KYBERNETIKA — VOLUME *53* (2017), NUMBER 5, PAGES 853–867

STABILIZATION OF NONLINEAR SYSTEMS WITH VARYING PARAMETER BY A CONTROL LYAPUNOV FUNCTION

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In this paper, we provide an explicit homogeneous feedback control with the requirement that a control Lyapunov function exists for affine in control systems with bounded parameter that satisfies an homogeneous condition. We use a modified version of the Sontag's formula to achieve our main goal. Moreover, we prove that the existence of an homogeneous control Lyapunov function for an homogeneous system leads to an homogeneous closed-loop system which is asymptotically stable by an homogeneous feedback control. In addition, we study the finite time stability for affine in control systems with varying parameter.

Keywords: feedback stabilization, homogeneous system, nonlinear control systems, Lyapunov function, finite time stability

Classification: 93D05, 93D15

1. INTRODUCTION

The stability problem of nonlinear control systems with uncertain parameters has witnessed an increasing interest in recent years. Owing to additional conditions that ensure stability and performance in presence of uncertainty on the physical parameters of the system, many processes such as the robust stabilization have been elaborated to study the control systems with uncertain parameters.

For smooth multi-input systems that are affine in the control

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{1}$$

where the state $x \in R^n$, the input $(u_1, \dots, u_m) \in R^m$, $f(0) = 0$, and $f,\ g_1,\ g_2, \dots, g_m$ are continuously differentiable vector fields, the basic stabilization Lyapunov condition provided in [1, 17, 19] and [20] can be expressed as follows:

there exists a positive definite real function $V:\ \mathbb{R}^n \longrightarrow \mathbb{R}$ (i. e., $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ near zero) such that for any $x \neq 0$, one has near zero

DOI: 10.14736/kyb-2017-5-0853

$$\text{if } \left\{ \begin{array}{c} \nabla V.g_1(x) = 0 \\ \vdots \\ \nabla V.g_m(x) = 0 \end{array} \right\}, \text{ then } \nabla V.f(x) < 0.$$

The concept of control Lyapunov functions (CLF) presented by Arstein [1] and Sontag [18] made enormous impact on stabilization theory. Sontag [17] introduced a universal feedback by involving the CLF. He proved that if (1) satisfies the above Lyapunov condition, then stabilization is possible by means of a feedback law that depends directly on the dynamics of the system. The CLF has been extensively adopted in diverse issues. Few authors tried to expand these results to uncertain nonlinear systems. However, how to extract CLF for nonlinear systems is still an open issue except in particular forms. Wang [22] designed a feedback controller related to the boundary of the uncertain parameters.

A study of the stabilization problem of a collection of multi-input nonlinear systems having common linear part with uncertain parameters is given in [3]. Their investigation is based on the existence of a quadratic common Lyapunov function which is not trivial. Moreover, they give a feedback which can globally asymptotically stabilize the collection of systems simultaneously. A new approach towards the stabilization of single-input non linear systems in triangular form was suggested in [4]. The authors show for any triangular system satisfying Coron-Praly sufficient condition, a continuous asymptotic stabilizing feedback can be designed using homogeneity approximation. Indeed in [11], it was already proved that if a nonlinear system has its first term in Taylor expansion asymptotically stable, then the nonlinear system is locally asymptotically stable. An extension of the Sontag's control to the stabilization of affine nonlinear systems with bounded parameter are stated in [21]. However, this control fails to preserve the homogeneity of the closed loop system. Therefore, our motivation is to design a modified version of Sontag's control that preserve the homogeneity of the closed loop system.

Another important subject in control theory is the finite time stabilization problem of continuous affine in control systems with parameter. The purpose of such study is to provide a controller that guarantee the finite time stabilization. New techniques developed in [2] for obtaining continuous finite-time stabilizing controllers. In [13], the authors propose an additional integral property on the Lyapunov function to prove the finite time convergence. Moreover, they develop some results about the regularity of the setting function outside the origin. However in [6], a back-stepping-like procedure proposed to construct an adaptive finite time controller to stabilize a class of nonlinear systems in normal form with parametric uncertainties.

In this paper, we consider the multi-input nonlinear systems in the form

$$\dot{x} = f(x,\theta) + g(x)u \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input and $\theta \in \mathbb{R}^d$ is a time invariant parameter. The main objective is to design a controller that make the nonlinear system (2) with time invariant parameter globally asymptotically stable. The purpose is to give a controller which maintain the homogeneity of the closed loop system under a necessary and sufficient stability condition.

