KYBERNETIKA — VOLUME *53* (2017), NUMBER 4, PAGES 578–594

BACKSTEPPING BASED NONLINEAR ADAPTIVE CONTROL FOR THE EXTENDED NONHOLONOMIC DOUBLE INTEGRATOR

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In this paper a steering control algorithm for the Extended Nonholonomic Double Integrator is presented. An adaptive backstepping based controller is proposed which yields asymptotic stabilization and convergence of the closed loop system to the origin. This is achieved by transforming the original system into a new system which can be globally asymptotically stabilized. Once the new system is stabilized, the stability of the original system can be easily established. Stability of the closed loop system is analyzed on the basis of Lyapunov theory. The effectiveness of the proposed control algorithm is verified through numerical simulation and the results are compared to existing methods.

Keywords: nonholonomic systems, feedback stabilization, systems with drift, adaptive backstepping, Lyapunov function

Classification: 93D15

1. INTRODUCTION

The problem of stabilization of nonholonomic systems has been the topic of active research for the last few decades. The reason for this is primarily threefold: i) Mechanical systems such as wheeled mobile robots, robot manipulators, space robots and underwater vehicles have non integrable constraints; ii) the formation of control law for systems which cannot be easily transformable into linear control problem in a meaningful way is quiet challenging and iii) these systems cannot be stabilized by static time invariant state feedback laws as pointed out by Brockett [6]. Different control strategies have been presented for the stabilization of nonholnomic systems to overcome the limitations of the Brockett result like: discontinuous time-invariant stabilization [4, 10, 12, 14], smooth time-varying stabilization [13, 15], adaptive techniques [9, 19] and sliding mode control [1, 5, 17].

Discontinuous feedback control approaches use the discontinuous change of coordinates and switching control strategy to overcome the difficulty of loss of controllability. The advantage is its simplicity and fast transient response, and the drawback is that the control input is discontinuous. However, those aforementioned works only considered the systems without drifts or with weak nonlinear drifts. The time-varying feedback

DOI: 10.14736/kyb-2017-4-0578

control, provides smooth/continuous controller and no switching is required. This approach introduces some persistent excitation signals in the control input to guarantee the convergence of the closed-loop signals. However, the convergence rate of this approach is slow.

In a number of published work [1, 5, 16], the authors have concentrated on the kinematic models of nonholonomic systems with their velocity as direct control inputs. Although this approach works well in many situation, the performance of some physical systems will degrade if their dynamics, like forces and torques which are the actual inputs, are neglected.

To solve the stabilization problem of nonholonomic systems, this paper presents a steering control algorithm for the Extended Nonholonomic Double Integrator (ENDI) system. The ENDI system is an extension of the nonholonomic integrator presented in [6]. The importance of ENDI system is that it features the dynamics and kinematics of a nonholonomic system with three states and two control inputs, for example the dynamics of a wheeled mobile robot.

Many control strategies [2, 3, 8] have been presented for ENDI system. This paper presents a control algorithm, based on adaptive backstepping technique [7, 11, 18], with the objective of steering the system from any arbitrary initial state to any desired state. A time varying transformation is constructed using adaptive backstepping technique and the original system is transformed into a new system which can be easily asymptotically stabilized. Once the stability of the transformed system is proven, the original system stability can be easily established.

The main contributions of this paper are: i) Comparing with the existing results in [16], the convergence rate is improved, ii) Successfully overcoming some essential difficulties, such as the weaker assumption on the system growth and the construction of a continuously differentiable Lyapunov Krasovskii functional, a new method for asymptotic stabilization of extended nonholonomic double integrator is given, which can lead to more general results.

The rest of the article is organized as follows. Section 2 presents the mathematical model of the Extended Nonholonomic Double Integrator. Section 3 presents problem formulation and Some Preliminaries. Section 4 presents the proposed control methodology in its general form. Section 5 presents the transformation of general system into new system. Section 6 presents simulation results for the application example. Section 7 represents the comparison of the proposed controller with the controller presented in [16] and finally section 8 concludes the paper.

2. MATHEMATICAL MODEL OF THE EXTENDED NONHOLONOMIC DOUBLE INTEGRATOR

In [6], Brockett introduced the following nonholonomic integrator system:

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_2 u_1 - x_1 u_2.\end{aligned} \tag{1}$$

Where $x = [x_1, x_2, x_3]^T \in \Re^3$ is the state vector and $u = [u_1, u_2]^T \in \Re^2$ is the control input vector. The above mentioned system has all the properties of nonholonomic systems and is known as a benchmark for control system design and analysis in the control literature [4, 6, 10, 12].

