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# CONSTRUCTION OF UNINORMS ON BOUNDED LATTICES

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In this paper, we propose the general methods, yielding uninorms on the bounded lattice $(L, \leq, 0, 1)$, with some additional constraints on $e \in L\backslash\{0,1\}$ for a fixed neutral element $e \in L\backslash\{0,1\}$ based on underlying an arbitrary triangular norm $T_e$ on $[0,e]$ and an arbitrary triangular conorm $S_e$ on $[e,1]$. And, some illustrative examples are added for clarity.

*Keywords:* bounded lattice, triangular norm, triangular conorm, uninorms

*Classification:* 03B52, 06B20, 03E72

## 1. INTRODUCTION

Triangular norms (t-norms) and triangular conorms (t-conorms) on the unit interval were systematically investigated by Schweizer and Sklar in [19]. These operators have been extensively used in many application in fuzzy set theory, fuzzy logics, multicriteria decision support and several branches of information sciences. For more details on t-norms, we refer to [1, 2, 17]. Uninorms were introduced in [22] and further investigated in [23] by Yager and Rybalov and in [14] by Fodor, Yager and Rybalov, which are also generalizations of t-norms and t-conorms. Uninorms on the real unit interval admit a neutral element $e$ to be an arbitrary point from $[0, 1]$ (if $e = 1$, we are in t-norm case, while if $e = 0$, we are in t-conorm case) and have to satisfy an additional condition. Such uninorms are interesting not only from a theoretical point of view (because of their structure, namely combination of a t-norm and a t-conorm), but also for their applications, since they have proved to be useful in several fields like expert systems, neural networks, fuzzy quantifiers. The uninorms were also studied by many authors in other papers [5, 6, 7, 9, 10, 11, 12, 13, 18, 20, 21].

Karaçal and Mesiar have shown the existence of uninorms on an arbitrary bounded lattice $L$, leaving the freedom for the neutral element $e \in L\backslash\{0,1\}$ in [15]. Their construction exploits the existence of a t-norm $T$ and a t-conorm $S$ for an arbitrary bounded lattice $L$, and as a by-product, existence of the smallest uninorm and of the greatest uninorm on $L$ with a fixed neutral element $e \in L\backslash\{0,1\}$ was shown.

In this paper, we study and discuss uninorms on an arbitrary bounded lattice $(L, \leq, 0, 1)$. We introduce the new methods of constructing uninorms on an arbitrary bounded lattice $(L, \leq, 0, 1)$ where some additional constraints on $e \in L\backslash\{0,1\}$ that is

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considered as neutral element are required by using the existence of t-norms on $[0, e]$ and t-conorms on $[e, 1]$. The construction methods to obtain uninorm on bounded lattices that is proposed in this study is different from the proposal of Karaçal and Mesiar in [15]. If both $x$ and $y$ are incomparable with $e$, then the construction method in Theorem 3.1 puts $x \vee y$ and on the remain domains these constructions coincide with the construction of the uninorm $U_t$ proposal in [13]. If both $x$ and $y$ are incomparable with $e$, then the construction method in Theorem 3.5 puts $x \wedge y$ and on the remain domains these construction coincide with the construction of the uninorm $U_s$ proposal in [13]. If both $x$ and $y$ are incomparable with $e$ or $x$ is from $[e, 1]$ and $y$ is incomparable with $e$ or $y$ is from $[e, 1]$ and $x$ is incomparable with $e$, then the construction method in Theorem 3.9 puts $x \vee y$ and on the remain domains these construction coincide with the construction of the uninorm $U_t$ proposal in [13]. If both $x$ and $y$ are incomparable with $e$ or $x$ is from $[0, e]$ and $y$ is incomparable with $e$ or $y$ is from $[0, e]$ and $x$ is incomparable with $e$, then the construction method in Theorem 3.12 puts $x \wedge y$ and on the remain domains these construction coincide with the construction of the uninorm $U_s$ proposal in [13]. In case of $e = 1$, we obtain already t-norms and in case of $e = 0$, we already obtain t-conorms. And, some illustrative examples are given to clearly understand these methods of characterizing uninorms on bounded lattices.

## 2. PRELIMINARIES

In this section, some preliminaries concerning bounded lattices and uninorms (t-norms, t-conorms) on them are recalled.

**Definition 2.1.** (Birkhoff [4]) A lattice $(L, \leq)$ is bounded lattice if $L$ has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

Let $L$ be a bounded lattice. An upper bound of the elements $x, y \in L$ is an element $a \in L$ containing the elements both $x$ and $y$. The least upper bound of the elements $x, y \in L$ is an upper bound contained by every other upper bound, it is denoted $\sup\{x, y\}$ or $x \vee y$ . An lower bound of the elements $x, y \in L$ is an element $b \in L$ contained by the elements both $x$ and $y$. The greatest lower bound of the elements $x, y \in L$ is an lower bound containing every other lower bound, it is denoted $\inf\{x, y\}$ or $x \wedge y$ .

**Definition 2.2.** (Birkhoff [4]) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if $a$ and $b$ are incomparable, in this case we use the notation $a \parallel b$.

**Definition 2.3.** (Birkhoff [4]) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of $L$ defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, we define $(a, b] = \{x \in L \mid a < x \leq b\}, [a, b) = \{x \in L \mid a \leq x < b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L$. Let $A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$ and $I_e = \{x \in L \mid x \parallel e\}$.

**Remark 2.4.** (Birkhoff [4]) Given a bounded lattice $(L, \leq, 0, 1)$, and $a, b \in L$, $a \leq b$. Subinterval $[a, b]$ of $L$ is a sublattice of $L$, but the rest of subinterval in Definition 2 is not necessary sublattice of $L$.

