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DIVERGENCE MEASURE BETWEEN FUZZY SETS USING CARDINALITY

Vladimír Kobza

In this paper we extend the concept of measuring difference between two fuzzy subsets defined on a finite universe. The first main section is devoted to the local divergence measures. We propose a divergence measure based on the scalar cardinalities of fuzzy sets with respect to the basic axioms. In the next step we introduce the divergence based on the generating function and the appropriate distances. The other approach to the divergence measure is motivated by class of the rational similarity measures between fuzzy subsets expressed using some set operations (namely intersection, complement, difference and symmetric difference) and their scalar cardinalities. Finally, this concept is extended into the fuzzy cardinality in the last part. Some open problems omitted in this paper are discussed in the concluding remarks section.

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1. INTRODUCTION, BASIC CONCEPTS

A way of measuring difference between two fuzzy sets by means of a function has been proposed in [9, 10]. This function is called a divergence. The measure of the difference of two fuzzy subsets is defined axiomatically in Definition 1.1.

The following general notation will be used. The universal set we will denote by X, where $X \neq \emptyset$. We define the membership function μ_A as the map $\mu_A : X \to [0,1]$ such that for each element $x \in X$ the membership degree $\mu_A(x)$ is assigned. The pair (X, μ_A) is said to be a fuzzy set.

The family of all fuzzy subsets defined on the universe X will be denoted by the symbol $\mathcal{F}(X)$. More formally,

$$\mathcal{F}(X) = \{\mu_A \mid \mu_A : X \to [0,1]\} = [0,1]^X.$$

Each fuzzy set A is uniquely determined by its membership function μ_A and vice versa. Therefore, instead of $\mu_A(x)$ we will write shortly $A(x)$ for a membership degree of an element $x \in X$ to the fuzzy set A.

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Also the universal set X, the empty set $\emptyset$ and the inclusion relation $\subseteq$ will be defined as follows:

$$\begin{aligned} A = X \quad &\Leftrightarrow \quad A(x) = 1 \text{ for all } x \in X, \\ A = \emptyset \quad &\Leftrightarrow \quad A(x) = 0 \text{ for all } x \in X, \\ A \subseteq B \quad &\Leftrightarrow \quad A(x) \leq B(x) \text{ for all } x \in X. \end{aligned}$$

We recall now that the intersection (union) of two fuzzy subsets is defined by means of a triangular norm T (triangular conorm S) as a function $T(S) : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties - commutativity, monotonicity, associativity and boundary condition $T(x, 1) = x$ and $S(x, 0) = x$, respectively. In this sense we define $(A \cap B)(x) = T(A(x), B(x))$ and $(A \cup B)(x) = S(A(x), B(x))$ for all $x \in X$.

Some important examples of t-norms are:

- the minimum t-norm: $T_M(a, b) = \min\{a, b\}$, for all $a, b \in [0, 1]$,
- the product t-norm: $T_P(a, b) = a \cdot b$, for all $a, b \in [0, 1]$,
- the Łukasiewicz t-norm: $T_L(a, b) = \max\{a + b - 1, 0\}$, for all $a, b \in [0, 1]$,
- the drastic t-norm:

$$T_D(a, b) = \begin{cases} \min\{a, b\}, & \text{if } \max\{a, b\} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

For these basic t-norms, it holds that $T_D \leq T_L \leq T_P \leq T_M$. In fact, for any t-norm T it is fulfilled that $T_D \leq T \leq T_M$.

We will use a general notation for the triple (X, T, S) in the following text, where $X \neq \emptyset$ and the t-norm T and t-conorm S are dual to each other, i.e. for all $a, b \in [0, 1]$ the equation $T(a, b) = 1 - S(1 - a, 1 - b)$ is fulfilled. The most widely used example is the triple (X, T_M, S_M) with the minimum t-norm T_M and the maximum t-conorm S_M. More details about triangular norms and their applications in fuzzy set operations are described in [6].

Definition 1.1. A map $D : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ is a divergence measure if and only if the function D satisfies the following conditions:

(1) for all $A \in \mathcal{F}(X)$; $D(A, A) = 0$;

(2) for all $A, B \in \mathcal{F}(X)$; $D(A, B) = D(B, A)$;

(3) for all $A, B, C \in \mathcal{F}(X)$; $\max\{D(A \cup C, B \cup C), D(A \cap C, B \cap C)\} \leq D(A, B)$.

Obviously any divergence D is a nonnegative and symmetric function and it becomes zero if the two fuzzy subsets coincide. Natural requirement is that the value of divergence measure is smaller if the two fuzzy subsets to be compared become more similar. This similarity can be formulated by means of a union or an intersection of these sets with another fuzzy set C which is expressed by the third condition.

We could replace the condition (1) by stronger one: for all $A, B \in \mathcal{F}(X); D(A,B) = 0 \Leftrightarrow A = B$. However, we do not require it in our considerations.

Further in the text we will consider only the case where the universal set X is finite.

In sense of the third condition of divergence measure if we add a singleton $\{x\}$ to the fuzzy subsets A, B, we get the following inequality:

$$D(A \cup \{x\}, B \cup \{x\}) \leq D(A, B).$$

Locality is the most important property of some divergence measures which allows us to compute the divergence point-by-point. The definition of local divergence measure was introduced in [9] as follows.

Definition 1.2. A divergence measure D is *local* if for all $A, B \in \mathcal{F}(X)$ and for all $x \in X$ we have:

$$D(A, B) - D(A \cup \{x\}, B \cup \{x\}) = h(A(x), B(x)),$$

where h is a function from $[0,1] \times [0,1]$ to $\mathbb{R}$.

Theorem 1.3. (Representation Theorem) Let (X, T, S) be a triple with X a finite universe and T and S any t-norm and t-conorm, respectively. Let D be a divergence associated to X. D is local if and only if

$$D(A, B) = \sum_{x \in X} h(A(x), B(x)),$$

where h is a map from $[0,1] \times [0,1]$ into $\mathbb{R}$ such that the following conditions are satisfied:

(1) for all $a \in [0,1]$; $h(a,a) = 0$;

(2) for all $a, b \in [0,1]$; $h(a,b) = h(b,a)$;

(3) for all $a, b, c \in [0,1]$; $h(a,b) \geq \max\{h(T(a,c), T(b,c)), h(S(a,c), S(b,c))\}$.

The proof can be found in [7].

We get a particular form of local divergence measure D if the function h will be constructed by means of a suitable distance in $[0,1]$.

Example 1.4. For any pair of fuzzy sets in X we define the function D using the Hamming distance as follows:

$$D(A, B) = \sum_{x \in X} |A(x) - B(x)|.$$

According to [7] the map D is a local divergence measure, if we work on (X, T_M, S_M), (X, T_P, S_P) or (X, T_L, S_L). However, the map D is not a divergence measure since (X, T_D, S_D) to be considered.

Now we introduce one example of divergence measure which is not local.

Example 1.5. Let us consider the map D defined as follows:

$$D(A,B) = \begin{cases} 0, & \text{if } A = B, \\ 1, & \text{if } A \neq B. \end{cases}$$

Then D is a divergence measure, but is not local. More details are discussed in [7].

