnegative definite and given by $\Omega\left(\underline{\theta}^0\right)=\sum_{v=1}^{s}\Omega_v\left(\underline{\theta}^0\right)$ and $J\left(\underline{\theta}^0\right)=\sum_{v=1}^{s}J_v\left(\underline{\theta}^0\right)$ where

$$\begin{aligned}\Omega_v\left(\underline{\theta}^0\right)&=E_{\underline{\theta}^0}\left\{\frac{\partial l_{st+v}\left(\underline{\theta}^0\right)}{\partial\underline{\theta}}+\frac{\partial l_{st+v-1}\left(\underline{\theta}^0\right)}{\partial\underline{\theta}}\right\}\left\{\frac{\partial l_{st+v}\left(\underline{\theta}^0\right)}{\partial\underline{\theta}}+\frac{\partial l_{st+v-1}\left(\underline{\theta}^0\right)}{\partial\underline{\theta}}\right\}'\\&\quad-E\left\{\frac{\partial l_{st+v-1}\left(\underline{\theta}^0\right)}{\partial\underline{\theta}}\frac{\partial l_{st+v-1}\left(\underline{\theta}^0\right)}{\partial\underline{\theta}'}\right\},\\J_v\left(\underline{\theta}^0\right)&=diag\left(E\left\{\frac{\partial^2 l_{st+v}\left(\underline{\theta}^0\right)}{\partial\underline{a}\partial\underline{a}'}\right\},E\left\{\frac{\partial^2 l_{st+v}\left(\underline{\theta}^0\right)}{\partial\underline{\phi}\partial\underline{\phi}'}\right\}\right),\end{aligned}$$

and $J\left(\underline{\theta}^0\right)$ is non-singular.

The proof of Theorem 3.2 rests classically on a Taylor-series expansion of $\frac{\partial L_t\left(\underline{\theta}\right)}{\partial\underline{\theta}}$ around $\underline{\theta}^0$. Indeed, for $\widehat{\underline{\theta}}_n\in\overset{o}{\Theta}$ we have almost surely

$$\underline{O}=\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial l_t}{\partial\underline{\theta}}\left(\widehat{\underline{\theta}}_n\right)=\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial l_t}{\partial\underline{\theta}}\left(\underline{\theta}^0\right)+\left(\frac{1}{n}\sum_{t=1}^{n}\frac{\partial^2 l_t}{\partial\underline{\theta}\partial\underline{\theta}'}\left(\underline{\theta}^*\right)\right)\sqrt{n}\left(\widehat{\underline{\theta}}_n-\underline{\theta}^0\right),$$

for some $\underline{\theta}^*$ where $\left\|\underline{\theta}^0-\underline{\theta}^*\right\|\leq\left\|\underline{\theta}^0-\widehat{\underline{\theta}}_n\right\|$. We will thus show that

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial l_t}{\partial\underline{\theta}}\left(\underline{\theta}^0\right)\rightsquigarrow\mathcal{N}\left(\underline{O},\Omega\right)\text{ and }p\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{\partial^2 l_t}{\partial\underline{\theta}\partial\underline{\theta}'}\left(\underline{\theta}^*\right)=J,$$

and hence the theorem will straightforwardly follow. The partial derivatives of $l_t\left(\underline{\theta}\right)$ are given by

$$\begin{aligned}\frac{\partial l_t}{\partial\underline{a}}\left(\underline{\theta}\right)&=-2\frac{\left(X_t-\mu_t\left(\underline{a}\right)\right)}{h_t\left(\underline{\phi}\right)}\frac{\partial\mu_t\left(\underline{a}\right)}{\partial\underline{a}},\\\frac{\partial l_t}{\partial\sigma^2}\left(\underline{\theta}\right)&=\left(1-\frac{\left(X_t-\mu_t\left(\underline{a}\right)\right)^2}{h_t\left(\underline{\phi}\right)}\right)\frac{1}{\sigma^2},\\\frac{\partial l_t}{\partial\underline{\phi}}\left(\underline{\theta}\right)&=\left(1-\frac{\left(X_t-\mu_t\left(\underline{a}\right)\right)^2}{h_t\left(\underline{\phi}\right)}\right)\frac{1}{h_t\left(\underline{\phi}\right)}\frac{\partial h_t\left(\underline{\phi}\right)}{\partial\underline{\phi}},\\\frac{\partial^2 l_t}{\partial\underline{a}\partial\underline{a}'}\left(\underline{\theta}\right)&=\frac{2}{h_t\left(\underline{\phi}\right)}\frac{\partial\mu_t\left(\underline{a}\right)}{\partial\underline{a}}\frac{\partial\mu_t\left(\underline{a}\right)}{\partial\underline{a}'}-2\frac{\left(X_t-\mu_t\left(\underline{a}\right)\right)}{h_t\left(\underline{\phi}\right)}\frac{\partial^2\mu_t\left(\underline{a}\right)}{\partial\underline{a}\partial\underline{a}'},\end{aligned}$$