The paper is organized as follows. In section 2, some preliminaries are introduced. Section 3 is divided in two parts. In the first one, we construct a stabilizing feedback for nonlinear systems affine in control. In the second, we study the stabilization of homogeneous systems with parameter by homogeneous feedback-law. In the last section, we investigate the problem of finite time stability for systems depending on a bounded parameter.

2. PRELIMINARIES

Let us consider a class of uncertain affine nonlinear systems described by

$$\dot{x} = f(x, \theta) + g(x)u \tag{3}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input, $g(x) = (g_1(x), \dots, g_m(x))$ and $\theta \in \mathbb{R}^d$ is a time invariant parameter vector. Without loss of generality, assume that the parameter θ varies in a given compact set $\Omega \subset \mathbb{R}^d$. The vector fields f and $g_1, \dots, g_m$ are assumed sufficient smooth on their arguments. We also assume that $f(0, \theta) = 0$ for all $\theta \in \Omega$, which means that the origin is an equilibrium point for the unforced system.

For the study of the stabilization of the nonlinear system (3), we recall the following results. Let $\{r_i,\ 1 \leq i \leq n\}$ a family of fixed positive reals, $r = (r_1, \dots, r_n)$. Let δ^r the dilation defined on $\mathbb{R}^n$ by $\delta^r_\varepsilon x = (\varepsilon^{r_1} x_1, \dots, \varepsilon^{r_n} x_n)$, for $\varepsilon > 0$, $x \in \mathbb{R}^n$.

Definition 2.1.

i) We say that a function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ is homogenous of degree k with respect to the dilation δ^r_ε, if
$$h(\delta^r_\varepsilon x) = \varepsilon^k h(x),\ \forall x \in \mathbb{R}^n,\ \forall \varepsilon > 0.$$

ii) We say that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is homogenous of degree k if each $f_i,\ \ i \in \{1, \dots, n\}$ is homogeneous of degree $k + r_i$.

iii) We say that a function $h : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, is homogenous of degree k with respect to δ^r_ε and the parameter θ, if
$$h(\delta^r_\varepsilon x, \theta) = \varepsilon^k h(x, \theta),\ \forall (x, \theta) \in \mathbb{R}^n \times \Omega,\ \forall \varepsilon > 0.$$

iv) We say that $f : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ is homogenous of degree k with respect to the parameter θ, if each $f_i,\ \ i \in \{1, \dots, n\}$ is homogeneous of degree $k+r_i$ with respect to the parameter θ.

Notations 2.2. Let $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n$. We introduce the following notations:

- $\langle x^T, y^T \rangle = \sum_{i=1}^{n} x_i y_i$ denotes the Euclidean inner product.
- $\|x\| = \sqrt{\langle x^T, x^T \rangle}$ denotes the Euclidean norm on $\mathbb{R}^n$.

- Let $M \in \mathcal{M}_{n,p}(\mathbb{R})$, M^T denotes the transpose matrix of M.
- Let $V: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a defferentiable map, we denote $\nabla V(x) = (\frac{\partial V}{\partial x_1}(x), \ldots, \frac{\partial V}{\partial x_n}(x))$.

Now, we recall the following result.

Lemma 2.3. (Rosier [14]) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. If V is homogeneous of degree k with respect to the dilation δ^r and if f is homogeneous of degree k_1, then $\langle \nabla V(x), f(x) \rangle$ is homogeneous of degree $k + k_1$.

3. CONSTRUCTION OF A STABILIZING FEEDBACK

Let the system given by

$$\dot{x} = f(x,\theta) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in \mathbb{R}^n \ , \ \ u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m \tag{4}$$

where $f \in C^1(\mathbb{R}^n \times \Omega)$, $\Omega \subset \mathbb{R}^d$, $g_i \in C^1(\mathbb{R}^n)$ for all $1 \leq i \leq m$ and $f(0,\theta) = 0$ for all $\theta \in \Omega$. In this section, we use control Lyapunov functions to construct a feedback control which stabilizes the control system (4). Then, we extend the given result in the case where the functions f and g, $(m = 1)$, are homogeneous with respect to the same dilation δ^r. We use a modified version of the Sontag feedback control.

In the following, we recall some classical definitions and results.

Definition 3.1. The control system (4) is said to be stabilizable (respectively continuously stabilizable) if there exists a non empty neighborhood of the origin $\mathcal{V}$ in $\mathbb{R}^n$ and a feedback control law $u \in C^0(\mathcal{V} \setminus \{0\}, \mathbb{R}^m)$ (respectively $u \in C^0(\mathcal{V}, \mathbb{R}^m)$) such that:

1) $u(0) = 0$,

2) the origin of the closed loop system (4) is asymptotically stable for all $\theta \in \Omega$.