2. CONCEPT OF DIVERGENCE MEASURE BASED ON SCALAR CARDINALITY

The concept of fuzzy cardinality was discussed by Dan Ralescu in [11]. Wygralak [13] introduced an axiomatic theory of scalar cardinality of fuzzy sets. J. Casasnovas and J. Torrens have given an axiomatic approach to fuzzy cardinalities of finite fuzzy sets in [2]. Our approach to a divergence measure will be based on both of them.

We recall the following definition from [11].

Definition 2.1. Let X be a universal set, let $a \in [0,1]$ and $x \in X$. A fuzzy singleton is a finite fuzzy set over X (denoted by x_a) such that:

$$x_a(y) = \begin{cases} a, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

Definition 2.2. A mapping $|\cdot| : \mathcal{F}(X) \to [0,\infty[$ is called *a scalar cardinality* of fuzzy set if the following conditions are satisfied:

(1) Coincidence: for all $x \in X$; $|x_a| = 1$ if and only if $a = 1$;

(2) Monotonicity: for all $a, b \in [0,1], x, y \in X$; $a \leq b \Rightarrow |x_a| \leq |y_b|$;

(3) Additivity: for all $A, B \in \mathcal{F}(X)$; $Supp(A) \cap Supp(B) = \emptyset \Rightarrow |A \cup B| = |A| + |B|$.

The following proposition is crucial for our purposes. The proof can be found in [15].

Proposition 2.3. A mapping $|\cdot| : \mathcal{F}(X) \to [0,\infty[$ is a scalar cardinality if and only if for each $A \in \mathcal{F}(X)$:

$$|A| = \sum_{x \in Supp(A)} f\left(A(x)\right),$$

where $f : [0,1] \to [0,1]$ is a function for which the following conditions are fulfilled:

(a) Boundary conditions: $f(0) = 0, f(1) = 1$;

(b) Monotonicity: for all $a, b \in [0,1]$; $a \leq b \Rightarrow f(a) \leq f(b)$.

In case of crisp finite sets the definition of cardinality leads to the classical formulation "number of elements" (having the membership values equal to 1).

2.1. Distance-based divergence measure

Our aim is to extend the previous approach to measure a difference of two fuzzy subsets discussed in [9] and [10] with new one based on cardinality. Motivation to it comes from real life situations as far as we want to compare two real objects on the basis of required criteria. Divergence measure given by suitable distance is well-founded however in the same situation it does not give us a complete information.

Let $A, B \in \mathcal{F}(X)$. We assume the following membership degrees of elements $x, y \in X$: $A(x) = 0.47$, $B(x) = 0.57$, $A(y) = 0.52$, $B(y) = 0.62$. We have the following question: What is a standard distance between the values $A(x), B(x)$ and $A(y), B(y)$? The answer should be 0.10 in both cases. For some reasons we can say that the elements having the membership degrees less than 0.5 are not significant. The definition of scalar cardinality allows us to modify by means of a function f as follows: we put $f(A(x)) = 0$ if $A(x) < 0.5$. Now we see that the elements $x, y \in X$ become more different in this sense.

Next, the definition of a distance for fuzzy sets will be introduced.

Definition 2.4. A map $d : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ is a distance if and only if the function d satisfies the following conditions:

(1) for all $A, B \in \mathcal{F}(X); d(A, B) \geq 0$ and $d(A, B) = 0$ if and only if $A = B$;

(2) for all $A, B \in \mathcal{F}(X); d(A, B) = d(B, A)$;

(3) for all $A, B, C \in \mathcal{F}(X); d(A, C) \leq d(A, B) + d(B, C)$.

The divergences and distances are not related in general. One example of the distance, which is not a divergence measure, will be shown in the following Example 2.5. More general example can be found in [8].

Example 2.5. Consider the map $d : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ defined by

$$d(A, B) = \begin{cases} 0, & \text{if } A = B, \\ 1, & \text{if } A \neq B, \text{ but } A = X \text{ or } B = X, \\ 0.5, & \text{otherwise.} \end{cases}$$

Let us prove that d is a distance. Positivity, the identity of indiscernibles and symmetry trivially hold. It remains to show the triangular inequality property. Let $A, B, C \in \mathcal{F}(X)$.

- If $d(A, C) = 0$, the inequality trivially holds.
- If $d(A, C) = 0.5$, then $A \neq C$, and therefore either $B \neq A$ or $B \neq C$, and consequently $d(A, C) = 0.5 \leq d(A, B) + d(B, C)$.
- Finally, if $d(A, C) = 1$, we can assume, without loss of generality, that $A = X$. If $B = A$, then $d(A, B) = 0$ and $d(B, C) = 1$, and therefore $d(A, C) = 1 = d(A, B) + d(B, C)$. Otherwise, if $B \neq A$, then $d(A, B) = 1$, and therefore $d(A, C) = 1 \leq d(A, B) + d(B, C)$.

We conclude that $d(A,C) \leq d(A,B)+d(B,C)$ in all cases. Thus, the map d is a distance.

However, the map d is not a divergence measure. To see this, consider the universe $X = \{x_1, x_2\}$ and the fuzzy subsets A and B defined as:

$$A = \{(x_1, 1), (x_2, 0)\}\ ,\ B = \{(x_1, 0), (x_2, 1)\}\,.$$

It holds that $d(A,B) = 0.5$. If we consider $C = B$, then $A \cup C = A \cup B = X$, and therefore:

$$d(A \cup C, B \cup C) = d(X, B) = 1 > 0.5 = d(A, B).$$

Thus, the map d is not a divergence measure.

Remark 2.6. It is necessary to clarify the notion of a suitable distance. As we have seen in Example 2.5, the set of distances and divergences are not comparable in general since neither concept of these can imply the other one. The distances which are also the divergence measures, are very important for our work and only this class of distances in the further text will be considered. Both of the divergence measures introduced in Example 1.4 and Example 1.5 are also the distances, simultaneously.

Let $d : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ be a distance and $A, B \in \mathcal{F}(X)$. Then we will define pointwise for each $x \in X$ as follows:

$$d(A,B) = d_0(A(x), B(x)),$$

where d_0 defined on $[0,1] \times [0,1]$ is a restricted distance of the original one d.

Proposition 2.7. Let $A, B \in \mathcal{F}(X)$ and $|\cdot|$ denotes a scalar cardinality. Let the map $D : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ be defined in the following way:

$$D(A,B) = |\Phi\,(A,B)|\,,$$

where $\Phi : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$ is a function defined axiomatically as:

$$A(x) \times B(x) \xmapsto{\Phi} C(x),$$

in which $C(x) = d_0(A(x), B(x))$ and $d_0 : [0,1] \times [0,1] \to [0,1]$ is a suitable distance (see Remark 2.6).

Then the map D is a divergence measure between fuzzy sets A and B. We say that the divergence measure D is generated by distance d_0.

Proof. Let us check that D is a divergence measure. If two fuzzy subsets A, B are the same, then $\Phi(A(x), B(x)) = 0$ and $f(0) = 0$, and therefore the divergence between A and B is zero. A reverse implication is not requested. It is evident that D is also commutative, since:

$$D(A,B) = |\Phi(A,B)| = \sum_{x \in X} f\left(\Phi(A(x), B(x))\right) = \sum_{x \in X} f\left(\Phi(B(x), A(x))\right) = D(B,A).$$

If the fuzzy subsets A and B become more similar in sense of union or intersection, then the value of divergence will decrease and the inequalities $D(A \cup C, B \cup C) \leq D(A, B), D(A \cap C, B \cap C) \leq D(A, B)$ are fulfilled since we have used only special class of distances (see Remark 2.6) and monotonicity of f from Proposition 2.3. □

We have introduced the class of distance-based divergence measure. It is obvious that not all distances can be considered and a utility of the concept distance-based divergences must be restricted to the special class of distances.