$$\frac{\partial^2 l_t}{\partial \underline{\phi} \partial \underline{\phi}'}(\underline{\theta}) = \left(\frac{\left(X_t - \mu_t\left(\underline{a}\right)\right)^2}{h_t\left(\underline{\phi}\right)} - 1 \right) \frac{1}{h_t^2\left(\underline{\phi}\right)} \frac{\partial h_t\left(\underline{\phi}\right)}{\partial \underline{\phi}} \frac{\partial h_t\left(\underline{\phi}\right)}{\partial \underline{\phi}'}$$
$$+ \left(1 - \frac{\left(X_t - \mu_t\left(\underline{a}\right)\right)^2}{h_t\left(\underline{\phi}\right)} \right) \frac{1}{h_t\left(\underline{\phi}\right)} \frac{\partial^2 h_t\left(\underline{\phi}\right)}{\partial \underline{\phi} \partial \underline{\phi}'},$$
$$\frac{\partial^2 l_t}{\partial \underline{a} \partial \underline{\phi}'}(\underline{\theta}) = 2 \frac{\left(X_t - \mu_t\left(\underline{a}\right)\right)}{h_t^2\left(\underline{\phi}\right)} \frac{\partial \mu_t\left(\underline{a}\right)}{\partial \underline{a}} \frac{\partial h_t\left(\underline{\phi}\right)}{\partial \underline{\phi}'},$$

in which with periodic notation for $j = 2, \ldots, p$,

$$\frac{\partial \mu_{st+v}\left(\underline{a}\right)}{\partial \underline{a}_1} = \underline{1}_v, \ \frac{\partial \mu_{st+v}\left(\underline{a}\right)}{\partial \underline{a}_j} = \underline{1}_v X_{st+v-j}, \ \frac{\partial h_{st+v}\left(\underline{\phi}\right)}{\partial \underline{\phi}_j} = 2\sigma^2 \alpha_{st+v}\left(\underline{\phi}\right) \underline{1}_v X_{st+v-j},$$

where $\underline{1}_v$ denotes a $s \times 1$ unit vector whose entries are all zero except for a one in the vth row. Note here that for a non Gaussian processes, neither $\Omega^{-1}\left(\underline{\theta}^0\right)$ nor $J^{-1}\left(\underline{\theta}^0\right)$ is an asymptotic covariance matrix (see for instance [33]) but well the so-called sandwich estimator $J^{-1}\left(\underline{\theta}^0\right) \Omega\left(\underline{\theta}^0\right) J^{-1}\left(\underline{\theta}^0\right)$. To prove the Theorem 3.2, we need to check the following intermediate results.

Lemma 3.5. Under Assumptions **A.1** – **A.4**, we have

1. $\sum_{v=1}^{s} E_{\underline{\theta}^0} \left\{ \sup_{\underline{\theta} \in \Theta} \left\| \frac{\partial l_{st+v}}{\partial \underline{\theta}}\left(\underline{\theta}^0\right) \right\|^2 \right\} < \infty,$
2. $\sum_{v=1}^{s} E_{\underline{\theta}^0} \left\{ \sup_{\underline{\theta} \in \Theta} \left\| \frac{\partial^2 l_{st+v}}{\partial \underline{\theta} \partial \underline{\theta}'}\left(\underline{\theta}^0\right) \right\| \right\} < \infty,$
3. $\sup_{\underline{\theta} \in \Theta} \left\| \frac{1}{sN} \sum_{t=1}^{N} \sum_{v=1}^{s} \left\{ \frac{\partial^2 l_{st+v}}{\partial \underline{\theta} \partial \underline{\theta}'}\left(\underline{\theta}^0\right) - E\left\{ \frac{\partial^2 \widetilde{l}_{st+v}}{\partial \underline{\theta} \partial \underline{\theta}'}\left(\underline{\theta}^0\right) \right\} \right\} \right\| = o(1).$
4. There exists a neighborhood $\mathcal{V}(\underline{\theta}^0)$ of $\underline{\theta}^0$ such that for all $i, j, k \in \{1, \ldots, d\}$, $\sum_{v=1}^{s} E_{\underline{\theta}^0} \left\{ \sup_{\underline{\theta} \in \mathcal{V}(\underline{\theta}^0)} \left| \frac{\partial^3 l_{st+v}(\underline{\theta})}{\partial \theta_i \partial \theta_j \partial \theta_k} \right| \right\} < +\infty.$