Notations 3.2. Let Ω a compact subset of $\mathbb{R}^d$, $\mathcal{V}$ be a neighborhood of the origin and $V : \mathcal{V} \to \mathbb{R}_+$ a continuously differentiable function. Let $(x,\theta) \in \mathcal{V} \times \Omega$, we denote the following:

$$\begin{aligned} a(x,\theta) &= \langle \nabla V(x), f(x,\theta) \rangle, \\ \bar{a}(x) &= \max_{\theta \in \Omega} a(x,\theta), \\ b_i(x) &= \langle \nabla V(x), g_i(x) \rangle, \ 1 \leq i \leq m, \\ B(x) &= (b_1(x), \ldots, b_m(x)), \\ b(x) &= \|B(x)\|^2. \end{aligned} \tag{5}$$

Definition 3.3.

i) A continuously differentiable positive definite function $V : \mathcal{V} \to \mathbb{R}_+$ is said to be a control Lyapunov function for the system (4) if for all $x \in \mathcal{V} \setminus \{0\}$, for all $\theta \in \Omega$ one has

$$\inf_{u \in \mathbb{R}^m} (a(x,\theta) + \langle B(x), u \rangle) < 0.$$

ii) We say that a control Lyapunov function V satisfies the small control property, if for each $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x\| < \delta$, then there exists some control u with $\|u\| < \epsilon$, satisfying $a(x,\theta) + \langle B(x), u\rangle < 0$ for all $\theta \in \Omega$.

Remark 3.4. If $m = 1$, the small control property is equivalent to

$$\limsup_{\|x\|\to 0} \frac{a(x,\theta)}{|B(x)|} \leq 0.$$

Lemma 3.5. (Cha et al. [21]) Let $g : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$, where Ω is a compact set of $\mathbb{R}^d$. If g is continuous on $\mathbb{R}^n \times \Omega$, then the function $\overline{g}(x) := \max_{\theta\in\Omega} g(x,\theta)$ is continuous on $\mathbb{R}^n$.

In the next, we give two results of stabilization. The first one in the case of affine in control systems depending on a parameter, the second in the homogeneous case.

3.1. Stabilization of affine systems depending on a parameter

Theorem 3.6. If there exists a continuously differentiable control Lyapunov function $V : \mathcal{V} \to \mathbb{R}_+$ for the control system (4), then it is stabilizable by means of the feedback $u(x) = (u_1(x), \ldots, u_m(x))$ defined, for $i \in \{1, \ldots, m\}$, by

$$u_i(x) = \begin{cases} 0 & \text{if } b(x) = 0 \\ -b_i(x)\dfrac{\bar{a}(x) + (\mid \bar{a}(x) \mid^p + b(x)^q)^{\frac{1}{p}}}{b(x)} & \text{if } b(x) \neq 0 \end{cases} \tag{6}$$

where $p > 1, \quad q > 1$ are positive real numbers.

If furthermore V satisfies the small control property, then the feedback control (6) is also continuous at the origin.

Proof. Suppose there exists a continuously differentiable control Lyapunov function $V : \mathcal{V} \to \mathbb{R}_+$ for the control system (4). Let $E = \{(x,y) \in \mathbb{R}^2 : x < 0 \text{ or } y > 0\}$ and φ the function defined on E by

$$\varphi(x,y) = \begin{cases} 0 & \text{if } y = 0 \\ \dfrac{x + (\mid x \mid^p + \mid y \mid^q)^{\frac{1}{p}}}{y} & \text{if } y \neq 0. \end{cases}$$

According to [12], φ is continuous on E. The fact that V is a control Lyapunov function implies that $(\bar{a}(x), b(x)) \in E$ for all $x \in \mathcal{V} \setminus \{0\}$. Moreover the functions $\bar{a}(x)$ and $b(x)$ are continuous.

Thus the feedback function $u(x) = (u_1(x), u_2(x), \ldots, u_m(x))$ defined by

$$u_i(x) = -b_i(x)\varphi(\bar{a}(x), b(x))$$

is continuous on $\mathcal{V}\setminus\{0\}$. We obtain for all $x\in\mathcal{V}\setminus\{0\}$, for all $\theta\in\Omega$

$$\langle\nabla V(x), f(x,\theta)+\sum_{i=1}^{m}u_i(x)g_i(x)\rangle=-(\mid\bar{a}(x)\mid^p+b(x)^q)^{\frac{1}{p}}<0.$$

We conclude that V is a definite Lyapunov function for the closed loop system (4) by the feedback $u(x)$ given below, that implies that the origin of the closed loop system (4) is locally asymptotically stable. □

3.2. Stabilization of homogeneous systems depending on a parameter

In the following, we give a result of stabilization of the nonlinear control system (4), in the case where $m=1$, f and g are homogeneous of degree k_0 and k_1 respectively with respect to the dilation δ^r, $r=(r_1,r_2,\dots,r_n)$ and $r_i>0$ for all i. To simplify the notations we denote $\delta=\delta^r$.