The nonholonomic integrator (1) exhibits, under suitable control and state transformations, the kinematics of the wheeled robot. However, when we take into account both the kinematics and dynamics of the wheel robot, the nonholonomic integrator model fails to capture all the features. To represent a more realistic case, we must use the extended nonholonomic integrator model. The dynamical equations of motion of a mobile robot of the unicycle type can be represented into the following form [6]:

$$\begin{aligned} \ddot{x}_1 &= u_1 \\ \ddot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1\dot{x}_2 - x_2\dot{x}_1. \end{aligned} \tag{2}$$

Defining the state variable $x = [x_1, x_2, x_3, x_4, x_5]^T = [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2]^T$, system (2) can be written as:

$$\begin{aligned} \dot{x}_1 &= x_4 \\ \dot{x}_2 &= x_5 \\ \dot{x}_3 &= x_1x_5 - x_2x_4 \\ \dot{x}_4 &= u_1 \\ \dot{x}_5 &= u_2. \end{aligned} \tag{3}$$

The above system (3) can be rewritten in the following general form:

$$\dot{x} = X_0(x) + X_1(x)u_1 + X_2(x)u_2 \tag{4}$$

where

$$X_0(x) = \begin{bmatrix} x_4 \\ x_5 \\ x_1x_5 - x_2x_4 \\ 0 \\ 0 \end{bmatrix}, \; X_1(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; X_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

As in [1], system (3) will be referred to as the Extended Nonholonomic Double Integrator (ENDI). The ENDI system (3) satisfies the following properties:

i. The vector fields X_0, X_1, X_2 are real, analytic and complete, additionally $X_0(0) = 0$.

ii. The ENDI system is locally strongly accessible for any $x \in \Re^5$ as this satisfies the Lie algebra rank condition (LARC) for accessibility, namely that $L(X_0, X_1, X_2)$, the Lie algebra of vector fields generated by $X_0(x), X_1(x), X_2(x)$ span R^5 at each point $x \in \Re^5$ i.e, $span\{X_1, X_2, X_3, X_4, X_5\} = R^5$, $\forall\, x \in R^5$, where:

$$X_3(x) = [X_0(x), X_1(x)] = \begin{bmatrix} 1 \\ 0 \\ -x_2 \\ 0 \\ 0 \end{bmatrix} \tag{5a}$$

$$X_4(x) = [X_0(x), X_2(x)] = \begin{bmatrix} 0 \\ 1 \\ x_1 \\ 0 \\ 0 \end{bmatrix} \tag{5b}$$

$$X_5(x) = [X_3(x), X_4(x)] = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}. \tag{5c}$$

3. THE CONTROL PROBLEM AND SOME PRELIMINARIES

This section presents the control problem and some preliminaries needed in proofs.

3.1. The control problem

Given the desired set point $x_{des} \in \Re^5$, construct a feedback law in terms of the controls $u_i : \Re^5 \to \Re,\ \ i = 1, 2$ such that the desired set point x_{des} is an attractive set for (3), so that there exists an $\varepsilon > 0$, such that $x(t; t_0, x_0) \to x_{des}$, as $t \to \infty$ for any initial condition $(t_0, x_0) \in \Re^+ \times \mathcal{B}(x_{des}; \varepsilon)$. It is assumed generally that $x_{des} = 0$, which can be obtained by a suitable transformation of the coordinate system.

3.2. Preliminaries

To characterize the stability of the solution of nonholonomic systems the following definitions are given.

Definition 3.1. An equilibrium state $x = 0$ is said to be:

- **Stable** if for any positive scalar ϵ there exists a positive scalar δ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq t_0$.
- **Asymptotic stable** if it is stable and if in addition $x(t) \to 0$ as $t \to \infty$.

Definition 3.2. The equilibrium point of system (1) is said to be globally asymptotically stable if it is locally stable in sense of Lyapunov and globally attractive. According to Lyapunov direct method, as we know, if there exists an appropriate Lyapunov function V which is positive definite and radially unbounded, such that the time-derivative of V along the trajectory of system (1) is negative definite, then the equilibrium point of system (1) is globally asymptotically stable.

Definition 3.3. The Lyapunov theorem says that a point of equilibrium of a system is stable if there exists a positive definite function V such that its time derivative $\dot{V}$ is non-positive for all trajectories of a system.

Theorem (Lyapunov Direct method).
Let G be the subset of $\Re^n$ containing x_0. Suppose $f \in C^1(G)$ and that $f(x_0) = 0$. Further there exists a real function $V \in C^1(G)$ satisfying $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$. Then

- **Stable** if $\dot{V}(x) \leq 0$ for all $x \in G$.
- **Asymptotic stable** if $\dot{V}(x) < 0$ for all $x \in G$.

