KYBERNETIKA — VOLUME *53* (2017), NUMBER 5, PAGES 747–764

DISTRIBUTED EVENT-TRIGGERED ALGORITHM FOR OPTIMAL RESOURCE ALLOCATION OF MULTI-AGENT SYSTEMS

Weiyong Yu, Zhenhua Deng, Hongbing Zhou and Xianlin Zeng

This paper is concerned with solving the distributed resource allocation optimization problem by multi-agent systems over undirected graphs. The optimization objective function is a sum of local cost functions associated to individual agents, and the optimization variable satisfies a global network resource constraint. The local cost function and the network resource are the private data for each agent, which are not shared with others. A novel gradient-based continuous-time algorithm is proposed to solve the distributed optimization problem. We take an event-triggered communication strategy and an event-triggered gradient measurement strategy into account in the algorithm. With strongly convex cost functions and locally Lipschitz gradients, we show that the agents can find the optimal solution by the proposed algorithm with exponential convergence rate, based on the construction of a suitable Lyapunov function. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed scheme.

Keywords: distributed optimization, event-triggered strategy, multi-agent systems, resource allocation

Classification: 37N40, 90C26, 93A14

1. INTRODUCTION

Recently, the distributed optimization problem of multi-agent systems has received considerable attention in various areas for its broad application background, such as machine learning, sensor networks, smart grids, and many significant results have been obtained (see [13, 14, 22, 18, 9]). Resource allocation is one of the important optimization problems (see [7]). In particular, various distributed algorithms for resource allocation optimization have been discussed in [1, 4, 11, 20]. For example, [1] considered the network utility maximization problem and proposed a fast distributed dual-based gradient method for solving the problem. Furthermore, [23] proposed a class of projected continuous-time distributed algorithms to solve resource allocation optimization problems with local feasibility constraint.

In order to solve this distributed optimization problem, each agent needs to update its protocol by frequently measuring its gradient information and exchanging its state

DOI: 10.14736/kyb-2017-5-0747

information with its neighbors. However, in some practical situations, these agents may have limited energy or capacity. In other words, people have to care about the cost of the communication between agents and the measurement of the gradients of local cost functions when we deal with distributed optimization problem. On the other hand, it is well known that event-triggered strategy, which has drawn much attention (such as [10, 17, 6, 12, 21]), provides an effective method in solving control problem with reducing communication cost and computation burden in multi-agent systems for leader-following or leaderless cases. Some results can be found in [2, 19, 3], where different distributed event-triggered optimization algorithms were presented for different situations. However, to the best of our knowledge, up to now, no result has been obtained for the reduction of communication and measurement in the study of the distributed resource allocation optimization problem.

The purpose of this paper is to design an algorithm to solve the distributed resource allocation optimization problem by multi-agent systems concerning both neighboring communication cost and gradient measurement/computation burden. The contributions of this paper can summarized as follows. (i) Compared with the algorithm presented in [23], we construct a distributed event-triggered optimization algorithm, where communication and gradient measurement are triggered by two different events, respectively. Our algorithm can greatly reduce the costs of communication and gradient measurement, which are supported by the simulation studies. (ii) Compared with the results of [9], which proposed a class of distributed continuous-time algorithms with discrete-time communication that solve network optimization problem, the proposed algorithm in this paper can achieve the exact optimal solution with exponential convergence rate, while the results of [9] only achieved a neighborhood of the optimal solution.

The reminder of this paper is organized as follows. In Section 2, our problem is formulated with related preliminaries. Then the distributed gradient-based optimization scheme is proposed and the convergence is proved in Section 3, while a simulation example is given in Section 4. Finally, we give some concluding remarks in Section 5.

Notations: $\mathbb{R}$ and $\mathbb{R}^n$ represent the set of real numbers and the set of real n-dimensional column vectors, respectively; $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix; 1_n (or 0_n) denotes an n dimensional column vector whose components are all 1 (or 0); for a vector or a matrix X, X^T represents its transpose, and $\|\cdot\|$ represents the Euclidean norm of a vector or the corresponding induced norm of a matrix; $\otimes$ denotes the Kronecker product.

2. PRELIMINARIES AND PROBLEM STATEMENT

In this section, we provide some basic definitions on graph theory used in the paper and introduce our research problem.

2.1. Graph theory

A weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ consists of a finite vertex set $\mathcal{V} = \{1, \dots, N\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ with $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. An edge (j, i) represents that i, j can obtain each other's information. $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$ denotes the set of neighbors of agent i. A path is a sequence of vertices connected by edges. An undirected graph $\mathcal{G}$ is connected if there is a path between any two vertices of $\mathcal{G}$. The Laplacian matrix of graph $\mathcal{G}$ is $L = B - A$ with $B = \mathrm{diag}\{b_1, \dots, b_N\}$, where $b_i = \sum_{j=1}^{N} a_{ij}$, $i = 1, \dots, N$. We define $b = \max_{i \in \mathcal{V}}\{b_i\}$ and $a = \max_{i,j \in \mathcal{V}}\{a_{ij}\}$. Note that $L1_N = 0_N$. Denote the eigenvalues of L by $\lambda_1, \dots, \lambda_N$ with $\lambda_i \leq \lambda_{i+1}$, $i = 1, \dots, N-1$. The undirected graph $\mathcal{G}$ is connected if and only if $\lambda_2 > 0$.

2.2. Problem statement

Consider a network of N agents interacting over an undirected graph $\mathcal{G}$. The objective of this paper is to solve the problem of distributed resource allocation optimization under the network resource constraint through the collaboration of N agents:

$$\begin{aligned} &\min_{x \in \mathbb{R}^{Nm}} F(x), \quad F(x) = \sum_{i=1}^{N} f_i(x_i), \\ &\text{subject to} \ \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} d_i, \end{aligned} \tag{1}$$

where $x = \left(x_1^T, \dots, x_N^T\right)^T \in \mathbb{R}^{Nm}$, agent i can decide its local allocation $x_i \in \mathbb{R}^m$, and access the local resource data $d_i \in \mathbb{R}^m$. The total network resource is $\sum_{i=1}^{N} d_i$, then the optimization problem should satisfy the network resource constraint: $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} d_i$. The cost function of agent i, $f_i : \mathbb{R}^m \to \mathbb{R}$, and its local resource data d_i are only known by itself.

We introduce the following well-known assumptions on the graph $\mathcal{G}$ and the local cost functions, respectively.

Assumption 2.1. The undirected graph $\mathcal{G}$ is connected.

Assumption 2.2. For $i = 1, \dots, N$, the local cost functions f_i are differentiable and ω_i-strongly convex, that is

$$(x - y)^T(\nabla f_i(x) - \nabla f_i(y)) \geq \omega_i \|x - y\|^2, \ \text{for} \ \forall \ x, y \in \mathbb{R}^m,$$

furthermore, their gradients are θ_i-Lipschitz, that is

$$\|\nabla f_i(x) - \nabla f_i(y)\| \leq \theta_i \|x - y\|, \ \text{for} \ \forall \ x, y \in \mathbb{R}^m,$$

where the constants $\omega_i, \theta_i > 0$.

Remark 2.3. Under Assumption 2.1, there exists an orthogonal matrix $Q = \left[\frac{1_N}{\sqrt{N}} \ R\right]$ with $R \in \mathbb{R}^{N \times (N-1)}, RR^T = I_N - \frac{1}{N} 1_N 1_N^T$ such that $Q^T L Q = \mathrm{diag}\{0, \lambda_2, \dots, \lambda_N\}$, then $\lambda_2 I_{N-1} \leq R^T L R \leq \lambda_N I_{N-1}$, see [5].