We recall the following.

Proposition 3.7. (Sepulchre and Aeyels [15]) Let $V:\mathbb{R}^n\to\mathbb{R}\to$ be a positive definite function, homogeneous of degree k with respect to the dilation δ and continuously differentiable. Then, for each $s>0$, the following properties are satisfied:

(a) The level set $V^s:=\{x \text{ such that } V(x)=s\}$ of V is homogeneous, i.e., $V^s=\delta_{s^{\frac{1}{k}}}(V^1)$.

(b) V^s is homeomorphic to S^{n-1}.

(c) For each $i\in\{1,\dots,n\}$, $\frac{\partial V}{\partial x_i}$ is homogeneous of degree $(k-r_i)$, i.e.

$$\frac{\partial V}{\partial x_i}(\delta_s(x))=s^{k-r_i}\frac{\partial V}{\partial x_i}(x).$$

Consider the single input system described by

$$\dot{x}=f(x,\theta)+g(x)u \tag{7}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}$, θ is a parameter in a bounded set Ω of $\mathbb{R}^d$, f (resp. g) are of class C^1 on $\mathbb{R}^n$ and homogeneous of degree k_0 (resp. k_1) with respect to the dilation δ.

Lemma 3.8. Let $V:\mathbb{R}^n\to\mathbb{R}$ a map of class C^1, positive definite and homogeneous of degree k with respect to the dilation δ. Let $y\in\mathbb{R}^n\setminus\{0\}$ and $\lambda>0$; Denote $x=\delta_\lambda(y)$, one has for all $\theta\in\Omega$

$$\langle\nabla V(x), f(x,\theta)\rangle=\lambda^{k+k_0}\langle\nabla V(y), f(y,\theta)\rangle$$

and

$$\langle\nabla V(x), g(x)\rangle=\lambda^{k+k_1}\langle\nabla V(y), g(y)\rangle.$$

P r o o f. Let $y \in \mathbb{R}^n \setminus \{0\}$ and $\lambda > 0$; denote $x = \delta_\lambda(y)$, we have

$$\begin{aligned}\nabla V(x) &= \nabla V(\delta_\lambda(y)) \\ &= (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n})(\delta_\lambda(y)) \\ &= (\lambda^{k-r_1}\frac{\partial V}{\partial x_1}(y), \dots, \lambda^{k-r_n}\frac{\partial V}{\partial x_n}(y)) \\ &= \lambda^k A_\lambda^{-1} \nabla V(y),\end{aligned}$$

where

$$A_\lambda^{-1} = \begin{pmatrix} \lambda^{-r_1} & 0 & . & . & 0 \\ 0 & \lambda^{-r_2} & 0 & . & 0 \\ . & & . & & . \\ . & & & . & . \\ 0 & . & . & 0 & \lambda^{-r_n} \end{pmatrix}$$

So for $\theta \in \Omega$, one has

$$\begin{aligned}\langle \nabla V(x), f(x,\theta)\rangle &= \langle \nabla V(\delta_\lambda(y)), f(\delta_\lambda(y),\theta)\rangle \\ &= \langle \lambda^k A_\lambda^{-1}\nabla V(y), \lambda^{k_0} A_\lambda f(y,\theta)\rangle \\ &= \lambda^{k+k_0}\langle \nabla V(y), f(y,\theta)\rangle.\end{aligned}$$

A similar computation gives

$$\begin{aligned}\langle \nabla V(x), g(x)\rangle &= \langle \nabla V(\delta_\lambda(y)), g(\delta_\lambda(y))\rangle \\ &= \langle \lambda^k A_\lambda^{-1}\nabla V(y), \lambda^{k_1} A_\lambda g(y)\rangle \\ &= \lambda^{k+k_1}\langle \nabla V(y), g(y)\rangle.\end{aligned}$$

□

Let $V : \mathbb{R}^n \to \mathbb{R}$ a map of class C^1 and denote

$$\begin{aligned}&a(x,\theta) = \langle \nabla V(x), f(x,\theta)\rangle, \\ &b(x) = \langle \nabla V(x), g(x)\rangle, \\ &\bar{a}(x) = \sup_{\theta\in\Omega} a(x,\theta).\end{aligned}$$

Theorem 3.9. If there exists a continuously differentiable control Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ for the control system (4) homogeneous of degree k with respect to the dilation δ, then the feedback control

$$v(x) = \begin{cases} 0 & \text{if } b(x) = 0 \\ -\dfrac{\bar{a}(x) + (\mid \bar{a}(x) \mid^p + b(x)^{2q})^{\frac{1}{p}}}{b(x)} & \text{if } b(x) \neq 0 \end{cases} \tag{8}$$

where $p = \frac{2q(k+k_1)}{k+k_0}$, stabilizes the system (7) and is homogeneous of degree $k_0 - k_1$ with respect to the dilation δ.