**Definition 2.5.** (Karaçal and Mesiar [15], Karaçal et al. [16]) Let $(L, \leq, 0, 1)$ be a bounded lattice. Operation $U : L^2 \to L$ is called a uninorm on $L$ (shortly a uninorm, if $L$ is fixed) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e \in L$.

**Definition 2.6.** (Aşıcı [3], Çaylı and Karaçal [8]) Operation $T : L^2 \to L$ $\big(S : L^2 \to L\big)$ is called a t-norm (t-conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e = 1$ $(e = 0)$.

**Proposition 2.7.** (Karaçal and Mesiar [15]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and $U$ be a uninorm on $L$ with the neutral element $e$.

Then

i) $T^* = U|[0, e]^2 : [0, e]^2 \to [0, e]$ is a t-norm on $[0, e]$,

ii) $S^* = U|[e, 1]^2 : [e, 1]^2 \to [e, 1]$ is a t-conorm on $[e, 1]$.

**Proposition 2.8.** (Karaçal and Mesiar [15]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and $U$ be a uninorm on $L$ with the neutral element $e$. The following properties hold:

i) $x \wedge y \leq U(x, y) \leq x \vee y$ for all $(x, y) \in A(e)$,

ii) $U(x, y) \leq x$ for $(x, y) \in L \times [0, e]$,

iii) $U(x, y) \leq y$ for $(x, y) \in [0, e] \times L$,

iv) $x \leq U(x, y)$ for $(x, y) \in L \times [e, 1]$,

v) $y \leq U(x, y)$ for $(x, y) \in [e, 1] \times L$.

## 3. UNINORMS WITH FIXED UNDERLYING T-NORMS AND T-CONORMS

**Theorem 3.1.** Let $(L, \leqslant, 0, 1)$ be a bounded lattice and fix $e \in L \backslash \{0, 1\}$. Suppose that either $x \vee y > e$ for all $x \parallel e$ and $y \parallel e$ or $x \vee y \parallel e$ for all $x \parallel e$ and $y \parallel e$. If $T_e$ is a t-norm on $[0, e]$, then the function $U_t : L \times L \to L$ defined as

$$U_t(x, y) = \begin{cases} T_e(x, y) & \text{if } (x, y) \in [0, e]^2, \\ x \vee y & if \ (x, y) \in A(e) \cup I_e \times I_e, \\ y & \text{if } (x, y) \in [0, e] \times I_e, \\ x & \text{if } (x, y) \in I_e \times [0, e], \\ 1 & \text{otherwise}, \end{cases} \tag{1}$$

is a uninorm on $L$ with the neutral element $e$.

Proof. i) Monotonicity: We prove that if $x \leq y$ then for all $z \in L$, $U_t(x,z) \leq U_t(y,z)$. The proof is split into all possible cases.

1. Let $x \leq e$.
   1.1. $y \leq e$,
   1.1.1. $z \leq e$,

$$U_t(x,z) = T_e\,(x,z) \leq T_e\,(y,z) = U_t(y,z)$$

   1.1.2. $z > e$ or $z \parallel e$,

$$U_t(x,z) = z = U_t(y,z)$$

   1.2. $y > e$,
   1.2.1. $z \leq e$,

$$U_t(x,z) = T_e\,(x,z) \leq x \leq y = U_t(y,z)$$

   1.2.2. $z > e$ or $z \parallel e$,

$$U_t(x,z) = z \leq 1 = U_t(y,z)$$

   1.3. $y \parallel e$,
   1.3.1. $z \leq e$,

$$U_t(x,z) = T_e\,(x,z) \leq x \leq y = U_t(y,z)$$

   1.3.2. $z > e$,

$$U_t(x,z) = z \leq 1 = U_t(y,z)$$

   1.3.3. $z \parallel e$,

$$U_t(x,z) = z \leq y \vee z = U_t(y,z)$$

2. Let $x > e$. Then $y > e$.
   2.1. $z \leq e$,

$$U_t(x,z) = x \leq y = U_t(y,z)$$

   2.2. $z > e$ or $z \parallel e$,

$$U_t(x,z) = 1 = U_t(y,z)$$

3. Let $x \parallel e$.
   3.1. $y > e$,
   3.1.1. $z \leq e$,

$$U_t(x,z) = x \leq y = U_t(y,z)$$

   3.1.2. $z > e$,

$$U_t(x,z) = 1 = U_t(y,z)$$

   3.1.3. $z \parallel e$,

$$U_t(x,z) = x \vee z \leq 1 = U_t(y,z)$$

   3.2. $y \parallel e$,

3.2.1. $z \leq e$,

$$U_t(x,z) = x \leq y = U_t(y,z)$$

3.2.2. $z > e$,

$$U_t(x,z) = 1 = U_t(y,z)$$

3.2.3. $z \parallel e$,

$$U_t(x,z) = x \vee z \leq y \vee z = U_t(y,z).$$

ii) Associativity: We demonstrate that $U_t(x, U_t(y,z)) = U_t(U_t(x,y),z)$ for all $x,y,z \in L$. Again the proof is split into all possible cases considering the relationships of the elements $x, y, z$ and $e$.