Remark 2.4. Under Assumption 2.2, the function F is strongly convex, then there exists a bounded optimal solution to problem (1), see [15].

3. DISTRIBUTED OPTIMIZATION ALGORITHM

In this section, we provide a distributed optimization algorithm with event-triggered communication and gradient measurement to cooperatively solve the optimization problem stated in (1) under the undirected graph.

To reduce the costs of communication and gradient measurement, the gradient-based event-triggered optimization algorithm is designed for agent $i \in \mathcal{V}$ as follows, which is a modified version of the algorithm given in [23]:

$$\begin{aligned} \dot{x}_i &= -\nabla f_i\big(x_i(t_{il}^1)\big) - y_i, \\ \dot{y}_i &= -\sum_{j\in\mathcal{N}_i} a_{ij}\big(y_i(t_{ik}^2) - y_j(t_{jk}^2)\big) + \sum_{j\in\mathcal{N}_i} a_{ij}\big(z_i(t_{ik}^2) - z_j(t_{jk}^2)\big) + (x_i - d_i), \\ \dot{z}_i &= -\sum_{j\in\mathcal{N}_i} a_{ij}\big(y_i(t_{ik}^2) - y_j(t_{jk}^2)\big), \qquad t \in \big[t_{il}^1, t_{i(l+1)}^1\big) \cap \big[t_{ik}^2, t_{i(k+1)}^2\big), \end{aligned} \tag{2}$$

where x_i is the estimation of agent i for the optimal solution of problem (1), y_i and z_i are the auxiliary variables of agent i.

Without loss of generality, we assume that $t_{i0}^1 = t_{i0}^2 = 0$. The triggering time sequences of the gradient information measurement $\{t_{il}^1\}_{l=0}^{\infty}$ and communication with its neighbors $\{t_{ik}^2\}_{k=0}^{\infty}$ for agent i are determined by

$$\begin{aligned} t_{i(l+1)}^1 &= \inf\big\{t : t > t_{il}^1, g_x^i > 0\big\}, \\ t_{i(k+1)}^2 &= \inf\big\{t : t > t_{ik}^2, g_y^i > 0 \text{ or } g_z^i > 0\big\}, \end{aligned} \tag{3}$$

with the trigger functions given as follows:

$$\begin{aligned} g_x^i &= \big\|e_x^i\big\| - \alpha\beta_1\Big\|\sum_{j\in\mathcal{N}_i} a_{ij}\big(x_i(t_{il}^1) - x_j(t_{jl}^1)\big)\Big\| - \beta_2 e^{-\gamma t}, \\ g_y^i &= \big\|e_y^i\big\| - \alpha\beta_3\Big\|\sum_{j\in\mathcal{N}_i} a_{ij}\big(y_i(t_{ik}^2) - y_j(t_{jk}^2)\big)\Big\| - \beta_4 e^{-\gamma t}, \\ g_z^i &= \big\|e_z^i\big\| - \alpha\beta_5\Big\|\sum_{j\in\mathcal{N}_i} a_{ij}\big(z_i(t_{ik}^2) - z_j(t_{jk}^2)\big)\Big\| - \beta_6 e^{-\gamma t}. \end{aligned} \tag{4}$$

The positive parameters α, β_i and γ are to be determined later. The measurement errors are defined by

$$e_x^i(t) = x_i(t_{il}^1) - x_i(t), \quad e_y^i(t) = y_i(t_{ik}^2) - y_i(t), \quad e_z^i(t) = z_i(t_{ik}^2) - z_i(t). \tag{5}$$

It is easy to see that $e_x^i(t_{il}^1) = 0$ and $e_y^i(t_{ik}^2) = e_z^i(t_{ik}^2) = 0$, for any $l, k \in \mathbb{N}$ and $i \in \mathcal{V}$.

Remark 3.1. Obviously, for any agent i, there exist two triggering time sequences $\{t_{il}^1\}_{l=0}^{\infty}$ and $\{t_{ik}^2\}_{k=0}^{\infty}$. By $t \in \big[t_{il}^1, t_{i(l+1)}^1\big) \cap \big[t_{ik}^2, t_{i(k+1)}^2\big)$, we can get a new time sequence denoted by $\{t_{ir}\}_{r=0}^{\infty}$, where

$$\{t_{i0}\} = 0, \ \bigcup_{r=0}^{\infty} \big[t_{ir}, t_{i(r+1)}\big) = [0, \infty) \text{ and } \big[t_{ir_1}, t_{i(r_1+1)}\big) \cap \big[t_{ir_2}, t_{i(r_2+1)}\big) = \emptyset \text{ for any } r_1 \neq r_2.$$

If $\left[t_{ir},t_{i(r+1)}\right) = \left[t^1_{il},t^1_{i(l+1)}\right) \cap \left[t^2_{ik},t^2_{i(k+1)}\right)$, then $\left[t_{i(r+1)},t_{i(r+2)}\right) = \left[t^1_{i(l+1)},t^1_{i(l+2)}\right) \cap \left[t^2_{ik},t^2_{i(k+1)}\right)$, or $\left[t^1_{il},t^1_{i(l+1)}\right) \cap \left[t^2_{i(k+1)},t^2_{i(k+2)}\right)$, or $\left[t^1_{i(l+1)},t^1_{i(l+2)}\right) \cap \left[t^2_{i(k+1)},t^2_{i(k+2)}\right)$.

Remark 3.2. It is worth mentioning that the conditions (3) and (4) are verified by agent i only based on information of itself and neighboring agents, thus the algorithm (2) is a distributed event-triggered scheme. Besides, we know from (4) that the continuous communication between neighboring agents is avoided.

The algorithm (2) can be rewritten as the following compact form

$$\begin{aligned} \dot{x} &= -\nabla F\big(x+e_x\big) - y, \\ \dot{y} &= -(L\otimes I_m)(y+e_y) + (L\otimes I_m)(z+e_z) + (x-d), \\ \dot{z} &= -(L\otimes I_m)(y+e_y), \end{aligned} \tag{6}$$

where $x=\big(x_1^T,\dots,x_N^T\big)^T \in \mathbb{R}^{Nm}, y=\big(y_1^T,\dots,y_N^T\big)^T \in \mathbb{R}^{Nm}, z=\big(z_1^T,\dots,z_N^T\big)^T \in \mathbb{R}^{mN}$, $e_x=\big({e_x^1}^T,\dots,{e_x^N}^T\big)^T \in \mathbb{R}^{Nm}, e_y=\big({e_y^1}^T,\dots,{e_y^N}^T\big)^T \in \mathbb{R}^{Nm}, e_z=\big({e_z^1}^T,\dots,{e_z^N}^T\big)^T \in \mathbb{R}^{Nm}$, $d=\big(d_1^T,\dots,d_N^T\big)^T \in \mathbb{R}^{Nm}$. L is the Laplacian matrix of graph $\mathcal{G}$.

The following lemma indicates the relationship between the equilibrium point of (6)(or (2)) and the optimal solution of problem (1).

Lemma 3.3. Suppose that Assumptions 2.1 and 2.2 hold. If (x^*,y^*,z^*) is the equilibrium point of (6), then x^* is the optimal solution of problem (1).