Proof. Notice that $X_t - \mu_t\left(\underline{a}\right) = e_t - \left(a_1(t) - a_1^0(t)\right) - \sum_{j=2}^{p} \left(a_j(t) - a_j^0(t)\right) X_{t-j} + \sum_{j=2}^{p} \phi_j^0(t) X_{t-j} e_{t-1}$, so we have

$$E_{\underline{\theta}^0} \left\{ \sup_{\underline{\theta} \in \Theta} \left\| \frac{\partial l_{st+v}}{\partial \underline{a}_1}\left(\underline{\theta}^0\right) \right\|^2 \right\} = 4 \left\| \underline{1}_v \right\|^2 E_{\underline{\theta}^0} \left\{ \sup_{\underline{\theta} \in \Theta} \frac{\left(X_{st+v} - \mu_{st+v}\left(\underline{a}\right)\right)^2}{h_{st+v}^2\left(\underline{\phi}\right)} \right\}$$
$$\leq 4 \left\| \underline{1}_v \right\|^2 \left\{ I(1) + I(2) \right\},$$

and for any $j \in \{2, \ldots, p\}$

$$E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\left\|\frac{\partial l_{st+v}}{\partial \underline{a}_j}\left(\underline{\theta}^0\right)\right\|^2\right\} = 4\left\|\underline{1}_v\right\|^2 E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\frac{\left(X_{st+v}-\mu_{st+v}\left(\underline{a}\right)\right)^2 X^2_{st+v-j}}{h^2_{st+v}\left(\underline{\phi}\right)}\right\}$$
$$\leq 4\left\|\underline{1}_v\right\|^2\left\{I(3)+I(4)\right\},$$

where the finiteness of the following expressions is obviously satisfied

$$I(1) = 2E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\left(\frac{(a_1(v)-a_1^0(v))^2}{h^2_{st+v}\left(\underline{\phi}\right)} + \frac{1}{h^2_{st+v}\left(\underline{\phi}\right)}\left(\sum_{j=2}^{p}\left(a_j(v)-a_j^0(v)\right)X_{st+v-j}\right)^2\right)\right\},$$

$$I(2) = E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\frac{h_{st+v}\left(\underline{\phi}^0\right)}{h^2_{st+v}\left(\underline{\phi}\right)}\right\},$$

$$I(3) = 2E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\left(\frac{(a_1(v)-a_1^0(v))^2 X^2_{st+v-j}}{h^2_{st+v}\left(\underline{\phi}\right)} + \frac{X^2_{st+v-j}}{h^2_{st+v}\left(\underline{\phi}\right)}\left(\sum_{j=2}^{p}\left(a_j(v)-a_j^0(v)\right)X_{st+v-j}\right)^2\right)\right\},$$

$$I(4) = E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\Theta}\frac{h_{st+v}\left(\underline{\phi}^0\right)X^2_{st+v-j}}{h^2_{st+v}\left(\underline{\phi}\right)}\right\}.$$

Using the same technique as in Ling et al. [27], we can show the uniform integrability of all partial derivatives. The assertion **3**, follows essentially from standard arguments, it suffices to replace the stationarity and the ergodicity arguments by periodic stationarity and periodic ergodicity ones and the assertion **4** follows essentially the same arguments as in Francq and Zakoîan [20]. □

It is worth noting that in spite $E\left\{\frac{\partial l_t}{\partial \underline{\theta}}\left(\underline{\theta}^0\right)|\Im_{t-2}\right\} = \underline{O}$, the process $\left(\frac{\partial l_t}{\partial \underline{\theta}}\left(\underline{\theta}^0\right), \Im_t\right)_{t\geq 1}$ is not a martingale difference sequence, so the asymptotic distribution of $n^{-1/2}\sum_{t=1}^{n}\frac{\partial l_t}{\partial \underline{\theta}}\left(\underline{\theta}^0\right)$ for the martingale difference sequence is not applicable here. To remedy this difficulty, use the representation (3.2) to define, for any $m \geq 1$,

$$\underline{U}_t(m) = \sum_{k=0}^{m}\left\{\prod_{i=0}^{k-1}\left(A_0(t-i)+e_{t-i-1}A_1(t-i)\right)\right\}\underline{H}\left(e_{t-k}+a_1(t-k)\right),$$
$$\underline{V}_t(m) = \left\{\prod_{i=0}^{m}\left(A_0(t-i)+e_{t-i-1}A_1(t-i)\right)\right\}\underline{X}_{t-m-1},$$

and consider the decompositions $\underline{X}_t(m) = \underline{U}_t(m) + \underline{V}_t(m)$ and $X_t(m) = \underline{H}'\underline{X}_t(m)$. Then, it is clear that $(\underline{U}_t(m))_{t\geq 0}$ is an $(m+1)-$dependent process and under the assumption **A.2**, $\underline{V}_t(m)$ (resp. $\underline{U}_t(m)$) converges in probability to $\underline{O}$ (resp. to $\underline{X}_t$) as $m \to +\infty$. Let $\underline{\lambda} := \left(\underline{\lambda}_1', \underline{\lambda}_2', \ldots, \underline{\lambda}_p', \mu_1, \underline{\mu}_2' \ldots, \underline{\mu}_p'\right)'$ be a constant vector in $\mathbb{R}^d$ such that $\underline{\lambda}'\underline{\lambda} \neq 0$ and let