1. Let $x \leq e$.

1.1. $y \leq e$,

1.1.1. $z \leq e$,

$$\begin{aligned} U_t(x,U_t(y,z)) &= U_t\left(x, T_e\left(y,z\right)\right) = T_e\left(x, T_e\left(y,z\right)\right) \\ &= T_e\left(T_e\left(x,y\right),z\right) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

1.1.2. $z > e$,

$$\begin{aligned} U_t(x,U_t(y,z) &= U_t\left(x, y \vee z\right) = U_t\left(x,z\right) = x \vee z = z \\ &= T_e(x,y) \vee z \\ &= U_t(T_e(x,y),z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

1.1.3. $z \parallel e$,

$$\begin{aligned} U_t(x,U_t(y,z)) &= U_t\left(x,z\right) = z \\ &= U_t(T_e(x,y),z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

1.2. $y > e$,

1.2.1. $z \leq e$,

$$\begin{aligned} U_t(x,U_t(y,z)) &= U_t\left(x, y \vee z\right) = U_t\left(x,y\right) = x \vee y = y \\ &= y \vee z \\ &= U_t\left(y,z\right) \\ &= U_t\left(x \vee y,z\right) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

1.2.2. $z > e$ or $z \parallel e$,

$$\begin{aligned} U_t(x,U_t(y,z)) &= U_t\left(x,1\right) = x \vee 1 = 1 \\ &= U_t\left(y,z\right) \\ &= U_t\left(x \vee y,z\right) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

1.3. $y \parallel e$,

1.3.1. $z \leq e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x,y) = y \\ &= U_t\,(y,z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

1.3.2. $z > e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x,1) = x \vee 1 = 1 \\ &= U_t\,(y,z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

1.3.3. $z > e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x, y \vee z) = y \vee z \\ &= U_t\,(y,z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

2. Let $x > e$.

2.1. $y \leq e$,

2.1.1. $z \leq e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t(x, T_e(y,z)) = x \vee T_e(y,z) = x \\ &= x \vee z \\ &= U_t\,(x,z) \\ &= U_t\,(x \vee y, z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

2.1.2. $z > e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x, y \vee z) = U_t\,(x,z) = 1 \\ &= U_t\,(x,z) \\ &= U_t\,(x \vee y, z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

2.1.3. $z \parallel e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x,z) = 1 \\ &= U_t\,(x,z) \\ &= U_t\,(x \vee y, z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

2.2. $y > e$,

2.2.1. $z \leq e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x, y \vee z) = U_t\,(x,y) = 1 \\ &= 1 \vee z \\ &= U_t\,(1,z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

2.2.2. $z > e$ or $z \parallel e$,

$$\begin{aligned} U_t(x, U_t(y, z)) &= U_t\,(x, 1) = 1 \\ &= U_t\,(1, z) \\ &= U_t(U_t(x, y), z) \end{aligned}$$

2.3. $y \parallel e$,

2.3.1. $z \leq e$,

$$\begin{aligned} U_t(x, U_t(y, z)) &= U_t\,(x, y) = 1 \\ &= 1 \vee z \\ &= U_t\,(1, z) \\ &= U_t(U_t(x, y), z) \end{aligned}$$

2.3.2. $z > e$,

$$\begin{aligned} U_t(x, U_t(y, z)) &= U_t\,(x, 1) = 1 \\ &= U_t\,(1, z) \\ &= U_t(U_t(x, y), z) \end{aligned}$$

2.3.2. $z \parallel e$,

$$\begin{aligned} U_t(x, U_t(y, z)) &= U_t\,(x, y \vee z) = 1 \\ &= U_t\,(1, z) \\ &= U_t(U_t(x, y), z) \end{aligned}$$

3. Let $x \parallel e$.

3.1. $y \leq e$,

3.1.1. $z \leq e$,

$$\begin{aligned} U_t(x, U_t(y, z)) &= U_t\,(x, T_e\,(y, z)) = x \\ &= U_t\,(x, z) \\ &= U_t(U_t(x, y), z) \end{aligned}$$

3.1.2. $z > e$,

$$\begin{aligned} U_t(x, U_t(y, z)) &= U_t\,(x, y \vee z) \;\; = U_t\,(x, z) = 1 \\ &= U_t\,(x, z) \\ &= U_t(U_t(x, y), z) \end{aligned}$$

3.1.3. $z \parallel e$,

$$\begin{aligned} U_t(x, U_t(y, z)) &= U_t\,(x, z) = x \vee z \\ &= U_t\,(x, z) \\ &= U_t(U_t(x, y), z) \end{aligned}$$

3.2. $y > e$,

3.2.1. $z \leq e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x, y \vee z) \ = U_t\,(x,y) = 1 \\ &= 1 \vee z \\ &= U_t\,(1,z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

3.2.2. $z > e$ or $z \parallel e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x,1) \ = 1 \\ &= U_t\,(1,z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

3.3. $y \parallel e$,

3.3.1. $z \leq e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x,y) \ = x \vee y \\ &= U_t\,(x \vee y, z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

3.3.2. $z > e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x,1) \ = 1 \\ &= U_t\,(x \vee y, z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

3.3.3. $z \parallel e$,

From hypotheses of Theorem 3.1, either $x \vee y > e$ for all $x \parallel e$ and $y \parallel e$ or $x \vee y \parallel e$ for all $x \parallel e$ and $y \parallel e$.

3.3.3.1. if $x \vee y > e$ for all $x \parallel e$ and $y \parallel e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x, y \vee z) \ = 1 \\ &= U_t\,(x \vee y, z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

3.3.3.2. if $x \vee y \parallel e$ for all $x \parallel e$ and $y \parallel e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x, y \vee z) \ = x \vee y \vee z \\ &= U_t\,(x \vee y, z) \\ &= U_t(U_t(x,y),z). \end{aligned}$$

iii) Commutativity: We show that for all $x, y \in L$, $U_t(x,y) = U_t(y,x)$. The proof is split into all possible cases.