If furthermore V satisfies the small control property, then the feedback control (8) continuously stabilizes the system (7).

P r o o f . Let $V : \mathbb{R}^n \to \mathbb{R}$ be an homogeneous control Lyapunov function for the system (7). By Theorem 3.6, the feedback control (8) continuously stabilizes the system (7). In the following, we verify that it is homogeneous of degree $k_0 - k_1$ with respect to the dilation δ. Let $y \in \mathbb{R}^n \setminus \{0\}$ and $\lambda > 0$, we denote $x = \delta_\lambda(y)$. By Lemma 3.8, for $\theta \in \Omega$ we have

$$\begin{aligned} a(x,\theta) &= \langle \nabla V(x), f(x,\theta)\rangle \\ &= \langle \nabla V(\delta_\lambda(y)), f(\delta_\lambda(y),\theta)\rangle \\ &= \lambda^{k+k_0} a(y,\theta). \end{aligned}$$

So

$$\begin{aligned} \bar{a}(x) &= \max_{\theta\in\Omega} a(x,\theta) \\ &= \max_{\theta\in\Omega} \lambda^{k+k_0} a(y,\theta) \\ &= \lambda^{k+k_0} \bar{a}(y). \end{aligned}$$

In addition, we have

$$\begin{aligned} b(x) &= b(\delta_\lambda(y)) \\ &= (\langle \nabla V(x), g(x)\rangle)^2 \\ &= \lambda^{2(k+k_1)} (\langle \nabla V(y), g(y)\rangle)^2 \\ &= \lambda^{2(k+k_1)} b(y) \end{aligned} .$$

Finilly, it is easy to verify that if we choose p and q satisfying $(k+k_0)p = 2(k+k_1)q$, then the feedback function u given in the form (8) below is homogeneous of degree k_0-k_1. We conclude that the homogeneous system (7) is stabilizable by an homogeneous feedback of degree $k_0 - k_1$. □

Example 3.10. Consider the system

$$\begin{cases} \dot{x}_1 = x_1^3(1+\sin(\theta)) + x_1 x_2^2 \cos(\theta) - u x_1^2 \\ \dot{x}_2 = -x_2^3 - x_1^2 x_2 \cos(\theta) - u x_3^4 \\ \dot{x}_3 = -x_3^5 + u x_2 x_3 \end{cases} \tag{9}$$

Denote $f(x,\theta) = \begin{pmatrix} x_1^3(1+\sin(\theta)) + x_1 x_2^2 \cos(\theta) \\ -x_2^3 - x_1^2 x_2 \cos(\theta) \\ -x_3^5 \end{pmatrix}$ and $g(x) = \begin{pmatrix} -x_1^2 \\ -x_3^4 \\ x_2 x_3 \end{pmatrix}$,
$r = (1, 1, \frac{1}{2})$. f and g are homogeneous of degree 2 and 1 respectively with respect to the dilation δ^r.

Let the control Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_3^4.$$

V is homogeneous of degree 2 with respect the dilation δ^r.

A simple computation gives

$$a(x,\theta) = \langle \nabla V(x), f(x,\theta)\rangle = x_1^4(1+\sin(\theta)) - x_2^4 - x_3^8 \le \bar{a}(x) = 2x_1^4 - x_2^4 - x_3^8$$
$$b(x) = \langle \nabla V(x), g(x)\rangle = -x_1^3.$$

Let $(p,q)=(3,2)$, by Theorem 3.9, the feedback

$$u(x)=\frac{(2x_1^4-x_2^4-x_3^8)+(\mid 2x_1^4-x_2^4-x_3^8\mid^3+x_1^{12})^{\frac{1}{3}}}{x_1^3},\ \text{if } x_1\neq 0$$

is homogeneous of degree 1 and stabilizes the system (9).