4. THE PROPOSED CONTROL ALGORITHM

Step 1: Transform the system (3) from x-domain into z-domain by using a time varying transformation: $z = T(x, \theta\phi(x_1))$ where θ is some unknown parameter estimated adaptively and $\phi(x_1)$ is some known function satisfying the condition $\phi(0) = 0$ and $\phi(1) = k$. Although the control algorithm works for constant values of k but for improved transient response we take $k = -\hat{\theta}$ as the dynamic gain, which is evident from simulation results. Let $\hat{\theta}$ be the estimate of θ and $\tilde{\theta} = \theta - \hat{\theta}$ be the estimation error.
The transformation has the following properties:

i. $z = T(x, \theta\phi(x_1)) = Ax + B\hat{\theta}\phi(x_1)$, where $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ constant matrices, where $n = 5$, $m = 2$.

ii. $x = \hat{T}(z, \theta\phi(x_1)) = \hat{A}z + \hat{B}\hat{\theta}\phi(x_1)$, where $\hat{A}A = I$ *i.e* $\hat{T}(z, \theta\phi(x_1))$ $= \text{inv}\{T(x, \theta\phi(x_1))\}$.

iii. The transformed dynamical system is: $\dot{z} = Mz + N\tilde{\theta}\phi(x_1)$, where M is negative definite.

Step 2: Choose the adaptive law for $\tilde{\theta}$ such that the transformed system is asymptotically stable by selecting a Lyapunov function:

$$V(z, \tilde{\theta}) = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^T\tilde{\theta}.$$

Step 3: $x = \hat{T}(z, \theta\phi(x_1)) = \hat{A}z + \hat{B}\hat{\theta}\phi(x_1) \to 0$ as $z \to 0$, $\hat{\theta}\phi(x_1) \to 0$.

5. CONSTRUCTION OF THE TRANSFORMATION

By choosing $u_1 = x_2,\ u_2 = x_3 + \theta\phi(x_1)$ system (3) can be written as:

$$\dot{x}_1 = x_4 \tag{6a}$$
$$\dot{x}_4 = x_2 \tag{6b}$$
$$\dot{x}_2 = x_5 \tag{6c}$$
$$\dot{x}_5 = x_3 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) \tag{6d}$$
$$\dot{x}_3 = x_1 x_5 - x_2 x_4. \tag{6e}$$

Consider equation (6a) and considering x_4 as virtual control, α_1 as the stabilizing function and $z_1 = x_4 - \alpha_1$ be the error variable, equation (6a) can be written as:

$$\dot{x}_1 = z_1 + \alpha_1. \tag{7}$$

To work out α_1, consider the Lyapunov Function $V_0 = \frac{1}{2}x_1^2$ for (7) then

$$\dot{V}_0 = x_1\dot{x}_1 = x_1(z_1 + \alpha_1).$$

By choosing $\alpha_1 = -x_1$, the above equation becomes

$$V_0 = -x_1^2 + x_1 z_1.$$

Equation (7) becomes:

$$\dot{x}_1 = z_1 - x_1. \tag{8}$$

Consider the equation (6b) and considering x_2 as the virtual control, α_2 as a stabilizing function and $z_2 = x_2 - \alpha_2$ be the error variable, equation (6b) can be rewritten as:

$$\dot{x}_4 = z_2 + \alpha_2.$$

Since the dynamic of first error variable $z_1 = x_4 - \alpha_1 = x_4 + x_1$ is:

$$\dot{z}_1 = \dot{x}_4 + \dot{x}_1 = z_2 + \alpha_2 + z_1 - x_1. \tag{9}$$

To work out α_2, consider the Lyapunov function: $V_1 = V_0 + \frac{1}{2}z_1^2$ for (8) and (9). Then

$$\dot{V}_1 = -x_1^2 + z_1\{z_2 + \alpha_2 + z_1\}.$$

By choosing $\alpha_2 = -2z_1$

$$\dot{V}_1 = -x_1^2 - z_1^2 + z_1 z_2.$$

Equation (9) becomes:

$$\dot{z}_1 = z_2 - z_1 - x_1. \tag{10}$$

Consider equation (6c) and considering x_5 as virtual control, α_3 as a stabilizing function and $z_3 = x_5 - \alpha_3$ be the error variable, equation (6c) can be written as:

$$\dot{x}_2 = x_5 = z_3 + \alpha_3.$$

The dynamic of the second error variable namely $z_2 = x_2 - \alpha_2 = x_2 + 2z_1$ is:

$$\dot{z}_2 = \dot{x}_2 + 2\dot{z}_1 = z_3 + \alpha_3 + 2z_2 - 2z_1 - 2x_1. \tag{11}$$

To compute α_3, consider the Lyapunov function: $V_2 = V_1 + \frac{1}{2}z_2^2$ for (8), (10) and (11). Then

$$\dot{V}_2 = -x_1^2 - z_1^2 + z_2\{z_3 + \alpha_3 + 2z_2 - z_1 - 2x_1\}.$$

By choosing $\alpha_3 = -3z_2 + z_1 + 2x_1$

$$\dot{V}_2 = -x_1^2 - z_1^2 + z_2^2 + z_2 z_3.$$

Equation (11) becomes:

$$\dot{z}_2 = z_3 - z_2 - z_1. \tag{12}$$

Consider equation (6d) and considering x_3 as a virtual control, α_4 as the stabilizing function and $z_4 = x_3 - \alpha_4$ be the error variable, equation (6d) can be written as:

$$\dot{x}_5 = x_3 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) = z_4 + \alpha_4 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1). \tag{13}$$