1. $x \leq e$,

1.1. $y \leq e$,

$$U_t(x,y) = T_e\,(x,y) = T_e\,(y,x) = U_t(y,x)$$

1.2. $y > e$,

$$U_t(x,y) = x \vee y = y \vee x = U_t(y,x)$$

1.3. $y \parallel e$,

$$U_t(x,y) = y = U_t(y,x).$$

2. $x > e$,

2.1. $y \leq e$,

$$U_t(x,y) = x \vee y = y \vee x = U_t(y,x)$$

2.2. $y > e$ or $y \parallel e$,

$$U_t(x,y) = 1 = U_t(y,x)$$

3. $x \parallel e$,

1.1. $y \leq e$,

$$U_t(x,y) = x = U_t(y,x)$$

1.2. $y > e$,

$$U_t(x,y) = 1 = U_t(y,x)$$

1.3. $y \parallel e$,

$$U_t(x,y) = x \vee y = y \vee x = U_t(y,x).$$

iv) Neutral element: We prove that for all $x \in L$, $U_t(x,e) = x$. The proof is split into all possible cases.

1. $x \leq e$,

$$U_t(x,e) = T_e\,(x,e) = x$$

2. $x > e$,

$$U_t(x,e) = x \vee e = x$$

3. $x \parallel e$,

$$U_t(x,e) = x.$$

□

**Example 3.2.** (i) The lattice $L_1$ in Figure 1 is a positive example satisfying constraint of Theorem 3.1 since $x \vee y > e$ for all $x \parallel e$ and $y \parallel e$ for neutral element $e$.

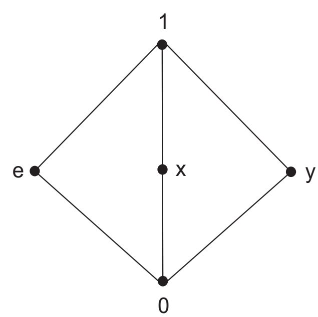


**Fig. 1.** The lattice $L_1$.

(ii) The lattice $L_2$ in Figure 2 satisfy constraint of Theorem 3.1 since $x \vee y \parallel e$ for all $x \parallel e$ and $y \parallel e$ for the indicated neutral element $e$.

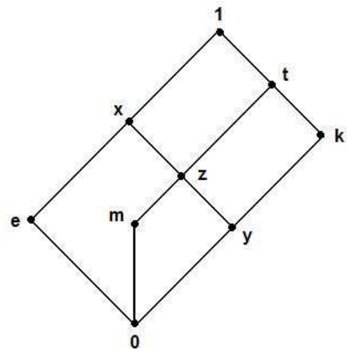


**Fig. 2.** The lattice $L_2$.

(iii) The next lattice $L_3$ is negative example, where, for a chosen neutral element $e$, constraints of Theorem 3.1 are violated. Because, $x \vee z = k > e$ for $x \parallel e$, $z \parallel e$ and $y \vee m = m \parallel e$ for $y \parallel e$, $m \parallel e$.

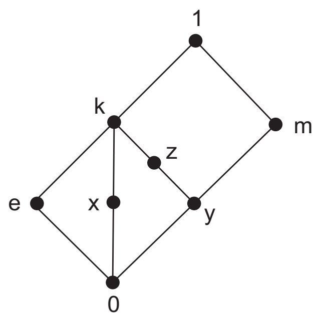


**Fig. 3.** The lattice $L_3$.

**Example 3.3.** Consider the lattice $L_2$ depicted in Figure 2. By using the construction method in Theorem 3.1 and [15, Theorem 1], taking the t-norm $T_e = T_\wedge$ (inf) on $[0, e]^2$, the uninorms $U_t$ on $L_2$ is defined, respectively, by Table 1 and Table 2

In the following example, we show that on any bounded lattice that does not satisfy constraints of Theorem 3.1, the operation $U$ defined by using (1) can not be a uninorm.

**Example 3.4.** Consider the lattice $L_3$ depicted in Figure 3. Define a mapping $U : L_3 \times L_3 \to L_3$ by Table 3. Then $U$ is constructed using (1), but $U$ is not a uninorm on $L_3$.

If we take elements $x, z \in L_3$, we have that $U(x, U(x, z)) = U(x, k) = 1$ and $U(U(x, x), z) = U(x, z) = k$. So, we obtain that $U$ is not a uninorm on $L_3$.

| $U_t$ | $0$ | $e$ | $m$ | $y$ | $z$ | $k$ | $t$ | $x$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $m$ | $y$ | $z$ | $k$ | $t$ | $x$ | $1$ |
| $e$ | $0$ | $e$ | $m$ | $y$ | $z$ | $k$ | $t$ | $x$ | $1$ |
| $m$ | $m$ | $m$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $y$ | $y$ | $y$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $z$ | $z$ | $z$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $k$ | $k$ | $k$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $t$ | $t$ | $t$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $x$ | $x$ | $x$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 1.** The uninorm $U_t$ on $L_2$ constructed using [15, Theorem 1].

| $U_t$ | $0$ | $e$ | $m$ | $y$ | $z$ | $k$ | $t$ | $x$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $m$ | $y$ | $z$ | $k$ | $t$ | $x$ | $1$ |
| $e$ | $0$ | $e$ | $m$ | $y$ | $z$ | $k$ | $t$ | $x$ | $1$ |
| $m$ | $m$ | $m$ | $m$ | $z$ | $z$ | $t$ | $t$ | $1$ | $1$ |
| $y$ | $y$ | $y$ | $z$ | $y$ | $z$ | $k$ | $t$ | $1$ | $1$ |
| $z$ | $z$ | $z$ | $z$ | $z$ | $z$ | $t$ | $t$ | $1$ | $1$ |
| $k$ | $k$ | $k$ | $t$ | $k$ | $t$ | $k$ | $t$ | $1$ | $1$ |
| $t$ | $t$ | $t$ | $t$ | $t$ | $t$ | $t$ | $t$ | $1$ | $1$ |
| $x$ | $x$ | $x$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 2.** The uninorm $U_t$ on $L_2$ constructed using Theorem 3.1.

