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ESTIMATION FOR HEAVY TAILED MOVING AVERAGE PROCESS

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In this paper, we propose two estimators for a heavy tailed MA(1) process. The first is a semi parametric estimator designed for MA(1) driven by positive-value stable variables innovations. We study its asymptotic normality and finite sample performance. We compare the behavior of this estimator in which we use the Hill estimator for the extreme index and the estimator in which we use the t-Hill in order to examine its robustness. The second estimator is for MA(1) driven by stable variables innovations using the relationship between the extremal index and the moving average parameter. We analyze their performance through a simulation study.

Keywords: extreme value theory, mixing processes, tail index estimation

Classification: 60G70, 62G32

1. INTRODUCTION

The modelling of extremes may be done in two different ways: modelling the *maximum* of a collection of random variables, and modelling the *largest values* over some high threshold.

The Fisher–Tippett theorem is one of two fundamental theorems in the EVT (Extreme Value theory). It plays the same role as the Central Limit Theorem plays in the studies of sums of random variables.

Theorem 1.1. (Fisher and Tippett [8]) Let (X_n) be a sequence of i.i.d. random variables with distribution F. Let $M_n = \max(X_1, \dots, X_n)$. If there exist norming constants $c_n > 0$ and $d_n \in \mathbf{R}$ and some non-degenerate distribution function G such that

$$\frac{M_n - d_n}{c_n} \xrightarrow{D} G, \tag{1}$$

then G is one of the following three types :

(i) Gumbel

$$\Lambda(x) = \exp(-e^{-x}), x \in \mathbf{R},$$

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(ii) Fréchet

$$\Phi_\alpha(x) = \begin{cases} 0, & x \leq 0 \\ \exp(-x^{-\alpha}), & x > 0, \alpha > 0, \end{cases}$$

(iii) Weibull

$$\Psi_\alpha(x) = \begin{cases} \exp(-(-x)^{\alpha}) & x \leq 0, \alpha > 0 \\ 1, & x > 0. \end{cases}$$

We say that F belongs to the max-domain of attraction of the distribution of extreme values G, and we note $F \in MDA(G)$.

Suppose we have observations $X_1, \ldots, X_n$ from the MA(1) model, i. e.,

$$X_t = \lambda\ Z_{t-1} + Z_t, \tag{2}$$

where $\{Z_t\}$ is a sequence of independent and identically distributed random variables with mean zero and finite variance. In case $-1 < \lambda < 1$, the maximum likelihood estimator $\widehat{\lambda}_{MLE}$ for λ has the following asymptotic limit:

$$\sqrt{n}(\widehat{\lambda}_{MLE} - \lambda) \xrightarrow{D} \mathcal{N}(0, 1 - \lambda^2),$$

(see Brockwell and Davis [1]) where $\xrightarrow{D}$ stands for convergence in distribution. Feigin et al. [6] study the nonnegative MA(1) process assuming either the right or the left tail of the innovations distribution is regularly varying. The estimator of the moving average parameter is proposed and its asymptotic non-normal distribution is established. The rest of this paper is organized as follows. In Section 2, we present the positive MA(1) stable process. In Section 3 the new semi parametric estimator of the moving average parameter for a positive MA(1) stable process is introduced and its properties examined. In Section 4, we study the performance of our estimator by some simulations. Section 5 is devoted to the proofs. In Section 6, we introduce the extremal index which is the key parameter for extending extreme value theory results from i.i.d. to stationary sequences and we propose a new estimator for the moving average parameter based on the relationship between this parameter and the extremal index.

2. POSITIVE MA(1) STABLE PROCESS

The α-stable family of distributions $X \sim S(\alpha, \beta, \sigma, \mu)$ includes the Gaussian one as a spacial case. However, this class of distributions allows in addition for asymmetry and heavy tails and we have $F_X \in MDA(\Phi_\alpha)$. In general, closed form density function of X are not known. The exceptions are for $\alpha = 2$ corresponding to normal distribution, $\alpha = 1$ and $\beta = 0$ yielding Cauchy distribution and $\alpha = 0.5$, $\beta = \pm 1$ for the Lévy distribution. The α-stable random variable is defined in terms of its characteristic function $\varphi_X(t)$ (see Samorodnitsky and Taqqu [22]) given by

$$\varphi_X(t) = \exp\left\{ i\mu t - \sigma^\alpha |t|^\alpha \left(1 + i\beta \frac{t}{|t|} w(t, \alpha)\right)\right\} \tag{3}$$

with

$$w(t,\alpha) = \begin{cases} tan(\frac{\alpha\pi}{2}) & if \quad \alpha \neq 1 \\ \\ \frac{2}{\pi}\ln|t| & if \quad \alpha = 1, \end{cases}$$

where the characteristic exponent (index of stability, tail exponent) $\alpha \in]0,2]$, a skewness parameter $\beta \in [-1,1]$, a scale parameter $\sigma > 0$, a location parameter $\mu \in \mathbf{R}$.