It is natural to ask if the distance-based divergence have a local property or not. The result is obtained in the following proposition.

Proposition 2.8. Let D be a divergence measure which is generated by distance d_0. Then D has a local property.

Proof. The divergence D can be rewritten by Proposition 2.3 and Proposition 2.7 as follows:

$$D(A, B) = |\Phi(A, B)| = \sum_{x \in X} f\left(\Phi(A(x), B(x))\right) = \sum_{x \in X} f\left(d_0(A(x), B(x))\right).$$

We can see that the divergence measure D can be expressed pointwise as a sum of suitable distances d_0. Each such distance d_0 satisfies all conditions for the function h from the Definition 1.2 and properties of local divergence given in Theorem 1.3. To show it, the distance d_0 takes the value 0 if both coordinates are the same. The map d_0 is also symmetric. The third axiom of divergence is fulfilled since we have considered only restricted class of divergence-generating distances d_0. Applying the function f from Proposition 2.3 all properties of the function h (see Representation Theorem 1.3) are kept without any change, more formally, for each $x \in X$ the following conditions hold:

(1) $f(\Phi(A(x), A(x))) = f(0) = 0$,

(2) $f(\Phi(A(x), B(x))) = f(\Phi(B(x), A(x)))$,

(3) $f(\Phi((A \cup C)(x), (B \cup C)(x))) \leq f(\Phi(A(x), B(x)))$ and $f(\Phi((A \cap C)(x), (B \cap C)(x))) \leq f(\Phi(A(x), B(x)))$.

Then we can define $h(a, b) = f(\Phi(a, b))$ for all $a, b \in [0, 1]$. It allows us to write:

$$D(A, B) = |\Phi(A, B)| = \sum_{x \in X} h(A(x), B(x)).$$

Thus, the divergence D has a local property. □

Example 2.9. Let $X = \{x_1, x_2, x_3\}$ and $A, B \in \mathcal{F}(X)$. Consider the following membership degrees: $A(x_1) = 0.2$, $A(x_2) = 0.8$, $A(x_3) = 0.7$, $B(x_1) = 0.3$, $B(x_2) = 0.3$, $B(x_3) = 0.3$ and the functions $f(x) = x$; $d_0(x, y) = |x - y|$. Then:

$$D(A, B) = |\Phi(A, B)| = \sum_{x \in X} f\left(d_0(A(x), B(x))\right) = \sum_{x \in X} |A(x) - B(x)| = 0.1 + 0.5 + 0.4 = 1.0.$$

2.2. Rational divergence measure

We have shown some other ways based on scalar cardinality and distances to express divergence measure $D(A,B)$ between fuzzy subsets A and B in the previous paragraph, while we still do not have more information about the function Φ. We have restricted only to known examples of divergence-generating distances already. Motivated by [3] and [4] we propose a class of the rational divergence measure fulfilling the conditions (1) − (3) for the function $\Phi(|A|,|B|)$.

We recall that the binary fuzzy relation R defined on $X \times X$ is a T-equivalence if and only if it satisfies the following properties for each $x, y, z \in X$ and the t-norm T:

(a) reflexivity: $R(x,x) = 1$,

(b) symmetry: $R(x,y) = R(y,x)$,

(c) T-transitivity: $T(R(x,y), R(y,z)) \leq R(x,z)$.

Sometimes the reflexivity condition can be replaced by weaker one named local reflexivity: $R(x,x) \geq R(x,y)$ for fixed $x \in X$ and each $y \in X$.

In [3] the following class of rational similarity measures is proposed:

$$S(A,B) = \frac{a.\alpha_{A,B} + b.\omega_{A,B} + c.\delta_{A,B} + d.\nu_{A,B}}{a'.\alpha_{A,B} + b'.\omega_{A,B} + c'.\delta_{A,B} + d'.\nu_{A,B}},$$

where:

$$\begin{aligned} \alpha_{A,B} &= \min\{|A \setminus B|, |B \setminus A|\}, \\ \omega_{A,B} &= \max\{|A \setminus B|, |B \setminus A|\}, \\ \delta_{A,B} &= |A \cap B|, \\ \nu_{A,B} &= |(A \cup B)^c|, \\ \text{and } & a, b, c, d, a', b', c', d' \in \{0,1\}. \end{aligned}$$

The reflexive similarity measures can be identified by the condition $c' = c, d' = d$.

Some of them have some special properties. The similarity measure R is self-complementary if and only if $c = d$ and $c' = d'$. For each similarity measure we can create a complementary similarity measure (denoted by R^c) by changing the coefficients $c \leftrightarrow d$ and $c' \leftrightarrow d'$, analogously, where $R^c(A,B) = R(A^c, B^c)$. It is true that $R^c = R$ if and only if R is self-complementary.

In [3] an explicit expression of 19 rational similarity measures is provided, while 16 of them are reflexive. Only the 10 reflexive rational similarity measures are self-complementary, too.

Each similarity measure was reviewed and for each of them the validity of some important properties has been verified such as boundary conditions, monotonicity and T-transitivity:

(a) the first boundary condition: $B_1 : R(A, \emptyset) = \frac{|A^c|}{n}$,

(b) the second boundary condition: $B_2 : R(A, X) = \frac{|A|}{n}$,

(c) the third boundary condition: $B_3 : R(A, A^c) = 0$,

(d) the monotonicity condition: $C_\theta : R(A \cup B, A \cap B)\ \theta\ R(A, B)$,

(e) the transitivity condition: $T(R(A, B), R(B, C)) \leq R(A, C)$ with the t-norm T as strong as possible.

Altogether 28 similarity measures were verified and 13 of them are T-transitive for at least one t-norm $T \in \{T_D, T_L, T_P, T_M\}$ (including also nonreflexive and complementary similarity measures) according to [3]. For all 9 reflexive T-transitive similarity measures were selected which are also T-equivalences. The reflexive T-transitive measures discussed in [1] and [3] are suitable candidates for fuzzification.

Since the denominator of the following rational measures can not be equal zero, some restrictions must be considered. In the following Example 2.10 we will consider measures only between two fuzzy sets A and B for which the conditions:

(i) $A \neq \emptyset$ or $B \neq \emptyset$ for the measures R_1, R_3,

(ii) $A \neq \emptyset$ and $B \neq \emptyset$ for the measure R_2,

are fulfilled.

Example 2.10. We show four examples of the reflexive rational similarity measures:

(a) $R_1(A, B) = \frac{|A \cap B|}{\max\{|A|, |B|\}}$.

The similarity measure R_1 fulfills the boundary conditions B_2, B_3, the monotonicity condition $C_{\leq}$ and it is transitive for the Łukasiewicz t-norm T_L (and for any weaker one).

(b) $R_2(A, B) = \frac{|A \cap B|}{\min\{|A|, |B|\}}$.

The similarity measure R_2 fulfills only the boundary condition B_3, the monotonicity condition $C_{\geq}$ and it is not transitive for any t-norm T.