| $U$ | $0$ | $e$ | $x$ | $y$ | $z$ | $m$ | $k$ | $1$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $x$ | $y$ | $z$ | $m$ | $k$ | $1$ |
| $e$ | $0$ | $e$ | $x$ | $y$ | $z$ | $m$ | $k$ | $1$ |
| $x$ | $x$ | $x$ | $x$ | $k$ | $k$ | $1$ | $1$ | $1$ |
| $y$ | $y$ | $y$ | $k$ | $z$ | $z$ | $m$ | $1$ | $1$ |
| $z$ | $z$ | $z$ | $k$ | $z$ | $z$ | $1$ | $1$ | $1$ |
| $m$ | $m$ | $m$ | $1$ | $m$ | $1$ | $m$ | $1$ | $1$ |
| $k$ | $k$ | $k$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 3.** The operation $U$ on $L_3$.

**Theorem 3.5.** Let $(L, \leqslant, 0, 1)$ be a bounded lattice and fix $e \in L \backslash \{0, 1\}$. Suppose that $x \wedge y < e$ for all $x \parallel e$ and $y \parallel e$ or $x \wedge y \parallel e$ for all $x \parallel e$ and $y \parallel e$. If $S_e$ is a t-conorm on $[e, 1]$, then the function $U_s : L \times L \to L$ defined as

$$U_s(x,y) = \begin{cases} S_e(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x \wedge y & \text{if } (x,y) \in A(e) \cup I_e \times I_e, \\ y & \text{if } (x,y) \in [e,1] \times I_e, \\ x & \text{if } (x,y) \in I_e \times [e,1], \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

is a uninorm on $L$ with the neutral element $e$.

It can be proved as dual of Theorem 3.1.

**Example 3.6.** (i) The lattice $L_4$ depicted in Figure 4 bring a positive example satisfying constraint of Theorem 3.5 since $x \wedge y \parallel e$ for all $x \parallel e$ and $y \parallel e$ for the indicated neutral element $e$. Note that the lattice $L_1$ given in Figure 1 is also positive example satisfying constraint of Theorem 3.5 since $x \wedge y < e$ for all $x \parallel e$ and $y \parallel e$.

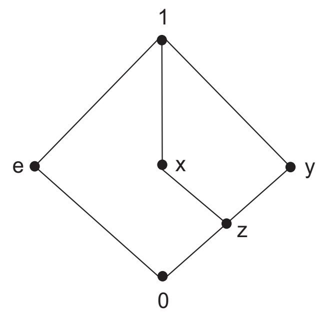


**Fig. 4.** The lattice $L_4$.

(ii) The next lattice $L_3$ is negative example, where, for a chosen neutral element $e$, constraint of Theorem 3.5 are violated. Because, $x \wedge z = m < e$ for $x \parallel e$, $z \parallel e$ and $y \wedge k = k \parallel e$ for $y \parallel e$, $k \parallel e$.

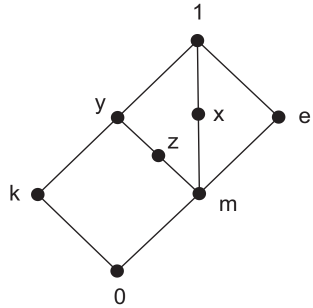


**Fig. 5.** The lattice $L_5$.

In the following example, we show that on any bounded lattice that does not satisfy constraints of Theorem 3.5, the operation $U$ constructed by using (2) can not be a uninorm.

**Example 3.7.** Consider the lattice $L_5$ depicted in Figure 5. Define a mapping $U : L_5 \times L_5 \to L_5$ by Table 4. Then $U$ is constructed using (2), but $U$ is not a uninorm on $L_5$.

| $U$ | $0$ | $m$ | $e$ | $k$ | $z$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $m$ | $0$ | $0$ | $m$ | $0$ | $0$ | $0$ | $0$ | $m$ |
| $e$ | $0$ | $m$ | $e$ | $k$ | $z$ | $x$ | $y$ | $1$ |
| $k$ | $0$ | $0$ | $k$ | $k$ | $0$ | $0$ | $k$ | $k$ |
| $z$ | $0$ | $0$ | $z$ | $0$ | $z$ | $m$ | $z$ | $z$ |
| $x$ | $0$ | $0$ | $x$ | $0$ | $m$ | $x$ | $m$ | $x$ |
| $y$ | $0$ | $0$ | $y$ | $k$ | $z$ | $m$ | $y$ | $y$ |
| $1$ | $0$ | $m$ | $1$ | $k$ | $z$ | $x$ | $y$ | $1$ |

**Tab. 4.** The operation $U$ on $L_5$.

If we take elements $x, z \in L_5$, we have that $U(x, U(x, z)) = U(x, m) = 0$ and $U(U(x, x), z) = U(x, z) = m$. So, we obtain that $U$ is not a uninorm on $L_5$.

**Example 3.8.** Consider a bounded lattice $L_1 = \{0, e, x, y, 1\}$ with given order in Figure 1 satisfying constraints of both Theorem 3.1 and Theorem 3.5.

(i) Define a mapping $U : L_1^2 \to L_1$ by Table 5 such that $U$ is constructed using the equality (1). Then, by Theorem 3.1, $U$ is a uninorm on $L$ with a neutral element $e$.

| $U$ | $0$ | $e$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $x$ | $y$ | $1$ |
| $e$ | $0$ | $e$ | $x$ | $y$ | $1$ |
| $x$ | $x$ | $x$ | $x$ | $1$ | $1$ |
| $y$ | $y$ | $y$ | $1$ | $y$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 5.** The uninorm $U$ on $L_1$.