The family of stable laws $S(\alpha,1,1,\mu)$ with $0 < \alpha < 1$, $\mu \geq 0$ define positive random variables with support $(\mu,\infty[$, such distributions have become a standard tool in modelling heavy tailed data in such diverse areas as finance, engineering and survival analysis. Quite often, one encounters insurance claims or lifetime data which display heavy tail behaviors which make positive stable laws good candidates for fitting this type of data.

In many applications, the desire to model the phenomena under study by non-negative dependent processes has increased. An excellent presentation of the classical theory concerning these models can be found, for example, in Brockwell and Davis [1]. Recently, advancements in such models have shifted focus to some specialized features of the model, e. g. heavy tail innovations of the model. Let the MA(1) process

$$X_t = \lambda\ Z_{t-1} + Z_t, \tag{4}$$

where $0 < \lambda < 1$ and $\sum_{j=0}^{\infty} \lambda^{j\omega} < \infty$ for $0 < \omega < \alpha$ and $Z_t \sim$ i.i.d. which, for simplicity, we take to be positive stable $S(\alpha,1,1,\mu)$, $0 < \alpha < 1$, $\mu \geq 0$. From Samorodnitsky and Taqqu [22] these random variables have the following approximate of the tail distribution for $x \to \infty$

$$1 - F_Z(x) \sim \frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)x^{-\alpha}, \tag{5}$$

and $X_t \sim S(\alpha,1,(\lambda^\alpha+1)^{1/\alpha},\mu(1+\lambda))$.

3. DEFINING THE ESTIMATOR AND MAIN RESULTS

We consider the MA(1) process in (4), we have

$$\lim_{x\to\infty}\frac{P(X_t > x)}{P(Z_t > x)} = 1 + \lambda^\alpha,$$

thus we have the following approximation

$$1 - F_X(x) \sim \frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)(1+\lambda^\alpha)x^{-\alpha}.$$

Hence we can estimate $\frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)(1+\lambda^\alpha)$ by $\frac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X}$, where

$$k = k(n) \to \infty, k/n \to 0$$

and

$$\widehat{\alpha}_X = \left(\frac{1}{k}\sum_{i=1}^{k}\log X_{n-i+1,n} - \log X_{n-k,n}\right)^{-1}$$

is the Hill estimator [11], with $X_{i,n}$ denoting the ith ascending order statistics $1 \leq i \leq n$, associated to the random sample $(X_1, X_2, \ldots, X_n)$. It follows that

$$\widehat{\lambda}_n = \left(\frac{\pi k X_{n-k,n}^{\widehat{\alpha}_X}}{2n\Gamma(\widehat{\alpha}_X) \sin\left(\frac{\pi\widehat{\alpha}_X}{2}\right)} - 1 \right)^{1/\widehat{\alpha}_X}. \tag{6}$$

We note that from Theorem 2.2 of Drees [4], we have

$$\sqrt{k}(\widehat{\alpha}_X - \alpha) \xrightarrow{D} \mathcal{N}\left(0, \sigma^2\right), \tag{7}$$

where

$$\sigma^2 = \alpha^2 \int_{(0,1]} \int_{(0,1]} (st)^{-(1+1/\alpha)} c(s,t) \nu(\mathrm{d}s) \nu(\mathrm{d}t),$$

ν being signed measure defined by $\nu(\mathrm{d}t) = t^{\alpha^{-1}}\mathrm{d}t - \delta_1(\mathrm{d}t)$, δ_1 is the Dirac measure with mass 1 at 1 and where

$$c(x,y) = \min(x,y) + \sum_{m=1}^{\infty} [c_m(x,y) + c_m(y,x)],$$

and

$$c_m(x,y) = \lim_{x\to\infty} \frac{n}{k} P\left[X_1 > F_X^{-1}\left(1 - \frac{k}{n}x\right), X_{1+m} > F_X^{-1}\left(1 - \frac{k}{n}y\right)\right]$$

for all $m \in \mathbf{N},\ x > 0,\ y \leq 1 + \varepsilon,\ \varepsilon > 0$ and F^{-1} denoting the inverse function of F.