P r o o f. Note that $e_x=e_y=e_z=0_{Nm}$ once the equilibrium of system (6) is achieved. Hence,

$$\begin{aligned} -\nabla F(x^*) - y^* &= 0_{Nm}, \\ -(L\otimes I_m)y^* + (L\otimes I_m)z^* + (x^*-d) &= 0_{Nm}, \\ -(L\otimes I_m)y^* &= 0_{Nm}. \end{aligned} \tag{7}$$

Since the undirected graph G is connected, using $L1_N=0_N$, we have

$$y^* = 1_{Nm}\otimes\lambda^*, \quad \lambda^*\in\mathbb{R}.$$

Substituting the above inequality into (7) and utilizing $1_N^T L = 0_N^T$, we find that

$$\begin{aligned} -\nabla F\big(x^*\big) - 1_{Nm}\otimes\lambda^* &= 0, \quad \lambda^*\in\mathbb{R}, \\ \sum_{i=1}^N x_i^* &= \sum_{i=1}^N d_i, \end{aligned} \tag{8}$$

which is exactly the optimality condition (KKT) for problem (1) by Theorem 3.34 in [16]. Thus, the conclusion follows. □

For (2), we have the following result.

Theorem 3.4. Under Assumptions 2.1 and 2.2, the optimization problem (1) is solved by the distributed algorithm (2) with event-triggered scheme (3). Concretely, for any initial conditions $x_i(0), y_i(0) \in \mathbb{R}^m$ and $\sum_{i=1}^N z_i(0) = 0_m$, the algorithm (2) with condition (3) makes $x_i(t) \to x_i^*$ exponentially as $t \to \infty$ for each $i \in \mathcal{V}$, where $x^* = \left(x_1^*, \dots, x_N^*\right)^T \in \mathbb{R}^{Nm}$ is the optimal solution of problem (1). Furthermore, the event-triggered scheme (3) is free of Zeno behavior.

Proof. For convenience, we show the proof in three steps.

Step 1: The changes of coordinates and the closed-loop system.

Let

$$\overline{x} = x - x^*, \quad \overline{y} = y - y^*, \quad \overline{z} = z - z^*, \tag{9}$$

where (x^*, y^*, z^*) is the equilibrium point of (6) and $x^* = \left(x_1^*, \dots, x_N^*\right)^T \in \mathbb{R}^{Nm}$, $y^* = \left(y_1^*, \dots, y_N^*\right)^T \in \mathbb{R}^{Nm}$, $z^* = \left(z_1^*, \dots, z_N^*\right)^T \in \mathbb{R}^{Nm}$.

From the initial condition $\sum_{i=1}^N z_i(0) = 0_m$, we obtain $(1_N^T \otimes I_m)\overline{z}(0) = 0_m$. Then the dynamic equations of $\overline{x}, \overline{y}$ and $\overline{z}$ are given by

$$\begin{aligned} \dot{\overline{x}} &= -h - \overline{y}, \\ \dot{\overline{y}} &= -(L \otimes I_m)(\overline{y} + e_y) + (L \otimes I_m)(\overline{z} + e_z) + \overline{x}, \\ \dot{\overline{z}} &= -(L \otimes I_m)(\overline{y} + e_y), \end{aligned} \tag{10}$$

where $h = h_1 + h_2$ with $h_1 = \nabla F(x + e_x) - \nabla F(x)$ and $h_2 = \nabla F(x) - \nabla F(x^*)$.

Define the following coordinate transformations

$$\begin{aligned} X &= \left[X_1^T \; X_{2:N}^T\right]^T = (Q^T \otimes I_m)\overline{x}, \quad & e_X &= \left[{e_X^1}^T \; {e_X^{2:N}}^T\right]^T = \left(Q^T \otimes I_m\right)e_x, \\ Y &= \left[Y_1^T \; Y_{2:N}^T\right]^T = \left(Q^T \otimes I_m\right)\overline{y}, \quad & e_Y &= \left[{e_Y^1}^T \; {e_Y^{2:N}}^T\right]^T = \left(Q^T \otimes I_m\right)e_y, \\ Z &= \left[Z_1^T \; Z_{2:N}^T\right]^T = \left(Q^T \otimes I_m\right)\overline{z}, \quad & e_Z &= \left[{e_Z^1}^T \; {e_Z^{2:N}}^T\right]^T = \left(Q^T \otimes I_m\right)e_z, \end{aligned} \tag{11}$$

where $X_1, e_X^1, Y_1, e_Y^1, Z_1, e_Z^1 \in \mathbb{R}^m$, $X_{2:N}, e_X^{2:N}, Y_{2:N}, e_Y^{2:N}, Z_{2:N}, e_Z^{2:N} \in \mathbb{R}^{(N-1)m}$ and Q is determined by Remark 2.3. Then we have from (10) and Remark 2.3 that

$$\begin{aligned} \dot{X}_1 &= -Y_1 - \left(\frac{1_N^T}{\sqrt{N}} \otimes I_m\right)h, & \dot{X}_{2:N} &= -Y_{2:N} - \left(R^T \otimes I_m\right)h, \\ \dot{Y}_1 &= X_1, & \dot{Y}_{2:N} &= X_{2:N} - \left(R^T L R \otimes I_m\right)\left(Y_{2:N} + e_Y^{2:N}\right) \\ & & &\quad + \left(R^T L R \otimes I_m\right)\left(Z_{2:N} + e_Z^{2:N}\right), \\ \dot{Z}_1 &= 0_m, & \dot{Z}_{2:N} &= -\left(R^T L R \otimes I_m\right)\left(Y_{2:N} + e_Y^{2:N}\right). \end{aligned} \tag{12}$$

Moreover, we obtain $Z_1 \equiv 0_m$ by $(1_N^T \otimes I_m)\overline{z}(0) = 0_m$.

Step 2: The choices of Lyapunov function and relevant parameters.

Construct the Lyapunov function $V = V_1 + V_2$, where

$$\begin{aligned} V_1 &= \frac{1}{2}k_1\big(X^TX + Y^TY\big) + \frac{1}{2}(k_1+k_2)Z^TZ + \frac{1}{2}k_2\big(Y_{2:N} - Z_{2:N}\big)^T\big(Y_{2:N} - Z_{2:N}\big), \\ V_2 &= \frac{1}{2}k_3(X+Y)^T(X+Y), \end{aligned} \tag{13}$$

with the positive parameters k_1, k_2 and k_3 are to be determined later.