(ii) Define a mapping $U : L_1^2 \to L_1$ by Table 6 such that $U$ is constructed using the equality (2). Then, by Theorem 3.5, $U$ is a uninorm on $L$ with a neutral element $e$.

| $U$ | $0$ | $e$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e$ | $0$ | $e$ | $x$ | $y$ | $1$ |
| $x$ | $0$ | $x$ | $x$ | $0$ | $x$ |
| $y$ | $0$ | $y$ | $0$ | $y$ | $y$ |
| $1$ | $0$ | $1$ | $x$ | $y$ | $1$ |

**Tab. 6.** The uninorm $U$ on $L_1$.

**Theorem 3.9.** Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $e \in L \backslash \{0, 1\}$. Suppose that $x \vee y \parallel e$ for all $x \parallel e$ and $y \parallel e$. If $T_e$ is a t-norm on $[0, e]$, then the function $U_T : L \times L \to L$ defined as

$$U_T(x,y) = \begin{cases} T_e(x,y) & \text{if } (x,y) \in [0,e]^2, \\ 1 & \text{if } (x,y) \in (e,1]^2, \\ y & \text{if } (x,y) \in [0,e] \times I_e, \\ x & \text{if } (x,y) \in I_e \times [0,e], \\ x \vee y & \text{otherwise.} \end{cases} \tag{3}$$

is a uninorm on $L$ with the neutral element $e$.

Proof. i) Monotonicity: We prove that if $x \le y$ then for all $z \in L$, $U_T(x,z) \le U_T(y,z)$. The proof is split into all possible cases.

1. Let $x \le e$.
   - 1.1. $y \le e$,
     - 1.1.1. $z \le e$,

$$U_T(x,z) = T_e(x,z) \le T_e(y,z) = U_T(y,z)$$

     - 1.1.2. $z > e$ or $z \parallel e$,

$$U_T(x,z) = z = U_T(y,z)$$

   - 1.2. $y > e$,
     - 1.2.1. $z \le e$,

$$U_T(x,z) = T_e(x,z) \le x \le y = U_T(y,z)$$

     - 1.2.2. $z > e$,

$$U_T(x,z) = z \le 1 = U_T(y,z)$$

     - 1.2.3. $z \parallel e$,

$$U_T(x,z) = z \le y \vee z = U_T(y,z)$$

   - 1.3. $y \parallel e$,
     - 1.3.1. $z \le e$,

$$U_T(x,z) = T_e(x,z) \le x \le y = U_T(y,z)$$

1.3.2. $z > e$ or $z \parallel e$,

$$U_T(x,z) = z \leq y \vee z = U_T(y,z)$$

2. Let $x > e$. Then $y > e$.
2.1. $z \leq e$,

$$U_T(x,z) = x \leq y = U_T(y,z)$$

2.2. $z > e$,

$$U_T(x,z) = 1 = U_T(y,z)$$

2.3. $z \parallel e$,

$$U_T(x,z) = x \vee z \leq y \vee z = U_T(y,z)$$

3. Let $x \parallel e$.
3.1. $y > e$,
3.1.1. $z \leq e$,

$$U_T(x,z) = x \leq y = U_T(y,z)$$

3.1.2. $z > e$,

$$U_T(x,z) = x \vee z \leq 1 = U_T(y,z)$$

3.1.3. $z \parallel e$,

$$U_T(x,z) = x \vee z \leq y \vee z = U_T(y,z)$$

3.2. $y \parallel e$,
3.2.1. $z \leq e$,

$$U_T(x,z) = x \leq y = U_T(y,z)$$

3.2.2. $z > e$ or $z \parallel e$,

$$U_T(x,z) = x \vee z \leq y \vee z = U_T(y,z)$$

ii) Associativity: We demonstrate that $U_T(x, U_T(y,z)) = U_T(U_T(x,y),z)$ for all $x,y,z \in L$. Again the proof is split into all possible cases considering the relationships of the elements $x,y,z$ and $e$.
1. Let $x \leq e$.
1.1.$y \leq e$,
1.1.1. $z \leq e$,

$$\begin{aligned} U_T(x,U_T(y,z)) &= U_T\left(x, T_e\left(y,z\right)\right) = T_e\left(x, T_e\left(y,z\right)\right) \\ &= T_e\left(T_e\left(x,y\right),z\right) \\ &= U_T(U_T(x,y),z) \end{aligned}$$

1.1.2. $z > e$,

$$\begin{aligned} U_T(x,U_T(y,z)) &= U_T\left(x, y \vee z\right) = U_T\left(x,z\right) = x \vee z = z \\ &= T_e(x,y) \vee z \\ &= U_T(T_e(x,y),z) \\ &= U_T(U_T(x,y),z) \end{aligned}$$

1.1.3. $z \parallel e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_t\,(x,z) = z \\ &= U_t(T_e(x,y),z) \\ &= U_T(U_T(x,y),z) \end{aligned}$$

1.2. $y > e$,

1.2.1. $z \leq e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x, y \vee z) = U_T\,(x,y) = x \vee y = y \\ &= y \vee z \\ &= U_T\,(y,z) \\ &= U_T\,(x \vee y, z) \\ &= U_T(U_T(x,y),z) \end{aligned}$$

1.2.2. $z > e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x,1) = x \vee 1 = 1 \\ &= U_t\,(y,z) \\ &= U_t\,(x \vee y, z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

1.2.3. $z \parallel e$,

$$\begin{aligned} U_t(x, U_t(y,z)) &= U_t\,(x, y \vee z) = x \vee (y \vee z) = y \vee z \\ &= U_t\,(y,z) \\ &= U_t\,(x \vee y, z) \\ &= U_t(U_t(x,y),z) \end{aligned}$$