We note from Dress [3] that:

$$\sigma^2 = \alpha^2 c(1,1). \tag{8}$$

In the case of MA(1) given by equation (4), de Haan et al. [9] showed that:

$$c(x,y) = \min(x,y) + (1+\lambda^\alpha)^{-1}(\min(x, y\lambda^\alpha) + \min(y, x\lambda^\alpha)).$$

Then we have

$$\sqrt{k}(\widehat{\alpha}_X - \alpha) \xrightarrow{D} \mathcal{N}\left(0, \alpha^2 \frac{(1+3\lambda^\alpha)}{(1+\lambda^\alpha)}\right). \tag{9}$$

The asymptotic normality of $\widehat{\lambda}_n$ is established in the following Theorem.

Theorem 3.1. Suppose the MA(1) process in (4) and $k = k_n$ be such that $k \to \infty$, $k/n \to 0$, then

$$\frac{\sqrt{k}}{\log(n/k)}(\widehat{\lambda}_n - \lambda) \xrightarrow{D} \mathcal{N}\left(0, \frac{\alpha^2(1+3\lambda^\alpha)\lambda^{2-2\alpha}}{(1+\lambda^\alpha)\left(\frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)\right)^2}\right).$$

We can estimate the extreme index α by the t-Hill estimator given by:

$$\widehat{\alpha}_X^{t-H} = \left(\left(\frac{1}{k} \sum_{j=1}^{k} \frac{X_{n-k,n}}{X_{n-j+1,n}} \right)^{-1} - 1 \right)^{-1}, \tag{10}$$

introduced in Fabián and Stehlík [5], its consistency for dependent data was proven in Jordanova et al. [13]. Other consistent estimator of α for the moving average process (4) is the t-lgHill estimator introduced in Jordanova et al. [14]. It is given by:

$$\widehat{\alpha}_X^{t-lgH} = \frac{M^{(2)} - (M^{(1)})^2}{M^{(1)}} \tag{11}$$

where

$$M^{(j)} = \frac{1}{k} \sum_{i=1}^{k} \left(\log \frac{X_{n-i+1,n}}{X_{n-k,n}} \right)^j, \quad j = 1, 2, \ldots$$

4. SIMULATION STUDY

Several approaches to the automated determination of an optimal sample fraction k for the Hill estimator have been studied (see e. g. Cheng and Peng [2], Neves and Fraga Alves [20]). An optimal bias/variance trade-off can be derived using the asymptotic mean squared error as the optimality criterion (see Hall and Welsh [10]), then we have:

$$k_{opt} = \arg\min_k E(\widehat{\alpha} - \alpha)^2. \tag{12}$$

To illustrate the performance of our estimator $\widehat{\lambda}_n$, we generate 100 replications of the time series $(X_1, \ldots, X_n)$ of sizes 1000 and 2000, where X_t is an MA(1) process satisfying

$$X_t = \lambda \, Z_{t-1} + Z_t, \; 1 < t < n, \tag{13}$$

with $0 < \lambda < 1$, and $Z_t \sim S(\alpha, 1, 1, 4)$, $0 < \alpha < 1$. Note that we use (12) to compute the values of the optimal fraction integer k_{opt}, the results are presented in Table 1 and Table 2, where *lb* and *ub* stand respectively for lower bound and upper bound of the confidence interval.

α	0.4		0.5	
n	1000	2000	1000	2000
$\widehat{\lambda}_n$	0.2158079	0.1949246	0.2496814	0.2311295
Bias	0.01580789	-0.005075427	0.0496814	0.03112951
RMSE	0.2789972	0.1793804	0.1825971	0.1562942
lb	0.1443343	0.1330465	0.08402811	0.09865995
ub	0.2872814	0.2568027	0.4153347	0.3635991
length	0.1429471	0.1237562	0.3313066	0.2649391

Tab. 1. Performance and 95% confidence intervals for $\lambda = 0.2$.