The dynamic of third error variable namely $z_3 = x_5 - \alpha_3 = x_5 + 3z_2 - z_1 - 2x_1$ is

$$\begin{aligned} \dot{z}_3 &= \dot{x}_5 + 3\dot{z}_2 - \dot{z}_1 - 2\dot{x}_1 \\ &= z_4 + \alpha_4 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1) + 3z_3 - 3z_2 - 3z_1 - z_2 + z_1 + x_1 - 2z_1 + 2x_1 \\ &= z_4 + \alpha_4 + 3z_3 - 4z_2 - 4z_1 + 3x_1 + \hat{\theta}\phi(x_1) + \tilde{\theta}\phi(x_1). \end{aligned} \tag{14}$$

To work out α_4, consider the Lyapunov function: $V_3 = V_2 + \frac{1}{2}z_3^2$ for (8), (10), (12) and (14). Then

$$\dot{V}_3 = \dot{V}_2 + z_3\dot{z}_3 = -x_1^2 - z_1^2 - z_2^2 + z_3\{z_4 + \alpha_4 + 3z_3 - 3z_2 - 4z_1 + 3x_1 + \hat{\theta}\phi(x_1)\} + z_3\tilde{\theta}\phi(x_1).$$

By choosing $\alpha_4 = -4z_3 + 3z_2 + 4z_1 - 3x_1 - \hat{\theta}\phi(x_1)$, we have

$$\dot{V}_3 = -x_1^2 - z_1^2 - z_2^2 - z_3^2 + z_3 z_4 + \tilde{\theta}\phi(x_1)z_3.$$

Equation (14) becomes:

$$\dot{z}_3 = z_4 - z_3 - z_2 + \tilde{\theta}\phi(x_1). \tag{15}$$

The stabilizing function α_4 can be written as:

$$\alpha_4 = -4z_3 + 3z_2 + 4z_1 - 3x_1 - \hat{\theta}\phi(x_1) = -4z_3 + 3z_2 + 4z_1 - 3x_1 - v + \hat{\theta}$$

where $v = \hat{\theta}(\phi(x_1) + 1)$.

The dynamics of fourth error variable namely $z_4 = x_3 - \alpha_4 = x_3 + 4z_3 - 3z_2 - 4z_1 + 3x_1 + v - \hat{\theta}$ is:

$$\begin{aligned} \dot{z}_4 &= \dot{x}_3 + 4\dot{z}_3 - 3\dot{z}_2 - 4\dot{z}_1 + 3\dot{x}_1 + \dot{v} - \dot{\hat{\theta}} \\ &= x_1x_5 - x_2x_4 + 4z_4 - 4z_3 - 4z_2 + 4\tilde{\theta}\phi(x_1) - 3z_3 + 3z_2 \\ &\quad + 3z_1 - 4z_2 + 4z_1 + 4x_1 + 3z_1 - 3x_1 + \dot{v} - \dot{\hat{\theta}} \\ &= \beta + 4z_4 - 7z_3 - 5z_2 + 10z_1 + x_1 + 4\tilde{\theta}\phi(x_1) + \dot{v} - \dot{\hat{\theta}} \end{aligned} \tag{16}$$

where

$$\beta = x_1x_5 - x_2x_4 = x_1(z_3 - 3z_2 + z_1 + 2x_1) - (z_2 - 2z_1)(z_1 - x_1).$$

Consider the Lyapunov function $V_4 = V_3 + \frac{1}{2}z_4^2 + \tilde{\theta}^2$ for (8), (10), (12), (15) and (16).

$$\dot{V}_4 = \dot{V}_3 + z_4\dot{z}_4 + 2\tilde{\theta}\dot{\tilde{\theta}} = -x_1^2 - z_1^2 - z_2^2 - z_3^2 + z_4\{\beta + 4z_4 - 6z_3 - 5z_2 + 10z_1 + x_1 + \dot{v} - \dot{\hat{\theta}}\}$$
$$+ \tilde{\theta}\{(z_3 + 4z_4)\phi(x_1) + \dot{\tilde{\theta}}\}.$$