1.3. $y \parallel e$,

1.3.1. $z \leq e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x,y) = y \\ &= U_T\,(y,z) \\ &= U_T(U_T(x,y),z) \end{aligned}$$

1.3.2. $z > e$ or $z \parallel e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_t\,(x, y \vee z) = y \vee z \\ &= U_T\,(y,z) \\ &= U_T(U_T(x,y),z) \end{aligned}$$

2. Let $x > e$.

2.1. $y \leq e$,

2.1.1. $z \leq e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_t(x, T_e(y,z)) = x \vee T_e(y,z) = x \\ &= x \vee z \\ &= U_T\,(x,z) \\ &= U_T\,(x \vee y, z) \\ &= U_T(U_T(x,y),z) \end{aligned}$$

2.1.2. $z > e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x, y \vee z) = U_T\,(x,z) = 1 \\ &= U_T\,(x,z) \\ &= U_T\,(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

2.1.3. $z \parallel e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x,z) = x \vee z \\ &= U_T\,(x,z) \\ &= U_T\,(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

2.2. $y > e$,

2.2.1. $z \leq e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_t\,(x, y \vee z) = U_t\,(x,y) = 1 \\ &= 1 \vee z \\ &= U_t\,(1,z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

2.2.2. $z > e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x,1) = 1 \\ &= U_T\,(1,z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

2.2.3. $z \parallel e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x, y \vee z) = 1 \\ &= 1 \vee z \\ &= U_T\,(1,z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

2.3. $y \parallel e$,

2.3.1. $z \leq e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x,y) = x \vee y \\ &= (x \vee y) \vee z \\ &= U_T\,(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

2.3.2. $z > e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x, y \vee z) = 1 \\ &= U_T\,(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

2.3.2. $z \parallel e$, then $y \vee z \parallel e$ from hypotheses of Theorem 3.9.

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T(x, y \vee z) = x \vee y \vee z \\ &= U_T(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

3. Let $x \parallel e$.

3.1. $y \leq e$,

3.1.1. $z \leq e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T(x, T_e(y,z)) = x \\ &= U_T(x, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

3.1.2. $z > e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T(x, y \vee z) \;\; = U_T(x, z) = x \vee z \\ &= U_T(x, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

3.1.3. $z \parallel e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T(x, z) = x \vee z \\ &= U_T(x, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

3.2. $y > e$,

3.2.1. $z \leq e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T(x, y \vee z) \;\; = U_T(x, y) = x \vee y \\ &= (x \vee y) \vee z \\ &= U_T(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

3.2.2. $z > e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T(x, 1) \;\; = x \vee 1 = 1 \\ &= U_T(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

3.2.3. $z \parallel e$,

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T(x, y \vee z) \;\; = x \vee y \vee z \\ &= U_T(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

3.3. $y \parallel e$,

3.3.1. $z \leq e$, then $x \vee y \parallel e$ from hypotheses of Theorem 3.9.

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T(x, y) \;\; = x \vee y \\ &= U_T(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

3.3.2. $z > e$, then $x \vee y \parallel e$ from hypotheses of Theorem 3.9.

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x, y \vee z) \ = x \vee y \vee z \\ &= U_T\,(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

3.3.3. $z \parallel e$, then $x \vee y \parallel e$ and $y \vee z \parallel e$ from hypotheses of Theorem 3.9.

$$\begin{aligned} U_T(x, U_T(y,z)) &= U_T\,(x, y \vee z) \ = x \vee y \vee z \\ &= U_T\,(x \vee y, z) \\ &= U_T(U_T(x,y), z) \end{aligned}$$

iii) Commutativity: We prove that for all $x, y \in L$, $U_T(x,y) = U_T(y,x)$. The proof is split into all possible cases.

1. $x \leq e$,

1.1. $y \leq e$,

$$U_T(x,y) = T_e\,(x,y) = T_e\,(y,x) = U_T(y,x)$$

1.2. $y > e$,

$$U_T(x,y) = x \vee y = y \vee x = U_T(y,x)$$

1.3. $y \parallel e$,

$$U_T(x,y) = y = U_T(y,x)$$

2. $x > e$,

2.1. $y \leq e$ or $y \parallel e$,

$$U_T(x,y) = x \vee y = y \vee x = U_T(y,x)$$

2.2. $y > e$,

$$U_T(x,y) = 1 = U_T(y,x)$$

3. $x \parallel e$,

3.1. $y \leq e$,

$$U_T(x,y) = x = U_T(y,x)$$

3.2. $y > e$ or $y \parallel e$,

$$U_T(x,y) = x \vee y = y \vee x = U_T(y,x)$$

iv) Neutral element: We show that for all $x \in L$, $U_T(x,e) = x$. The proof is split into all possible cases.

1. $x \leq e$,

$$U_T(x,e) = T_e\,(x,e) = x$$

2. $x > e$,

$$U_T(x,e) = x \vee e = x$$

2. $x \parallel e$,

$$U_T(x,e) = x$$

□

**Example 3.10.** (i) The lattice $L_6$ with given order in Figure 6 is a positive example providing restriction of Theorem 3.9, since $x \vee y \parallel e$ for all $x \parallel e$ and $y \parallel e$ for neutral element $e$.

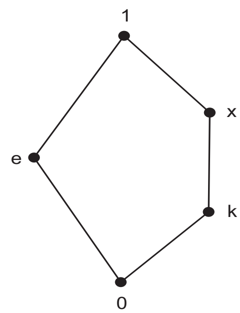


**Fig. 6.** The lattice $L_6$.

(ii) The next lattice $L_7$ give a negative examples that does not enable constraint of Theorem 3.9 for a chosen neutral element $e$. Because, $x \vee y = k > e$ for $x \parallel e$ and $y \parallel e$ for neutral element $e$.