Differentiating (13) along the trajectories of (12) yields

$$\begin{aligned} \dot{V}_1 &= k_1\big(X_1^T\dot{X}_1 + X_{2:N}^T\dot{X}_{2:N} + Y_1^T\dot{Y}_1 + +Y_{2:N}^T\dot{Y}_{2:N}\big) \\ &\quad +(k_1+k_2)Z_{2:N}^T\dot{Z}_{2:N} + k_2\big(Y_{2:N} - Z_{2:N}\big)^T\big(\dot{Y}_{2:N} - \dot{Z}_{2:N}\big) \\ &= k_1X^T\Big(-Y - \big(Q^T\otimes I_m\big)h\Big) + k_1Y^TX - k_1Y_{2:N}^T\big(R^TLR\otimes I_m\big)\big(Y_{2:N} + e_Y^{2:N}\big) \\ &\quad +k_1Y_{2:N}^T\big(R^TLR\otimes I_m\big)\big(Z_{2:N} + e_Z^{2:N}\big) - (k_1+k_2)Z_{2:N}^T\big(R^TLR\otimes I_m\big)\big(Y_{2:N} + e_Y^{2:N}\big) \\ &\quad +k_2\big(Y_{2:N} - Z_{2:N}\big)^T\Big(X_{2:N} + \big(R^TLR\otimes I_m\big)\big(Z_{2:N} + e_Z^{2:N}\big)\Big) \\ &= -k_1\overline{x}^Th - k_1Y_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N} + k_1Y_{2:N}^T\big(R^TLR\otimes I_m\big)\big(e_Z^{2:N} - e_Y^{2:N}\big) \\ &\quad -(k_1+k_2)Z_{2:N}^T\big(R^TLR\otimes I_m\big)e_Y^{2:N} + k_2\big(Y_{2:N}^T - Z_{2:N}^T\big)X_{2:N} \\ &\quad -k_2Z_{2:N}^T\big(R^TLR\otimes I_m\big)Z_{2:N} + k_2\big(Y_{2:N}^T - Z_{2:N}^T\big)\big(R^TLR\otimes I_m\big)e_Z^{2:N}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \dot{V}_2 &= k_3(X+Y)^T(\dot{X}+\dot{Y}) \\ &= k_3(X+Y)^T\begin{pmatrix} -Y_1 - \left(\frac{1_N^T}{\sqrt{N}}\otimes I_m\right)h + X_1 \\ \\ -Y_{2:N} - \big(R^T\otimes I_m\big)h + X_{2:N} - \big(R^TLR\otimes I_m\big)\big(Y_{2:N} + e_Y^{2:N}\big) \\ +\big(R^TLR\otimes I_m\big)\big(Z_{2:N} + e_Z^{2:N}\big) \end{pmatrix} \\ &= k_3(X_1+Y_1)^T\left(-Y_1 - \left(\frac{1_N^T}{\sqrt{N}}\otimes I_m\right)h + X_1\right) \\ &\quad +k_3(X_{2:N}+Y_{2:N})^T\Big(-Y_{2:N} - \big(R^T\otimes I_m\big)h + X_{2:N} - \big(R^TLR\otimes I_m\big)\big(Y_{2:N} + e_Y^{2:N}\big) \\ &\qquad + \big(R^TLR\otimes I_m\big)\big(Z_{2:N} + e_Z^{2:N}\big)\Big) \\ &= -k_3Y^TY - k_3\overline{x}^Th - k_3\overline{y}^Th + k_3X^TX \\ &\quad -k_3X_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N} - k_3Y_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N} \\ &\quad +k_3(X_{2:N}^T + Y_{2:N}^T)\big(R^TLR\otimes I_m\big)\big(Z_{2:N} + e_Z^{2:N} - e_Y^{2:N}\big). \end{aligned} \tag{15}$$

Recalling that $h = h_1 + h_2 = \big(\nabla F(x+e_x) - \nabla F(x)\big) + \big(\nabla F(x) - \nabla F(x^*)\big)$, using

Assumption 2.2 and $\|Q\| = 1$, we obtain the following estimation

$$\begin{aligned} -(k_1+k_3)\overline{x}^T h - k_3\overline{y}^T h &= -(k_1+k_3)\overline{x}^T\big(\nabla F(x+e_x)-\nabla F(x)\big) \\ &\quad -(k_1+k_3)\overline{x}^T\big(\nabla F(x)-\nabla F(x^*)\big) \\ &\quad -k_3\overline{y}^T\Big(\big(\nabla F(x+e_x)-\nabla F(x)\big)+\big(\nabla F(x)-\nabla F(x^*)\big)\Big) \\ &\le \theta(k_1+k_3)\|\overline{x}\|\|e_x\| - \omega(k_1+k_3)\|\overline{x}\|^2 + \theta k_3\|\overline{y}\|(\|e_x\|+\|\overline{x}\|) \\ &\le -\frac{\omega(k_1+k_3)-2\theta^2k_3^2}{2}\|X\|^2 + \frac{1}{2}\|Y\|^2 \\ &\quad +\frac{2\omega\theta^2k_3^2+\theta^2(k_1+k_3)}{2\omega}\|e_X\|^2, \end{aligned} \tag{16}$$

where $\omega = \min\limits_{i\in\mathcal{V}}\{\omega_i\}$ and $\theta = \max\limits_{i\in\mathcal{V}}\{\theta_i\}$.
From Assumption 2.1 and Remark 2.3, we get

$$\begin{aligned} k_1Y_{2:N}^T\big(R^TLR\otimes I_m\big)\big(e_Z^{2:N}-e_Y^{2:N}\big) &\le \frac{k_1}{2}Y_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N} + \lambda_N k_1\big(\|e_Y\|^2+\|e_Z\|^2\big), \\ -(k_1+k_2)Z_{2:N}^T\big(R^TLR\otimes I_m\big)e_Y^{2:N} &\le \lambda_N\|Z_{2:N}\|^2 + \lambda_N(k_1+k_2)^2\|e_Y\|^2, \\ k_2\big(Y_{2:N}^T-Z_{2:N}^T\big)X_{2:N} &\le k_2^2\|X\|^2 + \frac{1}{2}\|Y\|^2 + \frac{1}{2}\|Z\|^2, \\ k_2\big(Y_{2:N}^T-Z_{2:N}^T\big)\big(R^TLR\otimes I_m\big)e_Z^{2:N} &\le \frac{k_2}{2}Y_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N} + \lambda_N k_2\|e_Z\|^2 \\ &\quad +\frac{k_2}{2}Z_{2:N}^T\big(R^TLR\otimes I_m\big)Z_{2:N}. \end{aligned} \tag{17}$$

In addition, it is obvious that

$$\begin{aligned} -k_3X_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N} &\le \frac{\lambda_N k_3}{2}\|X\|^2 \\ &\quad +\frac{k_3}{2}Y_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N}, \\ k_3(X_{2:N}^T+Y_{2:N}^T)\big(R^TLR\otimes I_m\big)\big(Z_{2:N}+e_Z^{2:N}-e_Y^{2:N}\big) &\le \frac{\lambda_N k_3}{2}\|X\|^2 + 3\lambda_N k_3\|Z\|^2 \\ &\quad +\frac{k_3}{2}Y_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N} \\ &\quad +3\lambda_N k_3\big(\|e_Y\|^2+\|e_Z\|^2\big). \end{aligned} \tag{18}$$

Using (13) − (18), we have

$$\begin{aligned} \dot{V} \le\ & -\Big(\frac{\omega(k_1+k_3)}{2} - k_2^2 - k_3 - \lambda_N k_3 - \theta^2k_3^2\Big)\|X\|^2 - (k_3-1)\|Y\|^2 \\ & -\Big(\frac{\lambda_2k_2}{2} - 3\lambda_N k_3 - \lambda_N - \frac{1}{2}\Big)\|Z\|^2 - \frac{k_1-k_2}{2}Y_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N} \\ & +\frac{2\omega\theta^2k_3^2+\theta^2(k_1+k_3)}{2\omega}\|e_X\|^2 + \big(\lambda_N k_1 + \lambda_N(k_1+k_2)^2 + 3\lambda_N k_3\big)\|e_Y\|^2 \\ & +\big(\lambda_N k_1 + \lambda_N k_2 + 3\lambda_N k_3\big)\|e_Z\|^2. \end{aligned} \tag{19}$$

It results from the trigger conditions (3) and (4) that

$$\begin{aligned}\|e_x^i\| &\le \alpha\beta_1\Big\|\sum_{j\in\mathcal{N}_i} a_{ij}\big(x_i(t_l^i)-x_j(t_l^j)\big)\Big\| + \beta_2 e^{-\gamma t}\\ &\le \alpha\beta_1\left(\sum_{j\in\mathcal{N}_i} a_{ij}\|e_x^i\| + \sum_{j\in\mathcal{N}_i} a_{ij}\|x_i\| + \sum_{j\in\mathcal{N}_i} a_{ij}\|e_x^j\| + \sum_{j\in\mathcal{N}_i} a_{ij}\|x_j\|\right) + \beta_2 e^{-\gamma t}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|e_X\|^2 &\le \frac{6\alpha^2\beta_1^2(b^2+Na^2)}{(1-\alpha\beta_1 b)^2-6N\alpha^2\beta_1^2a^2}\|X\|^2 + \frac{2N\beta_2^2}{(1-\alpha\beta_1 b)^2-6N\alpha^2\beta_1^2a^2}e^{-2\gamma t},\\ \|e_Y\|^2 &\le \frac{6\alpha^2\beta_3^2(b^2+Na^2)}{(1-\alpha\beta_3 b)^2-6N\alpha^2\beta_3^2a^2}\|Y\|^2 + \frac{2N\beta_4^2}{(1-\alpha\beta_3 b)^2-6N\alpha^2\beta_3^2a^2}e^{-2\gamma t}, \quad (20)\\ \|e_Z\|^2 &\le \frac{6\alpha^2\beta_5^2(b^2+Na^2)}{(1-\alpha\beta_5 b)^2-6N\alpha^2\beta_5^2a^2}\|Z\|^2 + \frac{2N\beta_6^2}{(1-\alpha\beta_5 b)^2-6N\alpha^2\beta_5^2a^2}e^{-2\gamma t}.\end{aligned}$$