By choosing

$$\dot{v} = -\beta - 5z_4 + 6z_3 + 5z_2 - 10z_1 - x_1 + \dot{\hat{\theta}}$$

$$\dot{\tilde{\theta}} = -(z_3 + 4z_4)\phi(x_1) - \tilde{\theta} = -\dot{\hat{\theta}}$$

we have

$$\dot{V}_4 = -{x_1}^2 - {z_1}^2 - {z_2}^2 - {z_3}^2 - {z_4}^2 - \tilde{\theta}^2 \tag{17}$$

and the equation(16) become:

$$\dot{z}_4 = -z_4 - z_3 + 4\tilde{\theta}\phi(x_1). \tag{18}$$

The closed loop system becomes:

$$\begin{aligned} \dot{x_1} &= z_1 - x_1 \\ \dot{z_1} &= z_2 - z_1 - x_1 \\ \dot{z_2} &= z_3 - z_2 - z_1 \\ \dot{z_3} &= z_4 - z_3 - z_2 + \tilde{\theta}\phi(x_1) \\ \dot{z_4} &= -z_4 - z_3 + 4\tilde{\theta}\phi(x_1). \end{aligned} \tag{19}$$

By defining $z = [x_1, z_1, z_2, z_3, z_4]^T$ we have:

$$\dot{z} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \left[\ \tilde{\theta}\phi(x_1)\ \right]$$

$$= Mz + N\tilde{\theta}\phi(x_1). \tag{20}$$

It can be easily verified that M is negative definite. Since the derivative of Lyapunov function given by (17) is strictly negative therefore the transformed system (20) is asymptotically stable, therefore $x_1, z_1, z_2, z_3, z_4 \to 0$ and $\tilde{\theta} \to 0$.

By using the following relations

$$\begin{aligned} x_1 &= x_1 \\ z_1 &= x_4 - \alpha_1 = x_4 + x_1 \\ z_2 &= x_2 - \alpha_2 = x_2 + 2x_1 + 2x_4 \\ z_3 &= x_5 + 3z_2 - z_1 - 2x_1 = x_5 + 5x_4 + 3x_2 + 3x_1 \\ z_4 &= x_3 - \alpha_4 = x_3 + 4z_3 - 3z_2 - 4z_1 + 3x_1 + \hat{\theta}\phi(x_1) \\ &= 4x_5 + 10x_4 + x_3 + 9x_2 + 5x_1 + \hat{\theta}\phi(x_1) \end{aligned}$$

we can write:

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 0 \\ 3 & 3 & 0 & 5 & 1 \\ 5 & 9 & 1 & 10 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \hat{\theta}\phi(x_1) \end{bmatrix}$$

$$= Ax + B\hat{\theta}\phi(x_1) = T(x, \theta\phi(x_1)). \tag{21}$$

Also using the following relations

$$\begin{aligned} x_1 &= x_1 \\ x_4 &= z_1 - x_1 \\ x_2 &= z_2 - 2z_1 \\ x_5 &= z_3 - 3z_2 + z_1 + 2x_1 \\ x_3 &= z_4 - 4z_3 + 3z_2 + 4z_1 - 3x_1 - \hat{\theta}\phi(x_1) \end{aligned}$$

we have:

$$x = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ -3 & 4 & 3 & -4 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & 1 & -3 & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \hat{\theta}\phi(x_1) \end{bmatrix}$$

$$= \hat{A}z + \hat{B}\hat{\theta}\phi(x_1) = \hat{T}(z, \theta\phi(x_1)). \tag{22}$$

It can be checked that $A\hat{A} = I$ and $\hat{A}B = -\hat{B}$. As $z \to 0$ and $\hat{\theta}\phi(x_1) \to 0$, so $x \to 0$. From which conclude that the original system (6) will also converge asymptotically.

Theorem 5.1. By Choosing $u_1 = x_2$ and $u_2 = x_3 + \theta\phi(x_1)$ and the transformation as given in (21) the system (6) can be transformed into (20) which is asymptotically stable and therefore by (22) the original system is always asymptotically stable.

Proof. By considering a Lyapunov function $V(z) = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^2$ it follows that along the controlled system trajectories

$$\begin{aligned}\dot{V}_z &= x_1\dot{x}_1 + z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 + z_4\dot{z}_4 + \tilde{\theta}\dot{\tilde{\theta}} \\ &= x_1(z_1 - x_1) + z_1(z_2 - z_1 - x_1) + z_2(z_3 - z_2 - z_1) + z_3(z_4 - z_3 - z_2 + \tilde{\theta}\phi(x_1)) \\ &+ z_4(-z_4 - z_3 + 4\tilde{\theta}\phi(x_1)) + \tilde{\theta}\dot{\tilde{\theta}} \\ &= -x_1^2 - z_1^2 - z_2^2 - z_3^2 - z_4^2 + \tilde{\theta}\{\dot{\tilde{\theta}} + z_3\phi(x_1) + 4z_4\phi(x_1)\} \\ &= -x_1^2 - z_1^2 - z_2^2 - z_3^2 - z_4^2 - \tilde{\theta}^2 < 0.\end{aligned}$$

By choosing the Lyapunov function as: $V(z) = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^2$ and the adaptive law as: $\dot{\tilde{\theta}} = -z_3\phi(x_1) - 4z_4\phi(x_1) - \tilde{\theta}$, we will come up with the asymptotic stability of Lyapunov function $\dot{V} < 0$.