α	0.4		0.5	
n	1000	2000	1000	2000
$\widehat{\lambda}_n$	0.2911583	0.2930538	0.3609823	0.3101301
Bias	-0.008841675	-0.006946166	0.06098232	0.01013009
RMSE	0.1765576	0.1505765	0.252894	0.1779221
lb	0.2187662	0.2390450	0.1409462	0.1037048
ub	0.3635504	0.3470627	0.5810184	0.5165554
length	0.1447842	0.1080177	0.4400722	0.4128506

Tab. 2. Performance and 95% confidence intervals for $\lambda = 0.3$.

For an illustration of the behavior of $\widehat{\lambda}_n$ we made 100 samples of n = 4000 observations from the MA(1) in (13) for $\alpha = 0.4, \lambda = 0,2$. Then we plotted in Figure 1 the $\widehat{\lambda}_n$ and the $\widehat{\lambda}_n^{t-H}$ plots of the averages of the corresponding estimators together of λ for different k. We remark that:

1. Both estimators are biased because they are based on estimates of the tail index Hill and t-Hill which have optimality properties only when the underlying distribution is close to Pareto (see Jordanova et al. [12]). If the distribution is far from Pareto in particular the stable distribution, there may be outrageous errors and may perform very poorly.

2. Both estimators have similar behavior for fixed number of upper order statistics and show deviations from the true parameter $\lambda = 0.2$ as k is increased. Hence our estimator $\widehat{\lambda}_n$ for the moving average parameter is not robust using the two estimators Hill and t-Hill of extreme index. This result was expected because Jordanova et al. [12] have shown that the t-Hill and the Hill of extreme index applied to the moving average model are not robust with respect to large observations.

The same quality of the estimator is already noticed recently on the estimator of the autoregressive parameter (see Mami and Ouadjed [17]).

Now, we generate 100 replicates of sizes 10000 from the MA(1) in (13), we compare the bias and the root mean squared error (RMSE) of the three estimators of λ (our estimator $\widehat{\lambda}_n$, $\widehat{\lambda}_n^{t-H}$ and $\widehat{\lambda}_n^{t-lgH}$ in which we use the t-lgHill estimator in (11) for estimating the tail index α). The results are presented in Table 3. We remark that our estimator $\widehat{\lambda}_n$ has the smallest bias and the $\widehat{\lambda}_n^{t-H}$ estimator has the smallest RMSE.

α	0.4			
λ	0.2		0.3	
	Bias	RMSE	Bias	RMSE
$\widehat{\lambda}_n$	-0.0247184	0.1862913	-0.04036325	0.1053356
$\widehat{\lambda}_n^{t-H}$	-0.03595814	0.1073969	-0.05892177	0.08149884
$\widehat{\lambda}_n^{t-lgH}$	0.06373993	0.5579961	0.06922238	0.5446956

Tab. 3. Comparison of the estimators of moving average parameter.

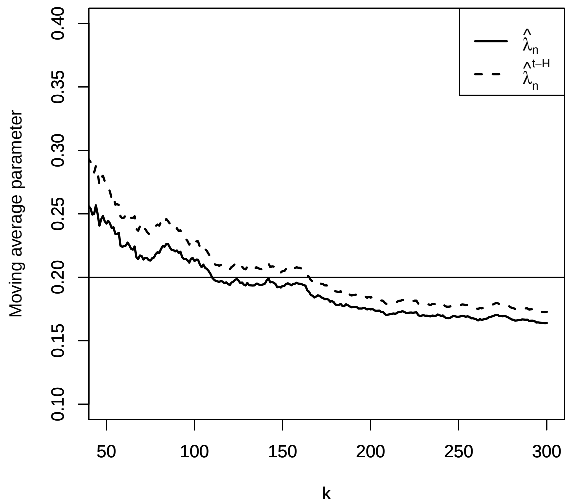

Fig. 1. Plot of the moving average estimators $\widehat{\lambda}_n$ and $\widehat{\lambda}_n^{t-H}$ for different k of the number of upper order statistics. The horizontal line is the true value $\lambda = 0.2$.