$$\begin{aligned}
S_n &= n^{-1/2}\underline{\lambda}'\sum_{t=1}^{n}\frac{\partial l_t}{\partial \underline{\theta}}\left(\underline{\theta}^0\right) \\
&= n^{-1/2}\sum_{t=1}^{n}\{\underline{\lambda}_1'\frac{\partial l_t}{\partial \underline{a}_1}\left(\underline{\theta}^0\right) + \underline{\lambda}_2'\frac{\partial l_t}{\partial \underline{a}_2}\left(\underline{\theta}^0\right) + \ldots + \underline{\lambda}_p'\frac{\partial l_t}{\partial \underline{a}_p}\left(\underline{\theta}^0\right) \\
&\quad + \mu_1\frac{\partial l_t}{\partial \sigma^2}\left(\underline{\theta}^0\right) + \underline{\mu}_2'\frac{\partial l_t}{\partial \underline{\phi}_2}\left(\underline{\theta}^0\right) + \ldots + \underline{\mu}_p'\frac{\partial l_t}{\partial \underline{\phi}_p}\left(\underline{\theta}^0\right)\} \\
&= n^{-1/2}\sum_{t=1}^{n}s_t,
\end{aligned}$$

where

$$\begin{aligned}
s_t &= \left(\underline{\lambda}_1'\frac{\partial \mu_t\left(\underline{a}^0\right)}{\partial \underline{a}_1} + \ldots + \underline{\lambda}_p'\frac{\partial \mu_t\left(\underline{a}^0\right)}{\partial \underline{a}_p}\right)\frac{\xi_t\left(1\right)}{\sqrt{h_t\left(\underline{\phi}^0\right)}} \\
&\quad + \left(\mu_1\frac{h_t\left(\underline{\phi}^0\right)}{\sigma^2} + \underline{\mu}_2'\frac{\partial h_t\left(\underline{\phi}^0\right)}{\partial \underline{\phi}_2} + \ldots + \underline{\mu}_p'\frac{\partial h_t\left(\underline{\phi}^0\right)}{\partial \underline{\phi}_p}\right)\frac{\xi_t\left(2\right)}{h_t\left(\underline{\phi}^0\right)},
\end{aligned} \tag{3.6}$$

with

$$\begin{aligned}
\xi_t\left(1\right) &= \frac{-2}{\sqrt{h_t\left(\underline{\phi}^0\right)}}\left(e_t + e_{t-1}\sum_{j=2}^{p}\phi_j^0\left(t\right)X_{t-j}\right), \\
\xi_t\left(2\right) &= 1 - \frac{1}{h_t\left(\underline{\phi}^0\right)}\left(e_t + e_{t-1}\sum_{j=2}^{p}\phi_j^0\left(t\right)X_{t-j}\right)^2.
\end{aligned}$$

Clearly, $(s_n, n \geq 1)$ is a centred, $2-$dependent, SPS process with $E\left\{s_n s_{n+k}\right\} = 0$ for any $|k| > 2$, so we have almost surely as $n \to \infty$,

$$Var\left\{\frac{1}{\sqrt{n}}\sum_{t=1}^{n}s_t\right\} \to \sum_{v=0}^{s-1}E\left\{s_{st+v}^2 + 2s_{st+v}s_{st+v-1}\right\} = \underline{\lambda}'\Omega\underline{\lambda} > 0.$$

Now, let us combine the arguments used to prove the central limit theorem for stationary $m-$dependent sequence with those used to prove the central limit theorem for sequence of independent but not identically distributed random variables. For this purpose we show the following lemma

Lemma 3.6. Under the conditions **A.2** – **A.4**, the sequence $(s_n)_{n\geq 1}$ defined by (3.6) is $\mathbb{L}_r$ near-epoch dependent (NED) process in the sense that $E\left\{\left|s_n - E\left\{s_n|\Im_n^{(m)}\right\}\right|^2\right\} = O\left(m^{-r}\right)$, for any $r > 2$ and m enough large where $\Im_n^{(m)} = \sigma\left\{X_{n-j}, 0 \leq j \leq m\right\}$.

P r o o f. The proof follows essentially the same arguments as in Ling et al. [27]. □

P r o o f of Theorem 3. First, all the hypotheses of Theorem 21.1 of Billingsley [12] are verified. Consequently, we have $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial l_t}{\partial \underline{\theta}}\left(\underline{\theta}^0\right) \rightsquigarrow \mathcal{N}\left(\underline{O}, \Omega\left(\underline{\theta}^0\right)\right)$. Second, the convergence $p\lim_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\frac{\partial^2 l_t}{\partial\underline{\theta}\partial\underline{\theta}'}\left(\underline{\theta}^*\right) = J$ follows upon the observation that for all $i, j \in \{1, \ldots, d\}$, almost surely