Thus the transformed system is asymptotically stable and by (22) the system (6a) – (6e) is also asymptotically stable. □

6. SIMULATION RESULTS

In this section, simulation results are presented to show the effectiveness of the proposed method in steering the ENDI system. The objective of the control design is to steer the states of the system to the origin. Figures 1 – 3 show simulation results for different initial conditions. Figures 4 - 8 show the effect of k on the transient response of the system for different values of k.

7. COMPARISON OF THE PROPOSED CONTROLLER WITH AN EXISTING CONTROLLER

The results of the proposed controller are compared to those of [16] in which a continuous and a discontinuous controller is presented. The initial condition are chosen to be the same as $x = [0.9, 0.7, 0.4, 0.8, 0.6]$ for each case. Figure 9 show the result of the proposed algorithm. Figure 10 and Figure 11 show the results of continuous and discontinuous controller respectively. The comparison shows that the response of the proposed controller is better than that of [16] in terms of settling time and is less oscillatory.

8. CONCLUSION

In this paper we presented a control algorithm for steering the Extended Nonholonomic Double Integrator, which is considered as a bench mark system in control system design and analysis. Based on adaptive backstepping technique, a time varying transformation was constructed and the original system was transformed into a new system which could be easily asymptotically stabilized. The effectiveness of the proposed method was verified through computer simulation.

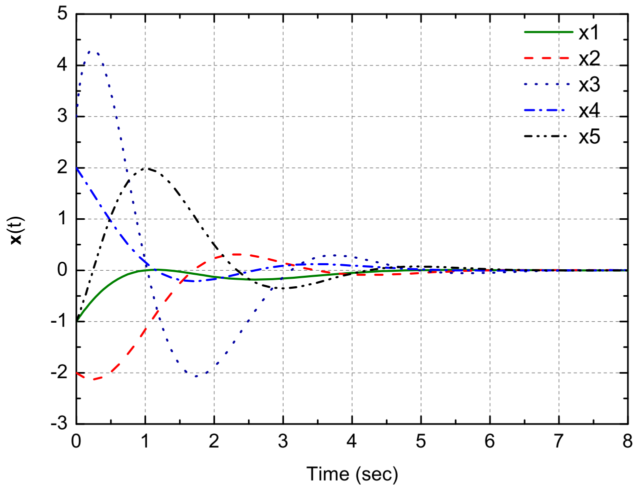

Fig. 1. Simulation results for initial conditions $x = [-1, -2, 3, 2, -1]^T$.

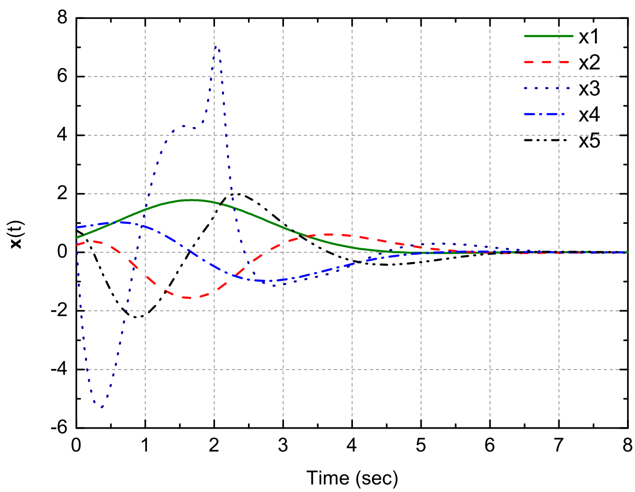

Fig. 2. Simulation results for initial conditions $x = [0.5, 0.25, 0.35, 0.85, 0.75]^T$.

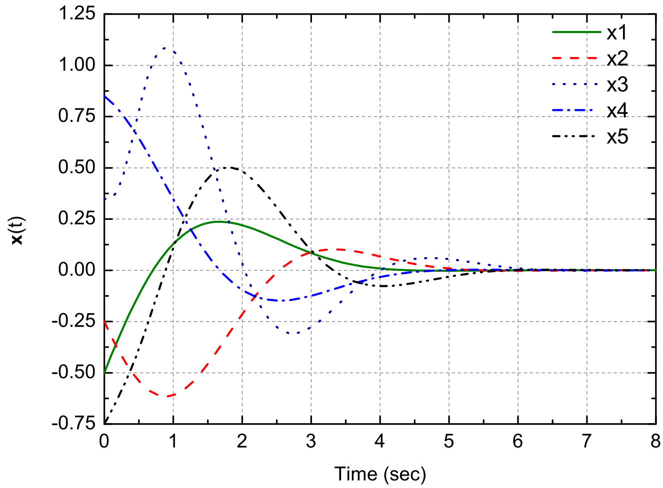

Fig. 3. Simulation results for initial conditions $x = [-0.5, -0.25, 0.35, 0.85, -0.75]^T$.