Substituting (20) into (19), we obtain

$$\begin{aligned}\dot V &\le -\mu_1\|X\|^2-\mu_2\|Y\|^2-\mu_3\|Z\|^2+\mu_4e^{-2\gamma t}\\ &\quad -\frac{k_1-k_2}{2}Y_{2:N}^T\big(R^TLR\otimes I_m\big)Y_{2:N}, \quad (21)\end{aligned}$$

where

$$\begin{aligned}\mu_1 &= \frac{\omega(k_1+k_3)}{2}-k_2^2-k_3-\lambda_N k_3-\theta^2k_3^2-\frac{6\alpha^2\beta_1^2(b^2+Na^2)\big(2\omega\theta^2k_3^2+\theta^2(k_1+k_3)\big)}{2\omega\big((1-\alpha\beta_1b)^2-6N\alpha^2\beta_1^2a^2\big)},\\ \mu_2 &= k_3-1-\frac{6\alpha^2\beta_3^2(b^2+Na^2)\big(\lambda_Nk_1+\lambda_N(k_1+k_2)^2+3\lambda_Nk_3\big)}{(1-\alpha\beta_3b)^2-6N\alpha^2\beta_3^2a^2},\\ \mu_3 &= \frac{\lambda_2k_2}{2}-3\lambda_Nk_3-\lambda_N-\frac{1}{2}-\frac{6\alpha^2\beta_5^2(b^2+Na^2)\big(\lambda_Nk_1+\lambda_Nk_2+3\lambda_Nk_3\big)}{(1-\alpha\beta_5b)^2-6N\alpha^2\beta_5^2a^2},\\ \mu_4 &= \frac{N\beta_2^2\big(2\omega\theta^2k_3^2+\theta^2(k_1+k_3)\big)}{\omega\big((1-\alpha\beta_1b)^2-6N\alpha^2\beta_1^2a^2\big)}+\frac{2N\beta_4^2\big(\lambda_Nk_1+\lambda_N(k_1+k_2)^2+3\lambda_Nk_3\big)}{(1-\alpha\beta_3b)^2-6N\alpha^2\beta_3^2a^2}\\ &\quad+\frac{2N\beta_6^2\big(\lambda_Nk_1+\lambda_Nk_2+3\lambda_Nk_3\big)}{(1-\alpha\beta_5b)^2-6N\alpha^2\beta_5^2a^2}.\end{aligned}$$

Choose the parameters k_i, α and β_j such that $k_1 > k_2$, $\mu_1, \mu_2, \mu_3, \mu_4 > 0$ (which obviously can be done). As a result,

$$\dot V \le -\mu\|P\|^2+\mu_4e^{-2\gamma t}, \quad (22)$$

where $\mu = \min\{\mu_1, \mu_2, \mu_3\}$ and $P^T = \big(X^T, Y^T, Z^T\big)$.

Note that $k_1\|P\|^2 \leq V \leq \frac{k_1+3k_2+2k_3}{2}\|P\|^2$. Choose $0<\gamma\leq \frac{\mu}{k_1+3k_2+2k_3}$, and thus,

$$\begin{aligned}\|P(t)\|^2 &\leq \frac{k_1+3k_2+2k_3}{2k_1}\big\|P(0)\big\|^2\exp\Big\{-\frac{2\mu}{k_1+3k_2+2k_3}t\Big\}\\ &\quad+\frac{\mu_4(k_1+3k_2+2k_3)}{2k_1\big(\mu-\gamma(k_1+3k_2+2k_3)\big)}\Big(\exp\big\{-2\gamma t\big\}-\exp\Big\{-\frac{2\mu}{k_1+3k_2+2k_3}t\Big\}\Big),\\ &\leq \gamma_0 e^{-2\gamma t}, \end{aligned} \tag{23}$$

where $\gamma_0=\frac{k_1+3k_2+2k_3}{2k_1}\big\|P(0)\big\|^2+\frac{\mu_4(k_1+3k_2+2k_3)}{2k_1\big(\mu-\gamma(k_1+3k_2+2k_3)\big)}$. This implies that x exponentially converges to x^*.

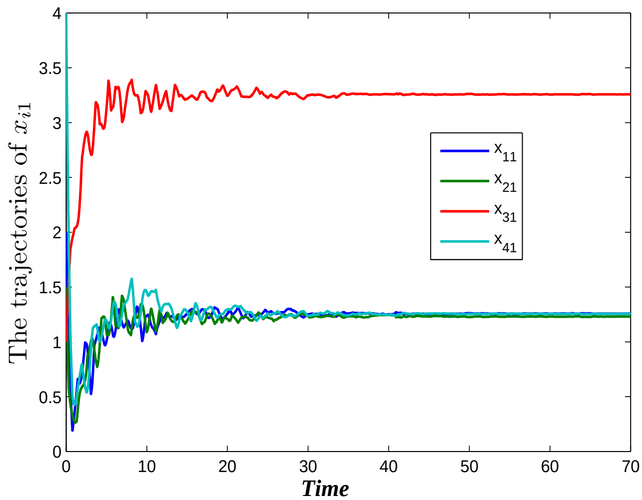

Fig. 1. The trajectories of x_{i1} in the algorithm (2).

Step 3: Analysis of Zeno behavior.

Firstly, we verify that the gradient information measurement is free of Zeno behavior. From (23), we have that $\|x(t)-x^*\|\leq\sqrt{\gamma_0}e^{-\gamma t}$, $\|y(t)-y^*\|\leq\sqrt{\gamma_0}e^{-\gamma t}$ and $\|z(t)-z^*\|\leq\sqrt{\gamma_0}e^{-\gamma t}$. Thus, according to (5), (2), and using Assumption 2.2, it follows that for $t\in\big[t_{il}^1,t_{i(l+1)}^1\big)$

$$\begin{aligned}\big\|\dot{e}_x^i(t)\big\| &= \|\dot{x}_i(t)\|\\ &\leq \Big(\big\|\big(\nabla f_i(x_i(t_{il}^1))-\nabla f_i(x_i^*)\big)\big\|+\|\overline{y}\|\Big)\\ &\leq \theta_i\sqrt{\gamma_0}e^{-\gamma t_{il}^1}+\sqrt{\gamma_0}e^{-\gamma t}.\end{aligned} \tag{24}$$

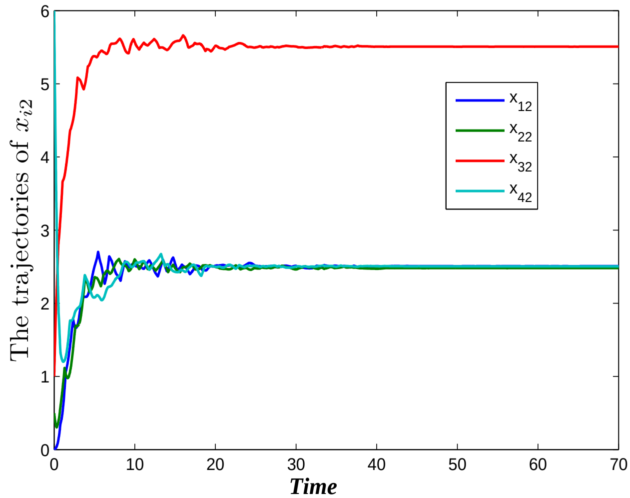

Fig. 2. The trajectories of x_{i2} in the algorithm (2).