$$\frac{1}{n}\sum_{t=1}^{n}\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\left(\underline{\theta}^*\right) = \frac{1}{n}\sum_{t=1}^{n}\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\left(\underline{\theta}^0\right) + \frac{1}{n}\sum_{t=1}^{n}\frac{\partial}{\partial\underline{\theta}'}\left\{\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\left(\widetilde{\underline{\theta}}\right)\right\}\left(\underline{\theta}^* - \underline{\theta}^0\right), \tag{3.7}$$

for some random vector $\widetilde{\underline{\theta}}$ such that almost surely $\left\|\underline{\theta}^0 - \widetilde{\underline{\theta}}\right\| \leq \left\|\underline{\theta}^0 - \underline{\theta}^*\right\| \leq \left\|\underline{\theta}^0 - \widehat{\underline{\theta}}_n\right\|$. From the strong consistency of $\widehat{\underline{\theta}}_n$, the periodic ergodicity and assertion **4** of Lemma 3.5 imply that

$$\begin{aligned}\limsup_{n\to\infty}\left\|\frac{1}{n}\sum_{t=1}^{n}\frac{\partial}{\partial\theta'}\left\{\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\left(\widetilde{\underline{\theta}}\right)\right\}\right\| &\leq \limsup_{n\to\infty}\frac{1}{n}\sum_{t=1}^{n}\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}^0)}\left\|\frac{\partial}{\partial\theta'}\left\{\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\left(\underline{\theta}\right)\right\}\right\| \\ &= E_{\underline{\theta}^0}\left\{\sup_{\underline{\theta}\in\mathcal{V}(\underline{\theta}^0)}\left\|\frac{\partial}{\partial\theta'}\left\{\frac{\partial^2 l_t}{\partial\theta_i\partial\theta_j}\left(\underline{\theta}\right)\right\}\right\|\right\} < +\infty.\end{aligned}$$

Therefor, since $\left\|\underline{\theta}^0 - \widehat{\underline{\theta}}_n\right\| \to 0$, $a.s.$ as $n \to +\infty$, the second term in the right-hand side of (3.7) converges $a.s.$ to 0, the first one converges to J and hence by Slutsky's theorem, the results of theorem follow. □

4. ASYMPTOTIC PROPERTIES OF $QMLE$ FOR $PARMA_S$ MODELS WITH PBL_S INNOVATIONS

As already stated in introduction, the PBL_s is a general class of non linear models. Among others, one of interesting model which is nested in PBL_s is defined by

$$e_t = \sum_{i=2}^{r}\phi_i\left(t\right)e_{t-i}\eta_{t-1} + \eta_t \tag{4.1}$$

where $(\eta_t)_{t\in\mathbb{Z}}$ is an $i.i.d.$ sequence with zero mean and variance $\sigma^2 > 0$, and $\phi_i\left(t\right)$ are periodic in t with period s. Hence, if $(e_t)_{t\in\mathbb{Z}}$ is a PC process, then we have $E\left\{e_t\right\} = 0$,

$Cov\left(e_t, e_{t-h}\right) = 0$ for all $h \neq 0$ and

$$E\left\{e_t^2|\Im_{t-2}^{(e)}\right\} = \sigma^2\left(1 + \left(\sum_{i=2}^{r}\phi_i\left(t\right)e_{t-i}\right)^2\right)$$

where $\Im_t^{(e)} = \sigma\left(e_{t-i}, i \geq 0\right)$, and hence the solution process $\left(e_t\right)_{t\in\mathbb{Z}}$ of (4.1) is martingale difference conditionally heteroscedastic and can therefore be used as the innovation of an $PARMA_s$ process ($PARMA_s - PBL_s$ for short) previously introduced by Francq [19] (in non periodic context). In this section, we focus on the estimation of the parameters of $PARMA_s - PBL_s$. Besides further tedious but simple technical manipulations, the extension of $ARMA - BL$ to its periodic counterpart doesn't entail any substantial difficulty. As for the $PARMA_s - PGARCH_s$ case (cf. Aknouche and Bibi [3]), the consistency of $QMLE$ for $PARMA_s - PBL_s$ will be proved without any additional moment restriction but for asymptotic normality, we need a fourth moment condition on the PBL_s component. For this purpose let $\{X_1, X_2, \ldots, X_n, n = sN\}$ be a realization of causal time series $\left(X_t\right)_{t\in\mathbb{Z}}$ generated by a $PARMA_s(p,q) - PBL_s(0,0,r,1)$ stochastic difference equation, i. e.,

$$\forall t \in \mathbb{Z} : X_t - \mu\left(t\right) = \sum_{i=1}^{p}\varphi_i\left(t\right)\left(X_{t-i} - \mu\left(t-i\right)\right) + e_t + \sum_{j=1}^{q}\psi_j\left(t\right)e_{t-j} \tag{4.2}$$