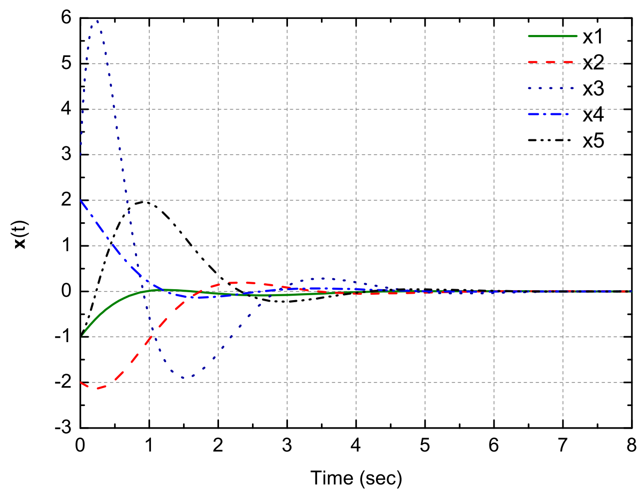

Fig. 4. Simulation results for gain $k = 1$ and initial conditions $x = [-1, -2, 3, 2, -1]^T$.

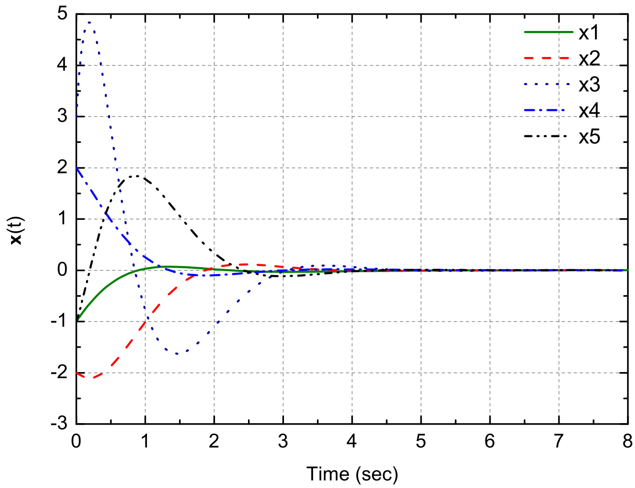

Fig. 5. Simulation results for gain $k = -1$ and initial conditions $x = [-1, -2, 3, 2, -1]^T$.

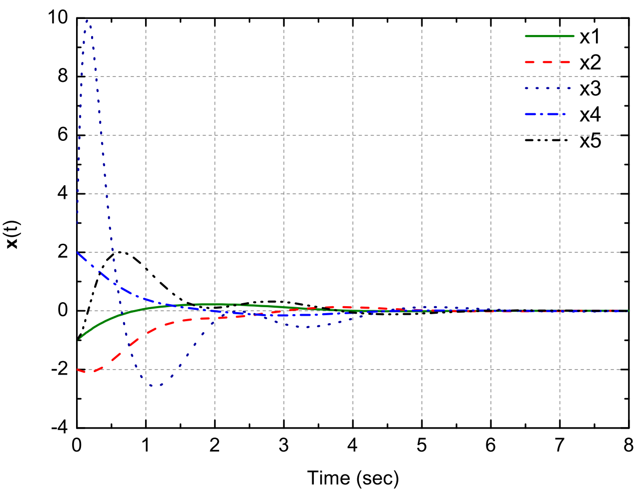

Fig. 6. Simulation results for gain $k = 2$ and initial conditions $x = [-1, -2, 3, 2, -1]^T$.

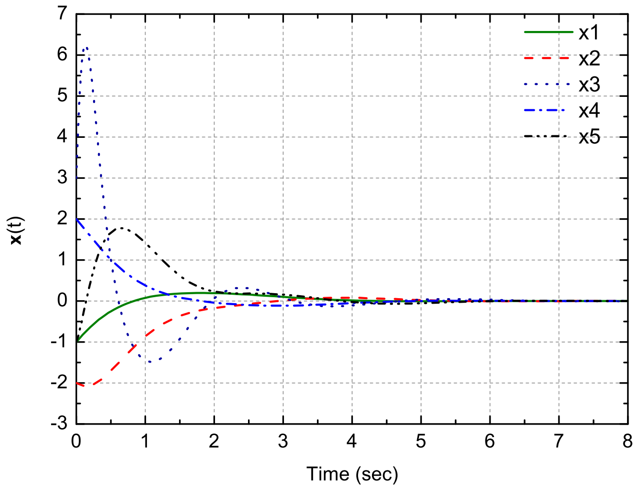

Fig. 7. Simulation results for gain $k = -2$ and initial conditions $x = [-1, -2, 3, 2, -1]^T$.