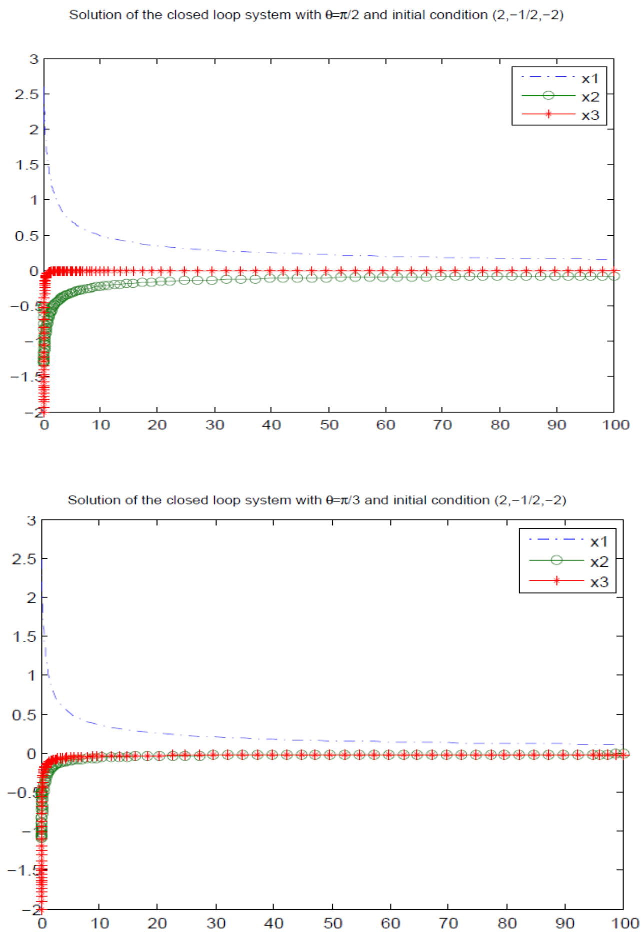

Fig. 1.

4. EXPONENTIALLY STABILIZING CLFS AND FINITE CLF

4.1. Exponentially stabilizing CLFs

Let the system described by

$$\dot{x} = f(x,\theta) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in \mathbb{R}^n \ , \ \ u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m \tag{10}$$

where $f \in C^1(\mathbb{R}^n \times \mathbb{R}^d)$, $g_i \in C^1(\mathbb{R}^n)$ for all $1 \leq i \leq m$ and $f(0,\theta) = 0$ for all θ. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, and $\Omega \subset \mathbb{R}^d$ is a compact set. Using the notations (5), we recall the following:

Definition 4.1. (Huang et al. [5]) Assume that $V : \mathbb{R}^n \to \mathbb{R}_+$ is a positive definite function and there exist positive constants α_1 and α_2 such that

$$0 < \alpha_1 \parallel x \parallel^{\alpha_2} \leq V(x).$$

The function V is said to be a e-CLF for the system (10) if it is smooth, proper and satisfies for all $x \neq 0$

$$b(x) = 0 \ \Rightarrow a(x,\theta) \leq -cV(x), \quad \forall \theta \in \Omega,$$

where c is a positive constant.

Theorem 4.2. If there exists a e-CLF for the system (4), then the following control feedback

$$u_i(x) = \begin{cases} 0 & \text{if } b(x) = 0 \\ -b_i(x) \dfrac{\bar{a}(x) + cV(x) + ((\bar{a}(x) + cV(x))^2 + b(x)^2)^{\frac{1}{2}}}{b(x)} & \text{if } b(x) \neq 0 \end{cases} \tag{11}$$

makes the closed loop system (10) exponentially stable.

Proof. Let $x \in \mathbb{R}^n \setminus \{0\}$, we have two cases:

First case: If $\langle \nabla V(x), g_i(x) \rangle = 0$, for all $1 \leq i \leq m$, then $b(x) = 0$. So

$$\dot{V}(x) = \langle \nabla V(x), f(x,\theta) \rangle \leq -cV(x).$$

Second case: If there exists $1 \leq i \leq m$ such that $\langle \nabla V(x), g_i(x) \rangle \neq 0$, then $b(x) \neq 0$. So

$$\begin{aligned} \dot{V}(x) &= \langle \nabla V(x), f(x,\theta) \rangle + \sum_{i=1}^{m} u_i(x) \langle \nabla V(x), g_i(x) \rangle \\ &= a(x,\theta) - b_i^2(x) \frac{\bar{a}(x) + cV(x) + (\mid \bar{a}(x) + cV(x) \mid^2 + b(x)^2)^{\frac{1}{2}}}{b(x)} \\ &\leq -cV(x). \end{aligned}$$

Thus, for $t \geq 0$,

$$\alpha_1 \parallel x(t) \parallel^{\alpha_2} \leq V(x) \leq e^{-c(t-t_0)} V(x_0)$$

which implies

$$\parallel x(t) \parallel \leq [\frac{V(x_0)}{\alpha_1}]^{\frac{1}{\alpha_2}} e^{\frac{-c(t-t_0)}{\alpha_2}}.$$