where $\left(e_t\right)_{t\in\mathbb{Z}}$ is defined by (4.1) and where $\mu\left(t\right)$, $\varphi_i\left(t\right)$, $\psi_j\left(t\right)$ and $\phi_i\left(t\right)$ are periodic in t with period s. For $\underline{\varphi} = \left(\underline{\varphi}'\left(1\right), \underline{\varphi}'\left(2\right), \ldots, \underline{\varphi}'\left(s\right)\right)'$ and $\underline{\phi} = \left(\underline{\phi}'(1), \ldots, \underline{\phi}'\left(s\right), \sigma^2\right)'$with $\underline{\varphi}(v) = \left(\mu\left(v\right), \varphi_1(v), \ldots, \varphi_p\left(v\right), \psi_1\left(v\right), \ldots, \psi_q(v)\right)'$ and $\underline{\phi}\left(v\right) = \left(\phi_2(v), \ldots, \phi_r(v)\right)', 1 \leq v \leq s$, denote by $\underline{\pi} = \left(\underline{\varphi}', \underline{\phi}'\right)'$ the model parameter and $\underline{\pi}^0 = \left(\underline{\varphi}^{0\prime}, \underline{\phi}^{0\prime}\right)'$ its true value which is supposed to belong to a same compact parameter space $\Pi \subset \mathbb{R}^{s(p+q+r)+1}$. Conditionally on the given initial values

$$X_0 = \ldots = X_{1-p} = e_0 = \ldots = e_{1-q-r} = \eta_0 = \eta_{-1} = 0, \tag{4.3}$$

the $QMLE$ of $\underline{\pi}^0$ denoted by $\widehat{\underline{\pi}}_n$ is defined as any measurable solution of $\widehat{\underline{\pi}}_n$ of

$$\widehat{\underline{\pi}}_n = Arg\max_{\underline{\pi}\in\Pi}\widetilde{L}_n\left(\underline{\pi}\right) = Arg\min_{\underline{\pi}\in\Pi}(-\widetilde{L}_n\left(\underline{\pi}\right)),$$

where $\widetilde{L}_n\left(\underline{\pi}\right)$ (ignoring the constants) is $\widetilde{L}_{sN}\left(\underline{\pi}\right) = -\frac{1}{sN}\sum_{t=1}^{N}\sum_{v=0}^{s-1}\widetilde{l}_{st+v}\left(\underline{\pi}\right)$ with $\widetilde{l}_t\left(\underline{\pi}\right) = \dfrac{\left(X_t - \widetilde{\mu}_t\left(\underline{\varphi}\right)\right)^2}{\widetilde{h}_t\left(\underline{\phi}\right)} + \log\widetilde{h}_t\left(\underline{\phi}\right)$ in which $\widetilde{\mu}_t\left(\underline{\varphi}\right)$ and $\widetilde{h}_t\left(\underline{\phi}\right)$ are constructed under the initial values (4.3) with

$$\widetilde{\mu}_t\left(\underline{\varphi}\right) = E\left\{\left(X_t - \mu\left(t\right)\right)|\Im_{t-1}^{(e)}\right\} = \sum_{i=1}^{p}\varphi_i\left(t\right)\left(X_{t-i} - \mu\left(t-i\right)\right) + \sum_{j=1}^{q}\psi_j\left(t\right)e_{t-j},$$

$$\widetilde{h}_t\left(\underline{\phi}\right) = Var\left\{\left(X_t - \mu\left(t\right)\right)|\Im_{t-1}^{(e)}\right\} = \sigma^2\left(1 + \beta_t^2\left(\underline{\phi}\right)\right) \text{ with } \beta_t\left(\underline{\phi}\right) = \sum_{i=2}^{r}\phi_i\left(t\right)e_{t-i},$$

and hence, the model (4.2) maybe rewritten as $X_t - \mu(t) = \widetilde{\mu}_t\left(\underline{\varphi}\right) + e_t$.

4.1. Strong consistency

For the strong consistency of $\widehat{\underline{\pi}}_{sN}$, we make the following additional assumptions.

A.5 For all $\underline{\pi} \in \Pi$, $\rho\left(\prod_{v=0}^{s-1} \Phi_{s-v}\right) < 1$ where Φ_v are the companion matrices associated with the $AR(p)$ part.

A.6 The polynomials $\varphi_v^0(z) = 1 - \sum_{i=1}^{p} \varphi_i^0(v) z^i$ and $\psi_v^0(z) = 1 - \sum_{j=1}^{q} \psi_j^0(v) z^j$ have no common roots and $\varphi_p(v)\psi_q(v) \neq 0$ for all $1 \leq v \leq s$.