(c) $R_3(A, B) = \frac{|A \cap B|}{|A \cup B|}$.

The similarity measure R_3 (Jaccard coefficient) fulfills the boundary conditions B_2, B_3, the monotonicity condition $C_=$ and it is transitive for the Łukasiewicz t-norm T_L.

(d) $R_4(A, B) = \frac{|(A \Delta B)^c|}{n}$.

The similarity measure R_4 (Simple matching coefficient) fulfills all the boundary conditions B_1, B_2, B_3, the monotonicity condition $C_=$ and it is transitive for the Łukasiewicz t-norm T_L.

The measure R_2 is not a T-equivalence.

We have studied all 9 rational similarity measures which are also T-equivalences. All of them have been changed so that the conditions (1) – (3) for the divergence measure are fulfilled. In the following text we introduce 9 rational measures $D^*(A,B)$ derived from T-equivalences, which will be fuzzified in the second step.

The explicit expression of this measures based on cardinalities are the following:

(1) $D_1^*(A,B) = \frac{\max\{|A\backslash B|,|B\backslash A|\}}{|A\cap B|}$, where $A\cap B \neq \emptyset$,

(2) $D_2^*(A,B) = \frac{\max\{|A\backslash B|,|B\backslash A|\}}{|(A\Delta B)^c|}$, where $(A\Delta B)^c \neq \emptyset$,

(3) $D_3^*(A,B) = \frac{|A\Delta B|}{|A\cap B|}$, where $A\cap B \neq \emptyset$,

(4) $D_4^*(A,B) = \frac{|A\Delta B|}{|(A\Delta B)^c|}$, where $(A\Delta B)^c \neq \emptyset$,

(5) $D_5^*(A,B) = \frac{\max\{|A|,|B|\}-\min\{|A|,|B|\}}{\min\{|A|,|B|\}}$, where $A\neq\emptyset$ and $B\neq\emptyset$,

(6) $D_6^*(A,B) = \frac{\max\{|A\backslash B|,|B\backslash A|\}}{\min\{|A\backslash B|,|B\backslash A|\}}$, where $A\setminus B\neq\emptyset$ and $B\setminus A\neq\emptyset$,

(7) $D_7^*(A,B) = \frac{\max\{|A\backslash B|,|B\backslash A|\}}{\min\{|A|,|B|\}}$, where $A\neq\emptyset$ and $B\neq\emptyset$,

(8) $D_8^*(A,B) = \frac{\max\{|A\backslash B|,|B\backslash A|\}}{\min\{|(A\backslash B)^c|,|(B\backslash A)^c|\}}$, where $A\setminus B\neq\emptyset$ and $B\setminus A\neq\emptyset$,

(9) $D_9^*(A,B) = 0$.

The introduced rational measures based on the cardinalities quantify the difference between the sets only in the crisp case. There is a question how the measures could be fuzzified. Some of the ideas are outlined in [1]. For our purposes we do it in the following way.

Moreover, only the parametric family of Frank t-norms with parameter p, for which $0\leq p\leq +\infty$, will be considered:

$$T_p^F(a,b) = \begin{cases} T_M(a,b) = \min\{a,b\}, & \text{if } p=0,\\ T_P(a,b) = a.b, & \text{if } p=1,\\ T_L(a,b) = \max\{a+b-1,0\}, & \text{if } p=+\infty,\\ \log_p\left(1+\frac{(p^a-1).(p^b-1)}{p-1}\right), & \text{otherwise.}\end{cases}$$

Let $A,B\in\mathcal{F}(X)$, let T be a t-norm from the family of Frank t-norms and S be a t-conorm such that T,S are dual. The complement of the set A we denote by A^c. For the membership values $A(x_i),B(x_i),C(x_i)$ we will write simply a_i,b_i,c_i, respectively, where $i\in\{1,\dots,n\}$ and $|X|=n$. Then we define the following:

$$|A\cap B| = \sum_{i=1}^{n} T(a_i,b_i),$$

$$|A\cup B| = \sum_{i=1}^{n} S(a_i,b_i) = \sum_{i=1}^{n}\left(a_i+b_i-T(a_i,b_i)\right),$$

$$|A \setminus B| = |A| - |A \cap B| = \sum_{i=1}^{n} (a_i - T(a_i, b_i)),$$

$$|B \setminus A| = |B| - |A \cap B| = \sum_{i=1}^{n} (b_i - T(a_i, b_i)),$$

$$|A \Delta B| = |A \cup B| - |A \cap B| = |A \setminus B| + |B \setminus A| = \sum_{i=1}^{n} (a_i + b_i - 2T(a_i, b_i)),$$

$$|A^c| = |X| - |A| = n - \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (1 - a_i).$$

In the following proposition we give an explicit expression of the divergence measures $D_1 - D_9$, which are adapted from the previous measures $D_1^* - D_9^*$ and also applicable in a fuzzy case. The divergence D_9 is quite trivial, however, we leave it on the list.

Proposition 2.11. Let (X, T, S) be a triple such that $|X| = n$, $T = T_M$ and $S = S_M$. Then the maps $D_i : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ for $i \in \{1, \dots, 9\}$ are divergence measures.

Moreover, some special cases assigning zero in the denominators of $D_1 - D_8$ for some triple (X, T, S) depending on the parameter p from Frank's family of t-norms T_p^F must be excluded, the complete list of these conditions is shown in Tables 1−3.

(D1)

$$D_1(A, B) = \frac{\max\left\{\sum_{i=1}^{n} (a_i - T(a_i, b_i)), \sum_{i=1}^{n} (b_i - T(a_i, b_i))\right\}}{\sum_{i=1}^{n} T(a_i, b_i)},$$

(D2)

$$D_2(A, B) = \frac{\max\left\{\sum_{i=1}^{n} (a_i - T(a_i, b_i)), \sum_{i=1}^{n} (b_i - T(a_i, b_i))\right\}}{\sum_{i=1}^{n} (1 - a_i - b_i + 2T(a_i, b_i))},$$

(D3)

$$D_3(A, B) = \frac{\sum_{i=1}^{n} (a_i + b_i - 2T(a_i, b_i))}{\sum_{i=1}^{n} T(a_i, b_i)},$$

(D4)

$$D_4(A, B) = \frac{\sum_{i=1}^{n} (a_i + b_i - 2T(a_i, b_i))}{\sum_{i=1}^{n} (1 - a_i - b_i + 2T(a_i, b_i))},$$

(D5)

$$D_5(A, B) = \frac{\left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\right|}{\min\left\{\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i\right\}},$$

(D6)

$$D_6(A, B) = \frac{\max\left\{\sum_{i=1}^{n} (a_i - T(a_i, b_i)), \sum_{i=1}^{n} (b_i - T(a_i, b_i))\right\}}{\min\left\{\sum_{i=1}^{n} (a_i - T(a_i, b_i)), \sum_{i=1}^{n} (b_i - T(a_i, b_i))\right\}},$$

(D7)

$$D_7(A,B) = \frac{\max\left\{\sum_{i=1}^n (a_i - T(a_i,b_i)), \sum_{i=1}^n (b_i - T(a_i,b_i))\right\}}{\min\left\{\sum_{i=1}^n a_i, \sum_{i=1}^n b_i\right\}},$$