Applying $e_x^i(t_{il}^1) = 0$ and integrating the above inequality from t_{il}^1 to t, we deduce that

$$\begin{aligned} \|e_x^i(t)\| &= \Big\| \int_{t_{il}^1}^t \dot{e}_x^i(s)\mathrm{ds}\Big\| \le \int_{t_{il}^1}^t \|\dot{e}_x^i(s)\|\mathrm{ds} \\ &\le \theta_i\sqrt{\gamma_0}e^{-\gamma t_{il}^1}\big(t - t_{il}^1\big) + \frac{\sqrt{\gamma_0}}{\gamma}\Big(e^{-\gamma t_{il}^1} - e^{-\gamma t}\Big), \quad \text{for} \quad t \in \big[t_{il}^1, t_{i(l+1)}^1\big). \end{aligned} \tag{25}$$

The next gradient measurement will not be executed until the first trigger function g_x^i crosses zero, that is

$$\begin{aligned} \alpha\beta_1\Big\| \sum_{j\in\mathcal{N}_i} a_{ij}\big(x_i(t_{il}^1) - x_j(t_{jl}^1)\big)\Big\| + \beta_2 e^{-\gamma t_{i(l+1)}^1} &= \big\|e_x^i(t_{i(l+1)}^1)\big\| \\ &\le \theta_i\sqrt{\gamma_0}e^{-\gamma t_{il}^1}\big(t_{i(l+1)}^1 - t_{il}^1\big) \\ &\quad + \frac{\sqrt{\gamma_0}}{\gamma}\Big(e^{-\gamma t_{il}^1} - e^{-\gamma t_{i(l+1)}^1}\Big). \end{aligned}$$

Accordingly, the following inequality holds

$$\frac{\sqrt{\gamma_0} + \beta_2\gamma}{\gamma}e^{-\gamma\left(t_{i(l+1)}^1 - t_{il}^1\right)} \le \theta_i\sqrt{\gamma_0}\big(t_{i(l+1)}^1 - t_{il}^1\big) + \frac{\sqrt{\gamma_0}}{\gamma}. \tag{26}$$

Since $e^\epsilon \ge 1 + \epsilon$ for any $\epsilon \in \mathbb{R}$, we have for any agent i

$$t_{i(l+1)}^1 - t_{il}^1 \ge \frac{\beta_2}{\sqrt{\gamma_0}(\theta_i + 1) + \beta_2\gamma} > 0. \tag{27}$$

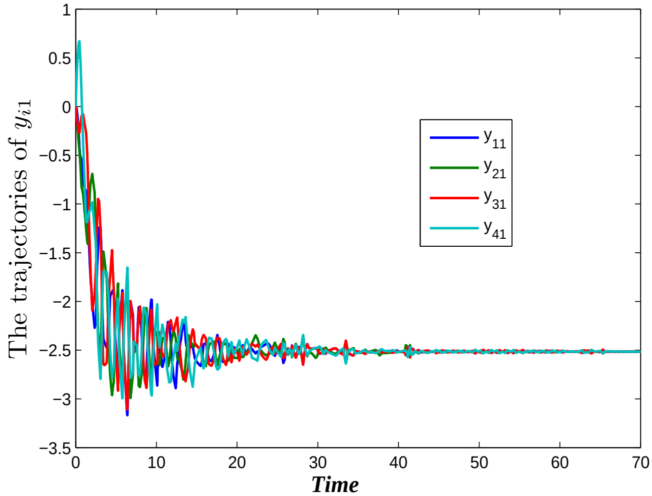

Fig. 3. The trajectories of y_{i1} in the algorithm (2).

Consequently, the Zeno behavior is excluded for any agent i.

Next, we prove that the communication with neighbors is free of Zeno behavior. Similar to (24), for $t \in \left[t_{ik}^2, t_{i(k+1)}^2\right)$, we get

$$\begin{aligned}
\|\dot{e}_y^i(t)\| &= \|\dot{y}_i(t)\| \\
&\leq \Big\| \sum_{j\in\mathcal{N}_i} a_{ij}\Big(\big(y_i(t_{ik}^2) - y_i^*\big) - \big(y_j(t_{jk}^2) - y_j^*\big)\Big)\Big\| \\
&\quad + \Big\| \sum_{j\in\mathcal{N}_i} a_{ij}\Big(\big(z_i(t_{ik}^2) - z_i^*\big) - \big(z_j(t_{jk}^2) - z_j^*\big)\Big)\Big\| + \|x_i - x_i^*\| \\
&\leq 4b\sqrt{\gamma_0} + \sqrt{\gamma_0}e^{-\gamma t}
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
\|\dot{e}_z^i(t)\| &= \|\dot{z}_i(t)\| \\
&= \Big\| \sum_{j\in\mathcal{N}_i} a_{ij}\Big(\big(y_i(t_{ik}^2) - y_i^*\big) - \big(y_j(t_{jk}^2) - y_j^*\big)\Big)\Big\| \\
&\leq 2b\sqrt{\gamma_0}.
\end{aligned} \tag{29}$$

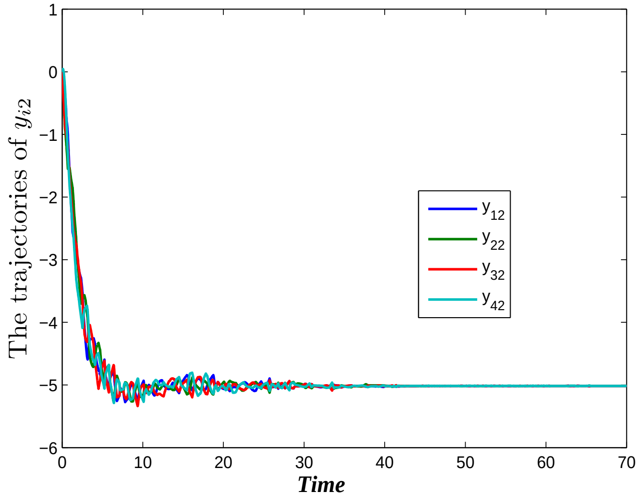

Fig. 4. The trajectories of y_{i2} in the algorithm (2).

Following the procedure of (25), we derive that

$$\begin{aligned}\alpha\beta_3\Big\|\sum_{j\in\mathcal{N}_i} a_{ij}\big(y_i(t^2_{ik})-y_j(t^2_{jk})\big)\Big\|+\beta_4 e^{-\gamma t^2_{i(k+1)}} &= \big\|e^i_y(t^2_{i(k+1)})\big\| \\ &\le 4b\sqrt{\gamma_0}\big(t^2_{i(k+1)}-t^2_{ik}\big) \\ &\quad +\frac{\sqrt{\gamma_0}}{\gamma}\Big(e^{-\gamma t^2_{ik}}-e^{-\gamma t^2_{i(k+1)}}\Big) \quad (30)\end{aligned}$$

and

$$\alpha\beta_5\Big\|\sum_{j\in\mathcal{N}_i} a_{ij}\big(z_i(t^2_{ik})-z_j(t^2_{jk})\big)\Big\|+\beta_6 e^{-\gamma t^2_{i(k+1)}} = \big\|e^i_z(t^2_{i(k+1)})\big\| \le 2b\sqrt{\gamma_0}\big(t^2_{i(k+1)}-t^2_{ik}\big). \quad (31)$$

Accordingly,

$$\frac{\sqrt{\gamma_0}+\beta_0\gamma}{\gamma}e^{-\gamma\left(t^2_{i(k+1)}-t^2_{ik}\right)} \le 4b\sqrt{\gamma_0}\big(t^2_{i(k+1)}-t^2_{ik}\big)e^{\gamma t^2_{ik}}+\frac{\sqrt{\gamma_0}}{\gamma}, \quad (32)$$

where $\beta_0=\min\{\beta_4,\beta_6\}$.