P r o o f . Let defining the tail quantile function of F_X as $U(t) = F_X^{-1}(1-1/t)$, for $t > 1$. Then for $x \to \infty$ we have

$$U(x) \sim \left(\frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)(1+\lambda^\alpha)\right)^{1/\alpha} x^{1/\alpha}. \tag{14}$$

Note that

$$\begin{aligned} \tfrac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X} - \tfrac{2}{\pi}\Gamma(\alpha)\sin\left(\tfrac{\alpha\pi}{2}\right)(1+\lambda^\alpha) \quad = \quad & \left(\tfrac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X} - \tfrac{k}{n}X_{n-k,n}^{\alpha}\right) \\ & + \tfrac{k}{n}U^\alpha(n/k)\left(\tfrac{X_{n-k,n}^\alpha}{U^\alpha(n/k)} - 1\right) \\ & + \tfrac{k}{n}U^\alpha(n/k) - \tfrac{2}{\pi}\Gamma(\alpha)\sin\left(\tfrac{\alpha\pi}{2}\right)(1+\lambda^\alpha). \end{aligned}$$

Using Mean-Value Theorem we find

$$\frac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X} - \frac{k}{n}X_{n-k,n}^{\alpha} = \left(\frac{k}{n}X_{n-k,n}^{\alpha}(\widehat{\alpha}_X - \alpha)\log X_{n-k,n}\right)(1+o_P(1)). \tag{15}$$

From Theorem 2.1 of Drees [4] we have

$$\frac{X_{n-k,n}^\alpha}{U^\alpha(n/k)} = 1 + O_P(1/\sqrt{k}). \tag{16}$$

From (14) we observe that for $n \to \infty$

$$\frac{k}{n}U^{\alpha}(n/k) \sim \frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)(1+\lambda^{\alpha}). \tag{17}$$

Combining (15), (16), (17) and using (9) we obtain

$$\frac{\sqrt{k}}{\log(n/k)}\left(\frac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X} - \frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)(1+\lambda^{\alpha})\right) \xrightarrow{D} \mathcal{N}\left(0, \alpha^4\frac{(1+3\lambda^{\alpha})}{(1+\lambda^{\alpha})}\right).$$

Using the map $f(x) = \left(\frac{x}{\frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)} - 1\right)^{1/\alpha}$, since

$$f\left(\frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)(1+\lambda^{\alpha})\right) = \lambda$$

and

$$f\left(\frac{k}{n}X_{n-k,n}^{\widehat{\alpha}_X}\right) = \left(\frac{\pi k X_{n-k,n}^{\widehat{\alpha}_X}}{2n\Gamma(\widehat{\alpha}_X)\sin\left(\frac{\pi\widehat{\alpha}_X}{2}\right)} - 1\right)^{1/\widehat{\alpha}_X} = \widehat{\lambda}_n$$

and applying the delta method, it follows that the estimator $\widehat{\lambda}_n$ defined in (6) satisfies the following result

$$\frac{\sqrt{k}}{\log(n/k)}(\widehat{\lambda}_n - \lambda) \xrightarrow{D} \mathcal{N}\left(0, \alpha^4\frac{(1+3\lambda^{\alpha})}{(1+\lambda^{\alpha})}\left[f'\left(\frac{2}{\pi}\Gamma(\alpha)\sin\left(\frac{\alpha\pi}{2}\right)(1+\lambda^{\alpha})\right)\right]^2\right).$$

This completes the proof of Theorem 3.1. □

Let now, the MA(1) process

$$X_t = \lambda\, Z_{t-1} + Z_t, \tag{18}$$

where $0 < \lambda < 1$ and $\sum_{j=0}^{\infty}\lambda^{j\omega} < \infty$ for $0 < \omega < \alpha$ and

$$Z_t \sim i.i.d.\ S(\alpha,\beta,\sigma,\mu),\ 0 < \alpha < 2.$$

In order to analyze the extremes in the process (18) we use the extremal index which is the key parameter extending extreme value theory from i.i.d. random processes to stationary time series and influences the frequency with which extreme events arrive as well as the clustering characteristics of an extreme event.

5. THE EXTREMAL INDEX

The main assumption in EVT is that the extreme observations are independent and identically distributed. This is not always fulfilled when working with real data.

Primary result incorporating dependence in the extremes is summarized in Leadbetter et al. [16]. For a strictly stationary time series (X_i) under some regularity conditions for the tail of F and for some suitable constants $c_n > 0$ and $d_n \in \mathbf{R}$, as the sample size $n \to \infty$

$$\frac{M_n - d_n}{c_n} \xrightarrow{D} (G)^\theta, \tag{19}$$

where $\theta \in]0,1[$ is the extremal index and G is the GEV distribution defined in (1). The quantity $1/\theta$ has a convenient heuristic interpretation, as it may be thought of as the mean cluster size of extreme values in a large sample.