We conclude that the closed-loop system (10) is exponentially stable. □

4.2. Finite time stabilization by f-CLF

Consider the system of differential equations

$$\dot{x}(t) = f(x(t)) \tag{12}$$

where $f : D \to \mathbb{R}^n$ is continuous on an open neighborhood $D \subset \mathbb{R}^n$ of the origin and $f(0) = 0$. A continuously differentiable function $x : I \to D$ is said to be a solution of (12) on the interval $I \subset \mathbb{R}$, if x satisfies (12) for all $t \in I$. The continuity of f implies that, for every $x \in D$, there exist $T_0 < 0 < T_1$ and a solution $x(t)$ of (12) defined on (T_0, T_1) such that $x(0) = x_0$. A solution x is said to be right maximally defined if x cannot be extended on the right (either uniquely or nonuniquely) to a solution of (12). Every solution of (12) has an extension that is right maximally defined. In this case, we denote by $\varphi_x(t)$ the unique solution of (12) on $[0, T(x))$ satisfying $\varphi_x(0) = x$.

Definition 4.3. The origin is said to be a finite-time stable equilibrium of (12) if there exists an open neighborhood $\mathcal{N} \subset D$ of the origin and a function $T : \mathcal{N} \setminus \{0\} \to (0 + \infty)$, called the settling-time function, such that the following statements hold:

(i) Finite-time convergence: For every $x \in \mathcal{N} \setminus \{0\}$, $\varphi_x(t)$ is defined on $[0, T(x))$, $\varphi_x(t) \in \mathcal{N} \setminus \{0\}$, for all $t \in [0, T(x))$, and $\lim_{t \to T(x)} \varphi_x(t) = 0$.

(ii) Lyapunov stability: For every open neighborhood U_ϵ of 0, there exists an open subset U_δ of $\mathcal{N}$ containing 0 such that, for every $x \in U_\delta \setminus \{0\}$, $\varphi_x(t) \in U_\epsilon$ for all $t \in [0, T(x))$.

The origin is said to be a globally finite-time-stable equilibrium if it is a finite-time stable equilibrium with $D = \mathcal{N} = \mathbb{R}^n$.

In the following, we consider the system described by

$$\dot{x} = f(x, \theta) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in \mathbb{R}^n \;, \;\; u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m \tag{13}$$

where $f \in C^0(\mathbb{R}^n \times \mathbb{R}^d)$, $g_i \in C^0(\mathbb{R}^n)$ for all $1 \leq i \leq m$ and $f(0, \theta) = 0$ for all θ. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, and $\Omega \subset \mathbb{R}^d$ a compact set. We use the notations (5).

Definition 4.4. (Huang et al. [5]) Let $c > 0$ and $0 < \beta < 1$. A positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a f-CLF of the system (13) if it is smooth, proper and satisfies for $x \neq 0$

$$b(x) = 0 \Rightarrow a(x, \theta) \leq -cV^\beta(x), \quad \forall \theta \in \Omega.$$

Lemma 4.5. (Huang et al. [5]) If there exists a smooth positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(0) = 0$ and a positive definite continuous function $r : \mathbb{R}^+ \to \mathbb{R}^+$ with $r(0) = 0$ such that

$$\dot{V}(x) \leq -r(V(x)) \tag{14}$$

and for all $\mu > 0$

$$\int_0^{\mu} \frac{\mathrm{d}z}{r(z)} < +\infty \tag{15}$$

then, the system (13) is finite time stable. Moveover, the setting time T satisfies the following inequality

$$T \leq t_0 + \int_0^{V(x_0)} \frac{\mathrm{d}z}{r(z)}. \tag{16}$$

Theorem 4.6. If there exists a f-CLF for system (13), then it can be stabilized in finite time by the controller

$$u_i(x) = \begin{cases} 0 & \text{if } b(x) = 0 \\ -b_i(x)\dfrac{\bar{a}(x) + cV^{\beta}(x) + (\mid \bar{a}(x) + cV^{\beta}(x) \mid^2 + b(x)^2)^{\frac{1}{2}}}{b(x)} | & \text{if } b(x) \neq 0. \end{cases} \tag{17}$$