(D8)

$$D_8(A,B) = \frac{\max\left\{\sum_{i=1}^n (a_i - T(a_i,b_i)), \sum_{i=1}^n (b_i - T(a_i,b_i))\right\}}{\min\left\{\sum_{i=1}^n (1 - a_i + T(a_i,b_i)), \sum_{i=1}^n (1 - b_i + T(a_i,b_i))\right\}},$$

(D9)

$$D_9(A,B) = 0.$$

	$p = 0$
$D_1(A,B)$	$(\forall i) \min\{a_i, b_i\} = 0$
$D_2(A,B)$	$(\forall i)(\min\{a_i, b_i\} = 0 \textit{ and } \max\{a_i, b_i\} = 1)$
$D_3(A,B)$	$(\forall i) \min\{a_i, b_i\} = 0$
$D_4(A,B)$	$(\forall i)(\min\{a_i, b_i\} = 0 \textit{ and } \max\{a_i, b_i\} = 1)$
$D_5(A,B)$	$A = \emptyset \textit{ or } B = \emptyset$
$D_6(A,B)$	$A \subseteq B \textit{ or } B \subseteq A$
$D_7(A,B)$	$A = \emptyset \textit{ or } B = \emptyset$
$D_8(A,B)$	$(A = X \textit{ and } B = \emptyset) \textit{ or } (A = \emptyset \textit{ and } B = X)$
$D_9(A,B)$	$-$

Tab. 1. Restriction conditions for rational divergence measures for $p = 0$ to be excluded.

	$0 < p < \infty$
$D_1(A,B)$	$(\forall i) \min\{a_i, b_i\} = 0$
$D_2(A,B)$	$(\forall i)(\min\{a_i, b_i\} = 0 \textit{ and } \max\{a_i, b_i\} = 1)$
$D_3(A,B)$	$(\forall i) \min\{a_i, b_i\} = 0$
$D_4(A,B)$	$(\forall i)(\min\{a_i, b_i\} = 0 \textit{ and } \max\{a_i, b_i\} = 1)$
$D_5(A,B)$	$A = \emptyset \textit{ or } B = \emptyset$
$D_6(A,B)$	$(\forall i)(a_i > 0 \Rightarrow b_i = 1) \textit{ or } (\forall i)(b_i > 0 \Rightarrow a_i = 1)$
$D_7(A,B)$	$A = \emptyset \textit{ or } B = \emptyset$
$D_8(A,B)$	$(A = X \textit{ and } B = \emptyset) \textit{ or } (A = \emptyset \textit{ and } B = X)$
$D_9(A,B)$	$-$

Tab. 2. Restriction conditions for rational divergence measures for $0 < p < \infty$ to be excluded.

	$p = \infty$
$D_1(A,B)$	$(\forall i)\ a_i + b_i \leq 1$
$D_2(A,B)$	$(\forall i)\ a_i + b_i = 1$
$D_3(A,B)$	$(\forall i)\ a_i + b_i \leq 1$
$D_4(A,B)$	$(\forall i)\ a_i + b_i = 1$
$D_5(A,B)$	$A = \emptyset\ or\ B = \emptyset$
$D_6(A,B)$	$(\forall i)(a_i > 0 \Rightarrow b_i = 1)\ or\ (\forall i)(b_i > 0 \Rightarrow a_i = 1)$
$D_7(A,B)$	$A = \emptyset\ or\ B = \emptyset$
$D_8(A,B)$	$(A = X\ and\ B = \emptyset)\ or\ (A = \emptyset\ and\ B = X)$
$D_9(A,B)$	$-$

Tab. 3. Restriction conditions for rational divergence measures for $p = \infty$ to be excluded.

P r o o f . In the first step, we check conditions for which the measures D_i having zero in the denominator, and therefore these fuzzy sets A, B must be excluded. For example, take D_1.

$$\sum_{i=1}^{n} T_p^F(a_i, b_i) = 0 \Leftrightarrow T_p^F(a_i, b_i) = 0 \ for\ all\ i \in \{1, \ldots, n\}\,.$$

- for $p = 0$: $T_p^F = T_M$ and $T_p^F(a_i, b_i) = 0 \Leftrightarrow \min\{a_i, b_i\} = 0$,
- for $p = 1$: $T_p^F = T_P$ and $T_p^F(a_i, b_i) = 0 \Leftrightarrow a_i \cdot b_i = 0 \Leftrightarrow \min\{a_i, b_i\} = 0$,
- for $0 < p < \infty$, $p \neq 1$: $T_p^F(a_i, b_i) = 0 \Leftrightarrow \log_p\left(1 + \frac{(p^{a_i}-1).(p^{b_i}-1)}{p-1}\right) = 0 \Leftrightarrow (p^{a_i} - 1).(p^{b_i} - 1) = 0 \Leftrightarrow p^{a_i} = 1$ or $p^{b_i} = 1 \Leftrightarrow a_i = 0$ or $b_i = 0 \Leftrightarrow \min\{a_i, b_i\} = 0$,
- for $p = \infty$: $T_p^F = T_L$ and $T_p^F(a_i, b_i) = 0 \Leftrightarrow \max\{a_i + b_i - 1, 0\} = 0 \Leftrightarrow a_i + b_i \leq 1$.

The other cases can be done similarly, the results are scheduled in Tables $1-3$.

In the second step, we must verify that the maps $D_1 - D_9$ are really divergences for a triple (X, T, S), where $T = T_M$ and $S = S_M$. It is evident that $D_i(A,B) = 0$ if and only if $A = B$ and $D_i(A,B) = D_i(B,A)$ for all $i \in \{1, \ldots, 9\}$. It remains to show the third condition from Definition 1.1. We will do it for the maps D_1 and D_5, the other can be proven in similar way. Obviously, the map D_9 is a divergence. Moreover, it is named as a minimum divergence measure.

To show that the map D_1 is a divergence we can divide it into 6 cases.

(i) If $a_i \leq b_i \leq c_i$, then $T(a_i, c_i) - T(T(a_i, c_i), T(b_i, c_i)) = a_i - T(a_i, b_i)$ and $T(T(a_i, c_i), T(b_i, c_i)) = T(a_i, b_i)$.
Therefore $\frac{T(a_i,c_i) - T(T(a_i,c_i),T(b_i,c_i))}{T(T(a_i,c_i),T(b_i,c_i))} = \frac{a_i - T(a_i,b_i)}{T(a_i,b_i)}$.

(ii) If $a_i \leq c_i < b_i$, then $T(a_i, c_i) - T(T(a_i, c_i), T(b_i, c_i)) = a_i - T(a_i, c_i) = a_i - a_i = 0 = a_i - T(a_i, b_i)$ and $T(T(a_i, c_i), T(b_i, c_i)) = T(a_i, c_i) = a_i = T(a_i, b_i)$.
Therefore $\frac{T(a_i,c_i)-T(T(a_i,c_i),T(b_i,c_i))}{T(T(a_i,c_i),T(b_i,c_i))} = \frac{a_i-T(a_i,b_i)}{T(a_i,b_i)}$.

(iii) If $c_i < a_i \leq b_i$, then $T(a_i, c_i) - T(T(a_i, c_i), T(b_i, c_i)) = c_i - T(c_i, c_i) = c_i - c_i = 0 = a_i - a_i = a_i - T(a_i, b_i)$ and $T(T(a_i, c_i), T(b_i, c_i)) = T(c_i, c_i) = c_i \leq a_i = T(a_i, b_i)$.
Therefore $\frac{T(a_i,c_i)-T(T(a_i,c_i),T(b_i,c_i))}{T(T(a_i,c_i),T(b_i,c_i))} = 0 = \frac{a_i-T(a_i,b_i)}{T(a_i,b_i)}$.