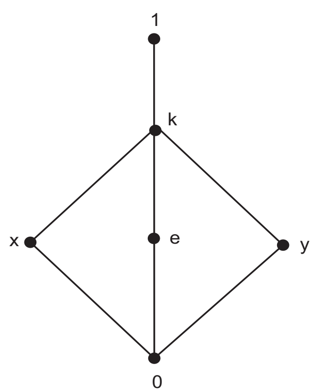


**Fig. 7.** The lattice $L_7$.

In the following example, we show that on any bounded lattice that does not satisfy constraint of Theorem 3.9, the operation $U$ defined by using (3) can not be a uninorm.

**Example 3.11.** Consider the lattice $L_7$ depicted in Figure 7. Define a mapping $U : L_7 \times L_7 \to L_7$ by Table 7. Then $U$ is constructed using (3), but $U$ is not a uninorm on $L_7$.

| $U$ | $0$ | $e$ | $x$ | $y$ | $k$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ |
| $e$ | $0$ | $e$ | $x$ | $y$ | $k$ | $1$ |
| $x$ | $0$ | $x$ | $x$ | $k$ | $k$ | $1$ |
| $y$ | $0$ | $y$ | $k$ | $y$ | $k$ | $1$ |
| $k$ | $0$ | $k$ | $k$ | $k$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 7.** The operation $U$ on $L_7$.

If we take elements $x, y, k \in L_7$, we have that $U(k, U(x, y)) = U(k, k) = 1$ and $U(U(k, x), y) = U(k, y) = k$. So, we obtain that $U$ is not a uninorm on $L_7$.

**Theorem 3.12.** Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $e \in L \backslash \{0, 1\}$. Suppose that $x \wedge y \parallel e$ for all $x \parallel e$ and $y \parallel e$. If $S_e$ is a t-conorm on $[e, 1]$, then the function $U_S : L \times L \to L$ defined as

$$U_S(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, e)^2, \\ S_e(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ y & \text{if } (x, y) \in [0, e] \times I_e, \\ x & \text{if } (x, y) \in I_e \times [0, e], \\ x \wedge y & \text{otherwise.} \end{cases} \tag{4}$$

is a uninorm on $L$ with the neutral element $e$.

It can be proved as dual of Theorem 3.9.

**Example 3.13.** (i) The lattice $L_8$ given in Figure 8 give a positive example providing for restraint of Theorem 3.12 since $x \wedge y \parallel e$ for all $x \parallel e$ and $y \parallel e$ for neutral element $e$.

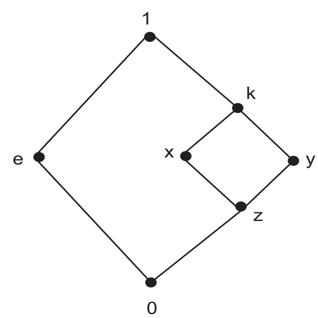


**Fig. 8.** The lattice $L_8$.

(ii) The next lattice $L_9$ bring negative examples that does not satisfying restraint of Theorem 3.12 for a chosen neutral element $e$. Because, $x \wedge y = s < e$ for $x \parallel e$ and $y \parallel e$ for neutral element $e$.

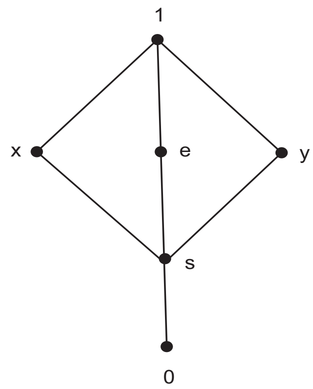


**Fig. 9.** The lattice $L_9$.

In the following example, we show that on any bounded lattice that does not satisfy constraint of Theorem 3.12, the operation $U$ defined by using (4) can not be a uninorm.

**Example 3.14.** Consider the lattice $L_9$ depicted in Figure 9. Define a mapping $U : L_9 \times L_9 \to L_9$ by Table 8. Then $U$ is constructed using (4), but $U$ is not a uninorm on $L_9$.

| $U$ | 0 | $s$ | $e$ | $x$ | $y$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $s$ | 0 | 0 | $s$ | $s$ | $s$ | 1 |
| $e$ | 0 | $s$ | $e$ | $x$ | $y$ | 1 |
| $x$ | 0 | $s$ | $x$ | $x$ | $s$ | 1 |
| $y$ | 0 | $s$ | $y$ | 1 | $y$ | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 |

**Tab. 8.** The operation $U$ on $L_9$.

If we take elements $x, y, s \in L_9$, we have that $U(s, U(x, y)) = U(s, s) = 0$ and $U(U(s, x), y) = U(s, y) = s$. So, we obtain that $U$ is not a uninorm on $L_9$.

## 4. CONCLUDING REMARKS

In this study, we have introduced and investigated characterization uninorms on bounded lattices. We give the new construction methods for building uninorms on an arbitrary bounded lattice $(L, \leq, 0, 1)$ with arbitrary zero element $e \in L\backslash\{0, 1\}$ with some additional constraints on $e \in L\backslash\{0, 1\}$ based on the knowledge of the existence of t-norms on and t-conorms on an arbitrary given bounded lattice $L$. If $L$ is a chain, then all elements in $L$ are comparable with $e$ indicated as neutral element. In this case, we consider only domains $[0,e]^2$, $[e,1]^2$, $[0,e]\times[e,1]$ and $[e,1]\times[0,e]$. So, by taking only these domains in our characterization methods to obtain uninorms on bounded lattices, these methods can be applied on chains without additional assumptions on $e \in L\backslash\{0,1\}$.

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