Assumption **A.5,** is made in order to ensure the causality of the $PARMA_s$ component given by (4.2). This condition is also equivalent to the associated top-Lyapunov exponent is strictly negative and hence the existence of finite moment of some order. Moreover, under this assumption we have $X_{st+v} = \mu(v) + \sum_{j=0}^{\infty} \delta_j(v) e_{st+v-j}$ where $(\delta_j(v))_{j\geq 0}$ satisfy $\sup_{0\leq v\leq s-1} |\delta_j(v)| = O(\rho^j)$ for some $0 < \rho < 1$. **A.6,** is an identifiability assumption.
As in the PBL_s case, we approximate $\left(\widetilde{l}_t(\underline{\pi})\right)_{t\geq 1}$ by its SPS version $(l_t(\underline{\pi}))_{t\in\mathbb{Z}}$ where

$$l_t(\underline{\pi}) = \frac{\left(X_t - \mu_t\left(\underline{\varphi}\right)\right)^2}{h_t\left(\underline{\phi}\right)} + \log h_t\left(\underline{\phi}\right). \tag{4.4}$$

The following theorem establishes the strong consistency of $\widehat{\underline{\pi}}_n$

Theorem 4.1. *Under* ***A.1, A.2, A.5*** *and* ***A.6,*** *we have* $\widehat{\underline{\pi}}_n \to \underline{\pi}^0$ *a.s. as* $n \to \infty$.

Up till now, the process $(e_t)_{t\in\mathbb{Z}}$ and hence $(X_t)_{t\in\mathbb{Z}}$ need not have a finite second order moment. For the case of $PARMA_s$ models with *i.i.d.* innovations our result is a complement to the asymptotic inference for $PARMA_s$ models given by Basawa and Lund [4] which have not established the strong consistency of the Gaussian MLE. Along the lines of proof of Theorem 3.1, Theorem 4.1 will be proved whenever establishing the following lemmas.

Lemma 4.1. *Under* ***A.2*** *and* ***A.5,*** *we have almost surely*

$$\sup_{\underline{\pi}\in\Pi} \left|L_{sN}(\underline{\pi}) - \widetilde{L}_{sN}(\underline{\pi})\right| \to 0, \text{as } N \to \infty.$$

Proof. The proof is similar to that of lemma 3.1 (see also Aknouche and Bibi [3]). □

Lemma 4.2. *Under* ***A.1, A.2, A.5*** and ***A.6,*** the model (4.2) is identifiable in the sense that, distinct values of $\underline{\pi}$ should correspond almost surely to distinct values of their associated log quasi-likelihood function $l_t(\underline{\pi})$ in (4.4) for some $t \geq 0$.

Proof. The proof follows essentially the same arguments used in proving Lemma 3.2. □

Lemma 4.3. *If* $\underline{\pi} \neq \underline{\pi}^0$ *then* $\sum\limits_{v=1}^{S} E_{\underline{\pi}^0}\{l_v(\underline{\pi})\} > \sum\limits_{v=1}^{S} E_{\underline{\pi}^0}\{l_v(\underline{\pi}^0)\}$.

Proof. We have

$$\sum_{v=1}^{s} E_{\underline{\pi}^0}\{l_v(\underline{\pi})\} - E_{\underline{\pi}^0}\{l_v(\underline{\pi}^0)\}$$
$$= \sum_{v=1}^{s}\left(E_{\underline{\pi}^0}\left\{\log\frac{h_v(\underline{\phi})}{h_v(\underline{\phi}^0)} + \frac{h_v(\underline{\phi}^0)}{h_v(\underline{\phi})} - 1\right\} + E_{\underline{\pi}^0}\left\{\frac{(X_v - \mu_v(\underline{\varphi}))^2}{h_v(\underline{\phi})} - \frac{(X_v - \mu_v(\underline{\varphi}^0))^2}{h_v(\underline{\phi}^0)}\right\}\right).$$

Using the inequality $x - 1 \geq \log x$ with equality if and only if $x = 1$, we obtain, $\sum\limits_{v=1}^{S} E_{\underline{\pi}^0}\{l_v(\underline{\pi})\} \geq \sum\limits_{v=1}^{S} E_{\underline{\pi}^0}\{l_v(\underline{\pi}^0)\}$ with equality if and only if $\mu_v(\underline{\varphi}) = \mu_v(\underline{\varphi}^0)$ and $h_v(\underline{\phi}) = h_v(\underline{\phi}^0)$ and hence $\underline{\pi} = \underline{\pi}^0$. □

Lemma 4.4. *For all* $\underline{\pi} \neq \underline{\pi}^0$ *there is a neighborhood* $\mathcal{V}(\underline{\pi})$ *such that*

$$\lim_{N\to\infty}\inf \inf_{\widetilde{\underline{\pi}} \in \mathcal{V}(\underline{\pi})}\left(-\widetilde{L}_{sN}(\widetilde{\underline{\pi}})\right) > \sum_{v=1}^{s} E_{\underline{\pi}^0}\{l_v(\underline{\pi}^0)\}.$$

Proof. The proof is similar to that of Lemma 3.4 and hence it will be omitted. □

4.2. Asymptotic normality

While strong consistency of $\widehat{\underline{\pi}}_{sN}$ follows irrespective of any moment requirements, this isn't the case for the asymptotic normality. Indeed, as in $PARMA_s - PGARCH_s$ case (cf. Aknouche and Bibi [3]), we will prove asymptotic normality of $\widehat{\underline{\pi}}_{sN}$ under the fourth moment condition on the $(e_t)_{t\in\mathbb{Z}}$. It can be shown that such an assumption is expressed by $\rho\left(\prod\limits_{v=0}^{s-1} E\left(\Phi_{s-v}^{\otimes 4}\right)\right) < 1$ where Φ_v are the companion random matrices associated with the state-space representation of $BL_s(0,0,r,1)$ model. Thus, we make the following assumptions.