(iv) If $b_i < a_i \leq c_i$, then $T(a_i, c_i) - T(T(a_i, c_i), T(b_i, c_i)) = a_i - T(a_i, b_i)$ and $T(T(a_i, c_i), T(b_i, c_i)) = T(a_i, b_i)$.
Therefore $\frac{T(a_i,c_i)-T(T(a_i,c_i),T(b_i,c_i))}{T(T(a_i,c_i),T(b_i,c_i))} = \frac{a_i-T(a_i,b_i)}{T(a_i,b_i)}$.

(v) If $b_i \leq c_i < a_i$, then $T(a_i, c_i) - T(T(a_i, c_i), T(b_i, c_i)) = c_i - T(c_i, b_i) = c_i - b_i \leq a_i - b_i = a_i - T(a_i, b_i)$ and $T(T(a_i, c_i), T(b_i, c_i)) = T(c_i, b_i) = b_i = T(a_i, b_i)$.
Therefore $\frac{T(a_i,c_i)-T(T(a_i,c_i),T(b_i,c_i))}{T(T(a_i,c_i),T(b_i,c_i))} \leq \frac{a_i-T(a_i,b_i)}{T(a_i,b_i)}$.

(vi) If $c_i < b_i < a_i$, then $T(a_i, c_i) - T(T(a_i, c_i), T(b_i, c_i)) = c_i - T(c_i, c_i) = c_i - c_i = 0 = a_i - a_i = a_i - T(a_i, b_i)$ and $T(T(a_i, c_i), T(b_i, c_i)) = T(c_i, c_i) = c_i \leq b_i = T(a_i, b_i)$.
Therefore $\frac{T(a_i,c_i)-T(T(a_i,c_i),T(b_i,c_i))}{T(T(a_i,c_i),T(b_i,c_i))} = 0 = \frac{a_i-T(a_i,b_i)}{T(a_i,b_i)}$.

We have shown the inequality

$$\frac{T(a_i, c_i) - T(T(a_i, c_i), T(b_i, c_i))}{T(T(a_i, c_i), T(b_i, c_i))} \leq \frac{a_i - T(a_i, b_i)}{T(a_i, b_i)}$$

in all six cases. Similarly can be proved the inequality

$$\frac{T(b_i, c_i) - T(T(a_i, c_i), T(b_i, c_i))}{T(T(a_i, c_i), T(b_i, c_i))} \leq \frac{b_i - T(a_i, b_i)}{T(a_i, b_i)}$$

and hence the following relationship is fulfilled:

$$\frac{\max\{T(a_i, c_i) - T(T(a_i, c_i), T(b_i, c_i)), T(b_i, c_i) - T(T(a_i, c_i), T(b_i, c_i))\}}{T(T(a_i, c_i), T(b_i, c_i))} \leq \frac{\max\{a_i - T(a_i, b_i), b_i - T(a_i, b_i)\}}{T(a_i, b_i)}.$$

Applying the previous result to all n elements of the universal set X we obtain:

$$\frac{\max\{\sum_{i=1}^{n}(T(a_i, c_i) - T(T(a_i, c_i), T(b_i, c_i))), \sum_{i=1}^{n}(T(b_i, c_i) - T(T(a_i, c_i), T(b_i, c_i)))\}}{\sum_{i=1}^{n} T(T(a_i, c_i), T(b_i, c_i))} \leq \frac{\max\{\sum_{i=1}^{n}(a_i - T(a_i, b_i)), \sum_{i=1}^{n}(b_i - T(a_i, b_i))\}}{\sum_{i=1}^{n} T(a_i, b_i)}.$$

We have proven that $D_1(A \cap C, B \cap C) \leq D_1(A, B)$. Similarly can be proved the inequality $D_1(A \cup C, B \cup C) \leq D_1(A, B)$. We conclude that the map D_1 is a divergence measure.

Now we are going to prove that the map D_5 is a divergence measure. Without loss of generality we can assume $b_i \leq a_i$. Three following cases must be considered: $b_i \leq a_i < c_i$ or $b_i \leq c_i \leq a_i$ or $c_i < b_i \leq a_i$. In all cases we have $|T(a_i, c_i) - T(b_i, c_i)| \leq |a_i - b_i|$ by Example 1.4. In detail, we have:

- if $b_i \leq a_i < c_i$, then $|T(a_i, c_i) - T(b_i, c_i)| = |a_i - b_i|$ and $\min\{T(a_i, c_i), T(b_i, c_i)\} = \min\{a_i, b_i\}$. Therefore $\frac{|T(a_i,c_i)-T(b_i,c_i)|}{\min\{T(a_i,c_i),T(b_i,c_i)\}} = \frac{|a_i-b_i|}{\min\{a_i,b_i\}}$.
- if $b_i \leq c_i \leq a_i$, then $|T(a_i, c_i) - T(b_i, c_i)| = |c_i - b_i| \leq |a_i - b_i|$ and $\min\{T(a_i, c_i), T(b_i, c_i)\} = \min\{c_i, b_i\} = b_i = \min\{a_i, b_i\}$. Therefore $\frac{|T(a_i,c_i)-T(b_i,c_i)|}{\min\{T(a_i,c_i),T(b_i,c_i)\}} \leq \frac{|a_i-b_i|}{\min\{a_i,b_i\}}$.
- if $c_i < b_i \leq a_i$, then $|T(a_i, c_i) - T(b_i, c_i)| = |c_i - c_i| = 0 \leq |a_i - b_i|$ and $\min\{T(a_i, c_i), T(b_i, c_i)\} = \min\{c_i, c_i\} = c_i \leq b_i = \min\{a_i, b_i\}$. Therefore $\frac{|T(a_i,c_i)-T(b_i,c_i)|}{\min\{T(a_i,c_i),T(b_i,c_i)\}} = 0 \leq \frac{|a_i-b_i|}{\min\{a_i,b_i\}}$.

We have shown the inequality

$$\frac{|T(a_i, c_i) - T(b_i, c_i)|}{\min\{T(a_i, c_i), T(b_i, c_i)\}} \leq \frac{|a_i - b_i|}{\min\{a_i, b_i\}}$$

in all three cases.

Applying the previous result to all n elements of the universal set X we obtain:

$$\frac{|\sum_{i=1}^n T(a_i, c_i) - \sum_{i=1}^n T(b_i, c_i)|}{\min\{\sum_{i=1}^n T(a_i, c_i), \sum_{i=1}^n T(b_i, c_i)\}} \leq \frac{|\sum_{i=1}^n a_i - \sum_{i=1}^n b_i|}{\min\{\sum_{i=1}^n a_i, \sum_{i=1}^n b_i\}}.$$

We have proven that $D_5(A \cap C, B \cap C) \leq D_5(A, B)$. Similarly can be proved the inequality $D_5(A \cup C, B \cup C) \leq D_5(A, B)$. We conclude that the map D_5 is a divergence measure. □

For the case $A = B = \emptyset$ we will define axiomatically $D(A, B) = 0$.

Let us study some important properties of these divergences focused on the monotonicity and local property.