Moreover, the setting time satisfies

$$T \leq t_0 + \frac{V(x_0)^{1-\beta}}{c(1-\beta)}.$$

Proof. Let $x \in \mathbb{R}^n \setminus \{0\}$, there is two cases:

i) If $\langle \nabla V(x), g_i(x) \rangle = 0$, for all $1 \leq i \leq m$, that means $b(x) = 0$, then

$$\dot{V}(x) = \langle \nabla V(x), f(x, \theta) \rangle \leq -cV^{\beta}(x).$$

ii) If there exists $1 \leq i \leq m$, such that $\langle \nabla V(x), g_i(x) \rangle \neq 0$, that means $b(x) \neq 0$, then by the same arguments given in the proof of theorem (4.2), we find

$$\dot{V}(x) \leq -cV^{\beta}(x)$$

This implies that the system (13) is finite time stable under the control (17).
Using (16), the setting time satisfies $T \leq t_0 + \int_0^{V(x_0)} \frac{\mathrm{d}z}{cz^{\beta}} = t_0 + \frac{V(x_0)^{1-\beta}}{c(1-\beta)}$. □

Example 4.7. Consider the system

$$\begin{cases} \dot{x}_1 = x_2^3 - sign(x_1)\theta(t) - x_1 \\ \dot{x}_2 = -x_1 + x_2 + x_2 u. \end{cases}$$

Let $V(x) = \frac{1}{2}(x_1^4 + x_2^2)$ and $\theta(t) = \frac{1}{2} + \frac{1}{t^2+\frac{1}{2}} \in [\frac{1}{2}, 2.5]$,

$$f(x,\theta) = \begin{pmatrix} x_2^3 - sign(x_1)\theta(t) - x_1 \\ -x_1 + x_2 \end{pmatrix} \text{ and } g(x) = \begin{pmatrix} 0 \\ x_2 \end{pmatrix},$$

$$\begin{aligned} a(x,\theta) &= \langle \nabla V(x) | f(x,\theta) \rangle = 2x_1x_2^3 - sign(x_1)x_1\theta(t) - x_1^2 - 2x_1x_2^3 + 2x_2^4 \\ &= - \mid x_1 \mid \theta(t) - x_1^2 + 2x_2^4 \\ &\leq \bar{a}(x) \\ &\leq -\frac{1}{2} \mid x_1 \mid - x_1^2 + 2x_2^4 \\ b(x) &= \langle \nabla V(x) | g(x) \rangle = x_2^2. \end{aligned}$$

So $b(x) = 0 \Rightarrow x_2 = 0$ then $a(x,\theta) = - \mid x_1 \mid \theta - x_1^2 \leq -\frac{1}{2}(x_1^2)^{\frac{1}{2}} = -\frac{1}{2}V^{\frac{1}{2}}(x)$.
Thus V is a f-CLF for the system (4). By the previous theorem, the feedback $u(x) =$

$$\begin{cases} 0 & \text{if } x_2 = 0 \\ -\dfrac{-\frac{1}{2} \mid x_1 \mid - x_1^2 + 2x_2^4 + \frac{1}{2}(\frac{1}{2}(x_1^4 + x_2^2))^{\frac{1}{2}} + \sqrt{[-\frac{1}{2} \mid x_1 \mid - x_1^2 + 2x_2^4 + \frac{1}{2}(\frac{1}{2}(x_1^4 + x_2^2))^{\frac{1}{2}}]^2 + x_2^4}}{x_2^2} & \text{if } x_2 \neq 0 \end{cases} \tag{18}$$

stabilizes the system in finite time.

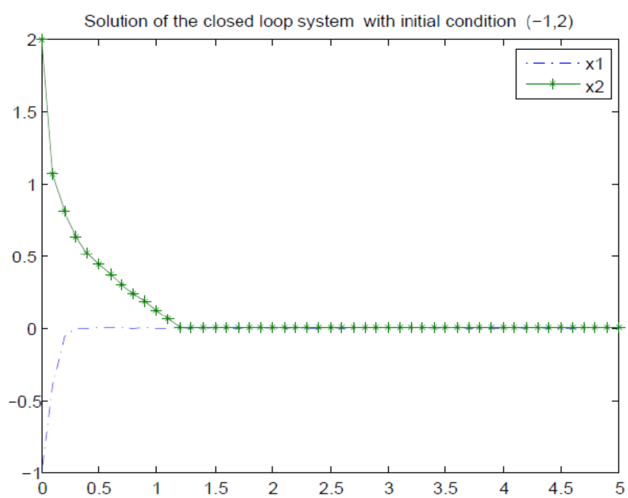

Fig. 2.

5. CONCLUSION

This paper raises some questions on the stabilization with varying parameter. Using the concept of CLF, sufficient conditions are derived to guarantee the robust stabilization. Moreover, homogeneous control were designed by using homogeneous CLF. Subsequently, we construct a control feedback that makes the system exponentially stable. Furthermore, we provide a controller using finite CLF which leads to stabilization in finite time.

(Received June 17, 2016)

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