We can see that the set $\Big\{t^2_{i(k+1)}-t^2_{ik}>0:\ 4b\sqrt{\gamma_0}\big(t^2_{i(k+1)}-t^2_{ik}\big)e^{\gamma t^2_{ik}}+\frac{\sqrt{\gamma_0}}{\gamma}-\frac{\sqrt{\gamma_0}+\beta_0\gamma}{\gamma}e^{-\gamma\left(t^2_{i(k+1)}-t^2_{ik}\right)}>0\Big\}$ is nonempty for any agent i. Hence, the Zeno behavior is avoided.

In the light of Lemma 3.3, we complete the proof of Theorem 3.4. □

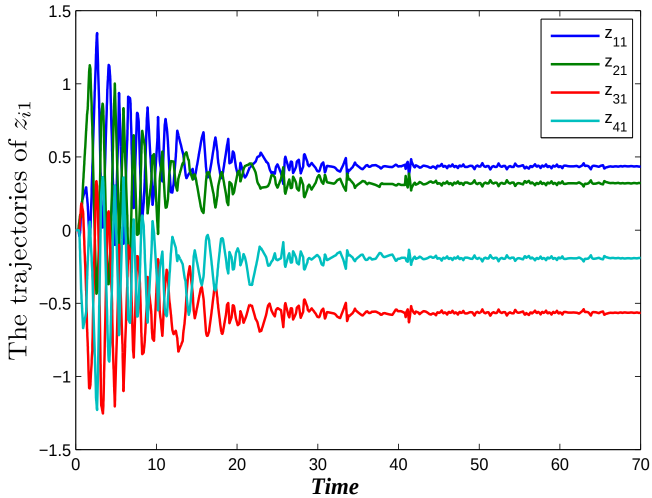

Fig. 5. The trajectories of z_{i1} in the algorithm (2).

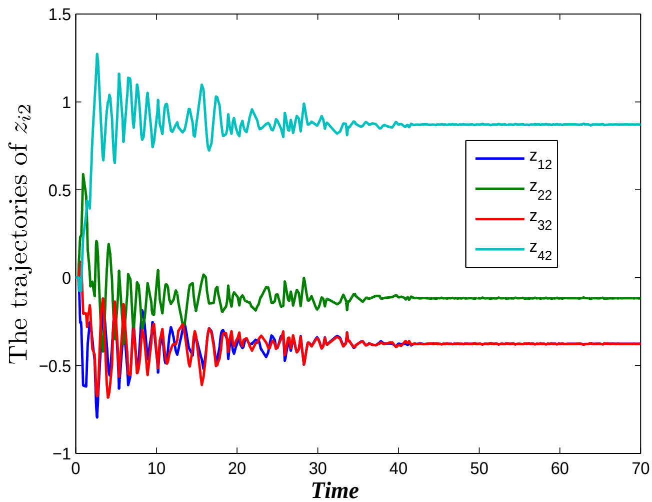

Fig. 6. The trajectories of z_{i2} in the algorithm (2).

Remark 3.5. Clearly, the result is consistent with the one given in [23] when the communication is continuous-time. The implementation of the proposed algorithm in [23] requires continuous-time communication among the agents, which is useful for analysis, but it is important to notice that communication is only available at discrete instants of time in practical scenarios. The main motivation of our study stems from this observation.

4. SIMULATION EXAMPLE

In this section, a simulation example is given for illustration.

Consider a network of five agents to minimize $F(x) = \sum_{i=1}^{4} f_i(x_i), x_i \in \mathbb{R}^2$ with the network resource constraint: $\sum_{i=1}^{4} x_i = \sum_{i=1}^{4} d_i$, where the local cost functions

$$\begin{aligned} f_1(x_1) &= \|x_1\|^2 \\ f_2(x_2) &= \frac{x_{21}^2}{20\sqrt{x_{21}^2+1}} + \frac{x_{22}^2}{20\sqrt{x_{22}^2+1}} + \|x_2\|^2 \\ f_3(x_3) &= \|x_3 - [2 \;\; 3]^T\|^2 \\ f_4(x_4) &= \ln(e^{-0.05x_{41}} + e^{0.05x_{41}}) + \ln(e^{-0.05x_{42}} + e^{0.05x_{42}}) + \|x_4\|^2, \end{aligned}$$

and the local resource data of these four agents are

$$d_1 = [2,1]^T, d_2 = [2,3]^T, d_3 = [2,4]^T, d_4 = [1,5]^T.$$

Obviously, for $i = 1, \dots, 4$, f_i are differentiable and strongly convex and the gradients of f_i are globally Lipschitz, and therefore f_i satisfy Assumption 2.2.

The optimal solution x^* can be calculated by MATLAB:

$$x^* = \left[x_1^{*T}, x_2^{*T}, x_3^{*T}, x_4^{*T}\right]^T = [1.2572, 2.5074, 1.2301, 2.4810, 3.2572, 5.5074, 1.2556, 2.5043]^T.$$

The weighted matrix of the undirected graph $\mathcal{G}$ is

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

which implies that $\mathcal{G}$ is connected.

According to Theorem 3.4, we can choose appropriate parameters α, β_i and γ such that the algorithm (2) solves our problem.

Let $\alpha = 0.0003, \beta_1 = \beta_2 = \beta_4 = 1, \beta_3 = \beta_6 = 2, \beta_5 = 0.5, \gamma = 0.1$, the initial values $x_1(0) = [2,0]^T, x_2(0) = [1.5, 0.5]^T$, $x_3(0) = [1,1]^T, x_4(0) = [4,6]^T$, $y_1(0) = y_2(0) = y_3(0) = y_4(0) = z_1(0) = z_2(0) = z_3(0) = z_4(0) = [0,0]^T$, The simulation results are shown in Figures 1-8.

From Figures 1- 2, it is clear that, the trajectories $x(t)$ converge to the global optimal solution x^*. Furthermore, Figures 3 − 6 show that the trajectories y_i and z_i of each agent i converge to some constants. Besides, for all circles in Figures 7 and 8, the ordinate of the circle center represents the time interval between the last sampling/communication time and the abscissa of the circle center denotes the latest sampling/communication time. From Figures 7 − 8, we can see that the costs of communication and gradient measurement are reduced and the Zeno behavior of triggering time can be avoided.

5. CONCLUSIONS

In this paper, a novel distributed event-triggered algorithm for continuous-time multi-agent systems has been designed to solve the network optimization problem. We have

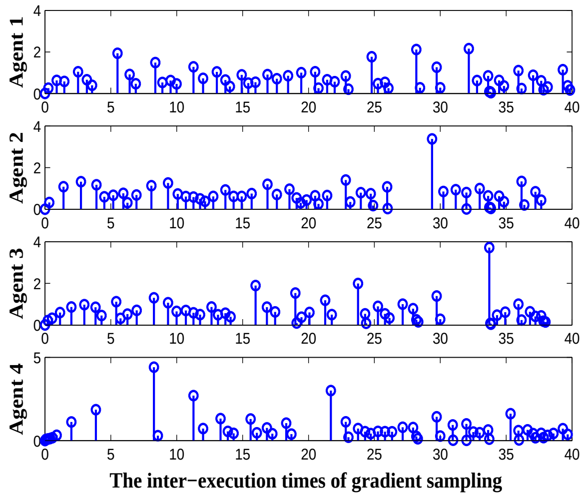

Fig. 7. The inter-execution times of gradient sampling.