A.7 $\rho\left(\prod\limits_{v=0}^{s-1} E\left(\Psi_{s-v}^{\otimes 4}\right)\right) < 1$ for all $\underline{\pi} \in \Pi$.

A.8 $\underline{\pi}^0$ is in the interior of Π.

Assumption **A.7,** clearly implies that $E\left(e_t^4\right) < \infty$ and **A.8,** is an adaptation of **A.3** to the $PARMA_s - PBL_s$. The following theorem establishes $\sqrt{n}$−consistency of $\widehat{\underline{\pi}}_n$.

Theorem 4.2. Under **A.2 – A.8,** we have

$$\sqrt{N}\left(\widehat{\underline{\pi}}_{sN}-\underline{\pi}^0\right)\rightsquigarrow\mathcal{N}\left(0,\Sigma\left(\underline{\pi}^0\right)\right),\text{ as } N\to\infty,$$

where $\Sigma\left(\underline{\pi}^0\right):=J^{-1}\left(\underline{\pi}^0\right)I\left(\underline{\pi}^0\right)J^{-1}\left(\underline{\pi}^0\right)$, the matrices $I\left(\underline{\pi}^0\right)$ and $J\left(\underline{\pi}^0\right)$ are given by

$$I\left(\underline{\pi}^0\right)=\sum_{v=1}^{s}E_{\underline{\pi}^0}\left\{\frac{\partial l_v\left(\underline{\pi}^0\right)}{\partial\underline{\pi}}\frac{\partial l_v\left(\underline{\pi}^0\right)}{\partial\underline{\pi}'}\right\},\quad J\left(\underline{\pi}^0\right)=\sum_{v=1}^{s}E_{\underline{\pi}^0}\left\{\frac{\partial^2 l_v\left(\underline{\pi}^0\right)}{\partial\underline{\pi}\partial\underline{\pi}'}\right\},$$

which may be partitioned as

$$I\left(\underline{\pi}^0\right)=\begin{pmatrix} I_{\underline{\varphi\varphi}} & I_{\underline{\varphi\phi}}\\ I_{\underline{\phi\varphi}} & I_{\underline{\phi\phi}}\end{pmatrix},\quad J\left(\underline{\pi}^0\right)=\begin{pmatrix} J_{\underline{\varphi\varphi}} & J_{\underline{\varphi\phi}}\\ J_{\underline{\phi\varphi}} & J_{\underline{\phi\phi}}\end{pmatrix},$$

such that the sub-matrices $I_{\underline{\varphi\varphi}}$, $I_{\underline{\varphi\phi}}$, $I_{\underline{\phi\phi}}$, $J_{\underline{\varphi\varphi}}$, $J_{\underline{\varphi\phi}}$ and $J_{\underline{\phi\phi}}$ are $s-$block diagonal.

Proof. The proof follows essentially the same arguments as in Aknouche and Bibi [3]. □

Remark 4.1. The fact that the submatrices $I_{\underline{\varphi\varphi}}$, $I_{\underline{\varphi\phi}}$, $I_{\underline{\phi\phi}}$, $J_{\underline{\varphi\varphi}}$, $J_{\underline{\varphi\phi}}$ *and* $J_{\underline{\phi\phi}}$ are s-block diagonal implies the asymptotic independence of the estimates for each season $1\leq v\leq s$ which isn't a surprising result in periodic time-varying models. Moreover, the asymptotic independence also appears for the estimates of the $PARMA_s$ component with symmetric distribution innovations.

Remark 4.2. It is clear from Theorem 4.2 that when $e_t=\eta_t$ for all t, our asymptotic results coincide with those for the pure $PARMA_s$ model with *i.i.d.* innovations (cf. Basawa and Lund [4]).

5. CONCLUDING REMARKS

This paper extends the superdiagonal standard bilinear (SBL) models to periodic one ($PSBL_s$) which allows the coefficients to vary periodically over the time. Hence, necessary and sufficient conditions ensuring the existence of such process are given. Our aim is the estimation of $PSBL_s$ models, this problem has been previously resolved in the statistical literature for the usual causal and invertible standard case using a distribution-free approach based on $QMLE$ which is also applied in some special $PSBL_s$ models. Hence, we have established under mild assumptions, the strong consistency and asymptotic normality of the $QMLE$ for causal and not necessarily invertible a $PSBL_s$ model. Moreover, it is observed that special case of $PSBL_s$ have some interesting properties that makes it able to use as a weak white noise. This finding leads us to consider the asymptotic properties of $QMLE$ of $PARMA_s$ models with $PSBL_s$ as innovation.

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