Proposition 2.12. For the divergences $D_1 - D_9$ the following relations are fulfilled for each t-norm $T \in \langle T_L, T_M \rangle$:

(i) $D_8 \leq D_2 \leq D_4 \leq D_3$ and $D_2 \leq D_1 \leq D_3$, but the divergences D_1 and D_4 are not comparable,

(ii) $D_7 \leq D_6$,

(iii) D_5 is not comparable with any other D_i,

(iv) $D_9 \leq D_i$ for all $i \in \{1, \ldots, 8\}$.

Proof.

(i) $T(a_i, b_i) \geq a_i + b_i - 1$ for a Łukasiewicz t-norm T_L, while $T_L(a_i, b_i) = 0$ if $a_i + b_i - 1 < 0$ and $T_L(a_i, b_i) = a_i + b_i - 1$ in other case. Since $T_L(a_i, b_i) \geq a_i + b_i - 1$ we have $T(a_i, b_i) \geq a_i + b_i - 1$ for each $T \in \langle T_L, T_M \rangle$. It is equivalent to $1 - a_i - b_i + 2T(a_i, b_i) \geq T(a_i, b_i)$. If we apply it to all elements of the universal set X, we get $\sum_{i=1}^{n} (1 - a_i - b_i + 2T(a_i, b_i)) \geq \sum_{i=1}^{n} T(a_i, b_i)$. Since the denominator of the divergence D_2 increases and the nominator we keep without change, we get $D_2 \leq D_1$. Similarly, we can show $D_4 \leq D_3$.

Since $T(a_i, b_i) \leq a_i$ and $T(a_i, b_i) \leq b_i$, adding $+b_i - 2T(a_i, b_i)$ to the first inequality and $+a_i - 2T(a_i, b_i)$ to the second inequality it follows that
$\max\{a_i - T(a_i, b_i), b_i - T(a_i, b_i)\} \leq a_i + b_i - 2T(a_i, b_i)$. It shows that $D_1 \leq D_3$ and $D_2 \leq D_4$.

Finally, from $a_i \geq T(a_i, b_i)$ and $b_i \geq T(a_i, b_i)$ adding $1 - a_i - b_i + T(a_i, b_i)$ to both inequalities we get $\min\{1 - a_i + T(a_i, b_i), 1 - b_i + T(a_i, b_i)\} \geq 1 - a_i - b_i + 2T(a_i, b_i)$ and then $D_8 \leq D_2$.

Now we show that the divergences D_1 and D_4 are not comparable. We give one counterexample. Consider the fuzzy sets given as follows:
$A_1 = 0.8/x + 1/y + 0.9/z; A_2 = 0.2/x + 0.7/y + 1/z;$
$B_1 = 0.5/x + 0.6/y + 0.4/z; B_2 = 0.9/x + 0/y + 0.6/z;$

We compute the divergences D_1 and D_4 for the minimum t-norm T_M:
$D_1(A_1, B_1) = \frac{1.2}{1.5} = 0.8$ and $D_4(A_1, B_1) = \frac{1.2}{1.8} = 0.67$, therefore $D_1 \geq D_4$ for the fuzzy sets A_1, B_1.
$D_1(A_2, B_2) = \frac{1.1}{0.8} = 1.38$ and $D_4(A_2, B_2) = \frac{1.8}{1.2} = 1.5$, therefore $D_1 \leq D_4$ for the fuzzy sets A_2, B_2.

(ii) $D_7 \leq D_6$ follows directly from the inequalities $a_i \geq a_i - T(a_i, b_i)$ and $b_i \geq b_i - T(a_i, b_i)$.

(iii) Consider again the fuzzy sets given in part (i) and for example, computing the divergences D_4 and D_5 for the minimum t-norm T_M we obtain:
$D_4(A_1, B_1) = 0.67$ by (i) and $D_5(A_1, B_1) = \frac{1.2}{1.5} = 0.8$, therefore $D_4 \leq D_5$ for the fuzzy sets A_1, B_1.
$D_4(A_2, B_2) = 1.5$ by (i) and $D_5(A_2, B_2) = \frac{0.4}{1.5} = 0.27$, therefore $D_4 \geq D_5$ for the fuzzy sets A_2, B_2.
We can see that the divergences D_4, D_5 are not comparable.

(iv) This holds trivially.

□

The divergences D_3, D_4 have a local property, since both of them can be expressed as a sum $\sum_{i=1}^{n} h(a_i, b_i)$, where the function $h : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ fulfills the conditions (1) – (3) from the definition of local divergence. The first and second are trivial, the third follows from the monotonicity of the t-norm T. The other divergences need not be local in general.

3. CONCEPT OF DIVERGENCE MEASURE BASED ON FUZZY CARDINALITY

In the last part we define the divergence in case of the concept of fuzzy cardinality according to [2]. We will do it through the α-cuts of the fuzzy cardinality $\mathbb{C}(A)$ represented as a fuzzy number. We recall that a fuzzy number is an arbitrary fuzzy subset $A \in \mathcal{F}(\mathbb{R})$ with universal set of real numbers $\mathbb{R}$, for which the following properties are fulfilled:

(i) A is normalized, i.e. there exists an element $x \in \mathbb{R}$ such that $A(x) = 1$,

(ii) the α-cuts $A^{(\alpha)} = \{x \in \mathbb{R}; A(x) \geq \alpha\}$ are closed intervals for each $\alpha \in (0, 1]$,

(iii) the support $Supp(A) = \{x \in \mathbb{R}; A(x) > 0\}$ is a bounded subset of the universe $\mathbb{R}$.

We assume that a fuzzy number is convex, see also [12]. The convexity condition may be written as

$$(\alpha_1 < \alpha_2) \Rightarrow \left(A^{(\alpha_2)} \subset A^{(\alpha_1)}\right).$$

Next we introduce an axiomatic definition of a fuzzy cardinality according to [2]. Let CFN denote the set of convex fuzzy numbers and let X be the universal set.

Definition 3.1. A function $\mathbb{C} : \mathcal{F}(X) \to CFN$ is a *fuzzy cardinality* if and only if it satisfies the following conditions:

(1) Additivity: $Supp(A) \cap Supp(B) = \emptyset \Rightarrow \mathbb{C}(A \cup B) = \mathbb{C}(A) \oplus \mathbb{C}(B)$;

(2) Variability: $A, B \in \mathcal{F}(X); i > |Supp(A)|, j > |Supp(B)| \Rightarrow \mathbb{C}(A)(i) = \mathbb{C}(B)(j)$;

(3) Consistency: $A \subset X \Rightarrow \forall i \in \mathbb{N}; \mathbb{C}(A)(i) \in \{0, 1\}$ and $\mathbb{C}(A)(n) = 1$ where $n = Supp(A)$;

(4) Monotonicity: If $x, y \in X, a, b \in [0, 1], a \leq b$ then: $\mathbb{C}(x_a)(0) \geq \mathbb{C}(y_b)(0)$ and $\mathbb{C}(x_a)(1) \leq \mathbb{C}(y_b)(1)$.

The additivity property seems quite natural and it is only an extension of the additivity of the scalar cardinality. If an element $x \in X$ does not belong to the support of A, i.e. $A(x) = 0$, then the element x does not affect the cardinality of A. It is expressed by variability property. Cardinality of the crisp sets must take only crisp values 0 or 1 as required by the consistency property. The last property requires that the cardinality of singletons at 0 must take a value which is decreasing and in case of singletons at 1 is increasing.