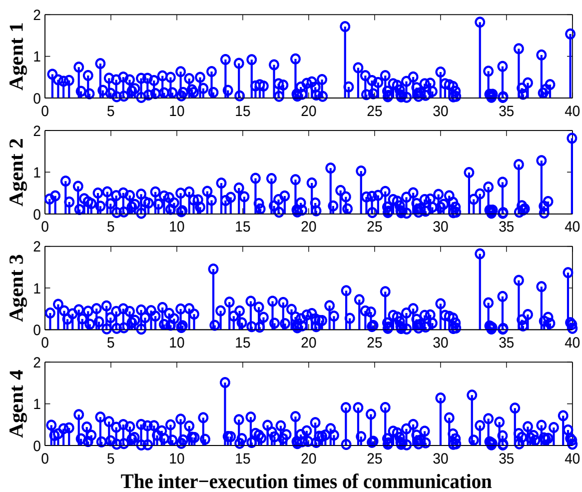

Fig. 8. The inter-execution times of communication.

proved that the proposed algorithm can achieve the exact optimization with exponential convergence rate, and moreover, the given two event-triggered strategies are free of Zeno behavior. Simulation results have been given to demonstrate the effectiveness of the algorithm. Future work may include the design of distributed optimization algorithms with various constraints.

ACKNOWLEDGEMENT

The first author would like to thank Dr. Xuefang Wang for her helpful suggestions. The work was partially supported by the National Key Research and Development Program of China (2016YFB0901902), National Natural Science Foundation of China (61333001, 61403231 and 61603378) and Beijing Natural Science Foundation (4152057) and China Postdoctoral Science Foundation (2015M581189).

(Received January 19, 2017)

REFERENCES

[1] A. Beck, A. Nedić, A. Ozdaglar, and M. Teboulle: An $O(1/k)$ gradient method for network resource allocation problems. IEEE Trans. Control Network Systems *1* (2014), 64–74. DOI:10.1109/tcns.2014.2309751

[2] W. Chen and W. Ren: Event-triggered zero-gradient-sum distributed consensus optimization over directed networks. Automatica *65* (2016), 90–97. DOI:10.1016/j.automatica.2015.11.015

[3] Z. Deng, X. Wang, and Y. Hong: Distributed optimization design with triggers for disturbed continuous-time multi-agent systems. IET Control Theory Appl. *11* (2017), 282–290. DOI:10.1049/iet-cta.2016.0795

[4] E. Ghadimi, I. Shames, and M. Johansson: Multi-step gradient methods for networked optimization. IEEE Trans. Signal Processing *61* (2013), 5417–5429. DOI:10.1109/tsp.2013.2278149

[5] C. Godsil and G. Royle: Algebraic Graph Theory. Springer-Verlag, New York 2001. DOI:10.1007/978-1-4613-0163-9

[6] Y. Gao and L. Wang: Sampled-data based consensus of continuous-time multi-agent systems with time-varying topology. IEEE Trans. Automat. Control *56* (2011), 1226–1231. DOI:10.1109/tac.2011.2112472

[7] Y. C. Ho, L. Servi, and R. Suri: A class of center-free resource allocation algorithms. Large Scale Systems *1* (1980), 51–62.

[8] H. K. Khalil: Nonlinear Systems. Third edition. Prentice Hall, New Jersey 2002.

[9] S. S. Kia, J. Cortés, and S. Martínez: Distributed convex optimization via continuous-time coordinate alorithms with discrete-time communication. Automatica *55* (2015), 254–264. DOI:10.1016/j.automatica.2015.03.001

[10] D. Lehmann and J. Lunze: Event-based control with communication delays and packet losses. Int. J. Control *85* (2012), 563–577. DOI:10.1080/00207179.2012.659760

[11] H. Lakshmanan and D. P. Farias: Decentralized resource allocation in dynamic networks of agents. SIAM J. Optim. *19* (2008), 911–940. DOI:10.1137/060662228

[12] J. Hu, G. Chen, and H. Li: Distributed event-triggered tracking control of leader-follower multi-agent systems with communication delays. Kybernetika *47* (2011), 630–643.

[13] A. Nedić, A. Ozdaglar, and P. A. Parrilo: Constrained consensus and optimization in multi-agent networks. IEEE Trans. Automat. Control *55* (2010), 922–938. DOI:10.1109/tac.2010.2041686

[14] Y. Lou, G. Shi, K. Johansson, and Y. Hong: Approximate projected consensus for convex interesection computation: convergence analysis and critical error angle. IEEE Trans. Automat. Control *59* (2014), 1722–1736. DOI:10.1109/tac.2014.2309261

[15] R. T. Rockafellar: Convex Analysis. Princeton University Press, Princeton 1972.

[16] A. P. Ruszczynski: Nonlinear Optimization. Princeton University Press, Princeton 2006.

[17] P. Tabuada: Event-triggered real-time scheduling of stabilizing control tasks. IEEE Trans. Automat. Control *52* (2007), 1680–1685. DOI:10.1109/tac.2007.904277

[18] X. Wang, P. Yi, and Y. Hong: Dynamic optimization for multi-agent systems with external disturbances. Control Theory Technol. *12* (2014), 132–138. DOI:10.1007/s11768-014-0036-y

[19] X. Wang, Z. Deng, S. Ma, and X. Du: Event-triggered design for multi-agent optimal consensus of Euler-Lagrangian systems. Kybernetika *1* (2017), 179–194.

[20] L. Xiao and S. Boyd: Optimal scaling of a gradient method for distributed resource allocation. J. Optim. Theory and Appl. *129* (2006), 469–488. DOI:10.14736/kyb-2017-1-0179

[21] Y. Zhang and Y. Hong: Distributed event-triggered tracking control of multi-agent systems with active leader. In: Proc. 10th IEEE World Congress on Intelligent Control and Automation 2012, pp. 1453–1458. DOI:10.1109/wcica.2012.6358108

[22] Y. Zhang, Y. Lou, Y. Hong, and L. Xie: Distributed projection-based algorithms for source localization in wireless sensor networks. IEEE Trans. Wireless Communications *14* (2015), 3131–3142. DOI:10.1109/twc.2015.2402672

[23] P. Yi, Y. Hong, and F. Liu: Initialization-free distributed algorithms for optimal resource allocation with feasibility constraints and application to economic dispatch of power systems. Automatica *74* (2016), 259–269. DOI:10.1016/j.automatica.2016.08.007

Weiyong Yu, Institute of Systems Science, Chinese Academy of Sciences, Beijing, 100190. P.R. China.
e-mail: y_wy2010@126.com

Zhenhua Deng, School of Information Science and Engineering, Central South University, Changsha, 410004, P.R. China. He is the corresponding author of this paper.
e-mail: zhdeng@amss.ac.cn

Hongbing Zhou, Institute of Systems Science, Chinese Academy of Sciences, Beijing, 100190. P.R. China.
e-mail: hbzhou@amss.ac.cn

Xianlin Zeng, Institute of Systems Science, Chinese Academy of Sciences, Beijing, 100190. P.R. China.
e-mail: xianlin.zeng@amss.ac.cn