The problem of estimating θ has received some attention in the literature (see Smith and Weissman [23], Weissman and Novak [25], Ferro and Segers [7]), Süveges [24] presents the maximum likelihood estimator as

$$\widehat{\theta}_{ML} = \frac{\sum_{i=1}^{N} qS_i + N - 1 + N_c - \left[\left(\sum_{i=1}^{N-1} qS_i N - 1 + N_c \right)^2 - 8N_c \sum_{i=1}^{N-1} qS_i \right]^{1/2}}{2 \sum_{i=1}^{N-1} qS_i} \tag{20}$$

where $S_i = T_i - 1$, T_i are the inter-exceedance times, N is the number of exceedances of a high threshold u and q is the estimate of $\bar{F}(u)$, and $N_C = \sum_{i=1}^{N-1} \mathbf{1}_{\{S_i \neq 0\}}$.

Leadbetter et al. [16] showed that the extremal index of the MA(1) process in (18) is

$$\theta = \frac{1}{1 + \lambda^\alpha}. \tag{21}$$

From (21) we have the following estimator of the moving average parameter

$$\widehat{\lambda}_n = \left(\frac{1}{\widehat{\theta}_n} - 1 \right)^{1/\widehat{\alpha}_n}. \tag{22}$$

Meerschaert and Scheffler [19] have proposed an alternative robust estimator for the tail index α based on the asymptotic of the sum. The method works for dependent data as the MA(1) process in (18) and performs about as well as Hill's estimator. The estimator is defined as follows

$$\widehat{\alpha}_{MS} = \left(\frac{\ln_+ \sum_{i=1}^{n} (X_i - \bar{X}_n)^2}{2 \ln n} \right)^{-1} \tag{23}$$

where $\bar{X}_n = n^{-1}(X_1 + \ldots + X_n)$ is the sample mean and $\ln_+(x) = \max(\ln x, 0)$.

Their estimator performs best in exactly those situations in which Hill's estimator is most likely to fail.

There exist other methods for estimating the stability index α. We can cite the empirical quantiles (see McCulloch [18]), the empirical characteristic function (see Koutrouvelis [15]) and the maximum likelihood (see Nolan [21]).

We are going to analyze the performance of the estimator in (22), through a simulation study based on the following different steps of the estimation procedure:

1. Estimate the extremal index θ using $\widehat{\theta}_{ML}$ for a high threshold u.
2. Estimate the tail index α using $\widehat{\alpha}_{MS}$ and the estimator of Koutrouvelis [15] denoted $\widehat{\alpha}_{KO}$.
3. Finally we estimate λ by

$$\widehat{\lambda}_{MS} = \left(\frac{1}{\widehat{\theta}_{ML}} - 1\right)^{1/\widehat{\alpha}_{MS}} \tag{24}$$

and

$$\widehat{\lambda}_{KO} = \left(\frac{1}{\widehat{\theta}_{ML}} - 1\right)^{1/\widehat{\alpha}_{KO}}. \tag{25}$$

We consider samples of size $n = 7000$ of the model MA(1) in (18) for $\lambda = 0.2$, $Z_t \sim S(0.4, 0.2, 1, 4)$ and $Z_t \sim S(1.2, 0.2, 1, 4)$, we use the threshold $u = q_{0.99}$ (the empirical quantile 0.99) for estimate $\widehat{\theta}_{ML}$ and compare the two estimators $\widehat{\lambda}_{MS}$ and $\widehat{\lambda}_{KO}$ using the bias (Bias) and the root mean square error (RMSE). We generate 100 replicates of the estimation procedures. The results are presented in the Table 4. We remark that $\widehat{\lambda}_{MS}$ has the smallest bias and RMSE.

λ	0.2			
α	0.4		1.2	
	Bias	RMSE	Bias	RMSE
$\widehat{\lambda}_{MS}$	0.005247667	0.08101369	0.001878179	0.058983
$\widehat{\lambda}_{KO}$	0.01595753	0.08549662	0.01972255	0.05910452

Tab. 4. Performance of $\widehat{\lambda}_{MS}$ and $\widehat{\lambda}_{KO}$.

6. CONCLUSION

In this work we propose two estimators for the moving average parameter. The first is based on the relationship between the tail of the process MA(1) and the innovations which have positive stable distribution. We establish its asymptotic normality and study its performance. The second estimator is for MA(1) process driven by α stable variables with $(0 < \alpha < 2)$ using the extremal index. We analyze their performance through a simulation study. For the future work we will try to find the asymptotic distribution of the second estimator.

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