The appropriate α-cuts of $\mathbb{C}(A)$ will be denoted by $\mathbb{C}(A)^\alpha = \left\langle \mathbb{C}(A)_L^{(\alpha)}, \mathbb{C}(A)_R^{(\alpha)} \right\rangle$, where $\mathbb{C}(A)$ is a fuzzy cardinality of the fuzzy set A and $\mathbb{C}(A)_L^{(\alpha)}, \mathbb{C}(A)_R^{(\alpha)}$ are the left (right) boundaries of the respective α-cuts.

Proposition 3.2. Let $A, B \in \mathcal{F}(X)$ and $\mathbb{C}(A), \mathbb{C}(B)$ be the fuzzy cardinalities of the fuzzy sets A, B. Then the map $D : \mathcal{F}(X) \times \mathcal{F}(X) \to \mathbb{R}$ defined as

$$D(A, B) = \max \left\{ \sup_{\alpha \in [0,1]} \left| \mathbb{C}(A)_L^{(\alpha)} - \mathbb{C}(B)_L^{(\alpha)} \right| ; \sup_{\alpha \in [0,1]} \left| \mathbb{C}(A)_R^{(\alpha)} - \mathbb{C}(B)_R^{(\alpha)} \right| \right\}.$$

is a divergence measure between fuzzy sets A and B.

P r o o f. We must verify the axioms of divergence measure given in [9]. Obviously, if two fuzzy sets A, B coincide, then also their α-cuts coincide and hence $D(A, B) = \max\{0, 0\} = 0$. Even in this case also the reverse implication is satisfied, i.e. if for $A, B \in \mathcal{F}(X)$ is $D(A, B) = 0$, then we get $A = B$ although the Definition 1.1 does not require it. Next, the function D is commutative, too. Now let us check the last property.

Let $A, B, C \in \mathcal{F}(X)$ and $\mathbb{C}(A), \mathbb{C}(B), \mathbb{C}(C)$ denote their fuzzy cardinalities. From the fuzzy set theory described in [6] we get the following relations: $\left[\mathbb{C}(A)^{(\alpha)} \cap \mathbb{C}(C)^{(\alpha)}\right] \subset \mathbb{C}(A)^{(\alpha)}$ and $\left[\mathbb{C}(B)^{(\alpha)} \cap \mathbb{C}(C)^{(\alpha)}\right] \subset \mathbb{C}(B)^{(\alpha)}$.

Since $\mathbb{C}(A), \mathbb{C}(B), \mathbb{C}(C)$ are convex sets we get the inequalities for all $\alpha \in [0, 1]$:
$|\mathbb{C}(A \cap C)_L^{(\alpha)} - \mathbb{C}(B \cap C)_L^{(\alpha)}| \le |\mathbb{C}(A)_L^{(\alpha)} - \mathbb{C}(B)_L^{(\alpha)}|$ and $|\mathbb{C}(A \cap C)_R^{(\alpha)} - \mathbb{C}(B \cap C)_R^{(\alpha)}| \le |\mathbb{C}(A)_R^{(\alpha)} - \mathbb{C}(B)_R^{(\alpha)}|$.
Hence $D(A \cap C, B \cap C) \le D(A, B)$, as required. □

4. CONCLUDING REMARKS

We give an alternative approach how can the difference of two fuzzy subsets be measured. In our considerations we have restricted ourselves to the case when the universe X is finite. In case of an infinite universe X we cannot use the concept of cardinality as above. It would be appropriate to find another approach to define a divergence measure that could be used also in case of an infinite universe X.

A complete characterization of the set of all suitable functions Φ is still an open problem. We have shown that the local property of the divergence based on the cardinality need not be fulfilled. Therefore our aim is to reduce the set of the functions Φ such that for each element of them a local divergence can be defined. In future work we will specify a class of some interesting properties which could be fulfilled by divergence D.

We have considered the value of divergence as a real number. A question how should this concept be extended so that the values of divergence could be represented as fuzzy numbers may be a quite interesting problem, but requiring a new approach.

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REFERENCES

[1] S. Ashraf and T. Rashid: Fuzzy similarity measures. LAP LAMBERT Academic Publishing, 2010.

[2] J. Casasnovas and J. Torrens: An axiomatic approach to fuzzy cardinalities of finite fuzzy sets. Fuzzy Sets and Systems *133* (2003), 193–209. DOI:10.1016/s0165-0114(02)00345-7

[3] B. De Baets, H. De Meyer, and H. Naessens: A class of rational cardinality-based similarity measures. J. Comput. Appl. Math. *132* (2001), 51–69. DOI:10.1016/s0377-0427(00)00596-3

[4] B. De Baets, S. Janssens, and H. De Meyer: On the transitivity of a parametric family of cardinality-based similarity measures. In. J. Approx. Reasoning *50* (2009), 104–116. DOI:10.1016/j.ijar.2008.03.006

[5] G. Deschrijver and P. Král': On the cardinalities of interval-valued fuzzy sets. Fuzzy Sets and Systems *158* (2007), 1728–1750. DOI:10.1016/j.fss.2007.01.005

[6] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, London 2000. DOI:10.1007/978-94-015-9540-7

[7] V. Kobza, V. Janiš, and S. Montes: Generalizated local divergence measures between fuzzy subsets. J. Intelligent and Fuzzy Systems (2017), accepted, in press.

[8] I. Montes: Comparison of Alternatives under Uncertainty and Imprecision. PhD Thesis, University of Oviedo 2013.

[9] S. Montes, I. Couso, P. Gil, and C. Bertoluzza: Divergence measure between fuzzy sets. Int. J. Approx. Reasoning *30* (2002), 91–105. DOI:10.1016/s0888-613x(02)00063-4

[10] S. Montes and P. Gil: Some classes of divergence measures between fuzzy subsets and between fuzzy partitions. Mathware and Soft Computing *5* (1998), 253–265.

[11] D. Ralescu: Cardinality, quantifiers and the aggregation of fuzzy criteria. Fuzzy Sets and Systems *69* (1995), 355–365. DOI:10.1016/0165-0114(94)00177-9

[12] G. Shang, Z. Zhang, and C. Cao: Multiplication operation on fuzzy numbers. J. Software *4* (2009), 331–338. DOI:10.4304/jsw.4.4.331-338

[13] M. Wygralak: Cardinalities of Fuzzy Sets. Springer, Berlin, Heidelberg, New York 2003. DOI:10.1007/978-3-540-36382-8

[14] M. Wygralak: Fuzzy sets with triangular norms and their cardinality theory. Fuzzy Sets and Systems *124* (2001), 1–24. DOI:10.1016/s0165-0114(00)00108-1

[15] M. Wygralak: Questions of cardinality of finite fuzzy sets. Fuzzy Sets and Systems *102* (1999), 185–210. DOI:10.1016/s0165-0114(97)00097-3

[16] L. Zadeh: Fuzzy sets and systems. System Theory, Brooklyn, Polytechnic Press (1965), 29–39.

[17] L. Zadeh: Fuzzy logic and its application to approximate reasoning. Inform. Process. *74* (1974), 591–594.

Vladimír Kobza, Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica. Slovak Republic.
e-mail: vladimir.kobza@umb.sk