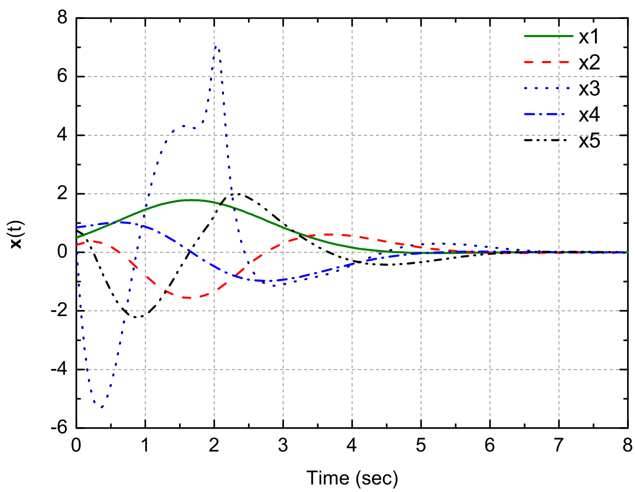

Fig. 8. Simulation results for gain $k = -\hat{\theta}$ and initial conditions $x = [-1, -2, 3, 2, -1]^T$.

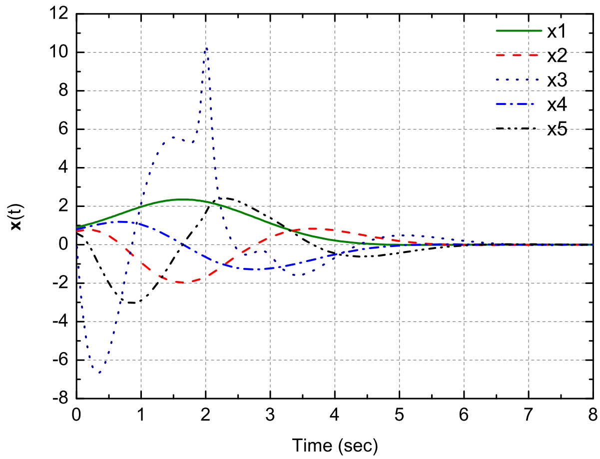

Fig. 9. Simulation results with the proposed control algorithm for initial conditions $x = [0.9, 0.7, 0.4, 0.8, 0.6]^T$.

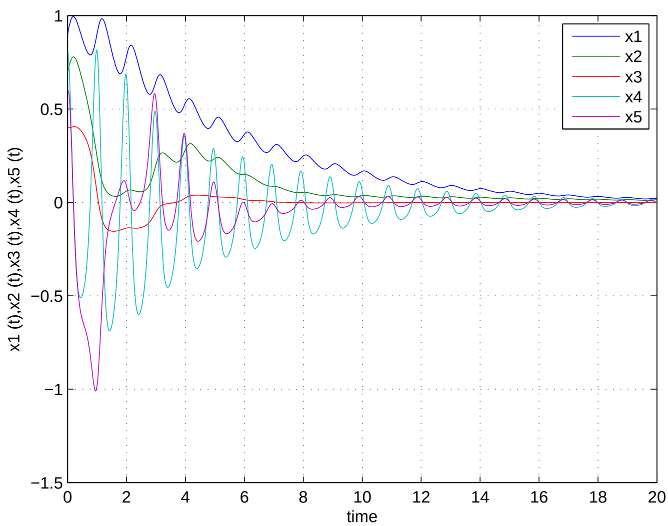

Fig. 10. Simulation results for continuous controller [16] for initial conditions $x = [0.9, 0.7, 0.4, 0.8, 0.6]^T$.

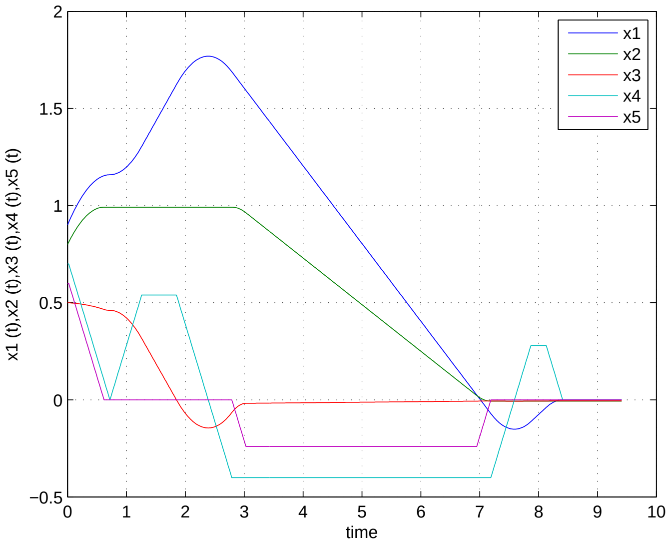

Fig. 11. Simulation results for discontinuous controller [16] for initial conditions $x = [0.9, 0.7, 0.4, 0.8, 0.6]^T$.

(Received September 21, 2016)

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