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NEW CRITERION FOR ASYMPTOTIC STABILITY OF TIME-VARYING DYNAMICAL SYSTEMS

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In this paper, we establish some new sufficient conditions for uniform global asymptotic stability for certain classes of nonlinear systems. Lyapunov approach is applied to study exponential stability and stabilization of time-varying systems. Sufficient conditions are obtained based on new nonlinear differential inequalities. Moreover, some examples are treated and an application to control systems is given.

Keywords: nonlinear time-varying systems, asymptotic stability, stabilization

Classification: 93Cxx, 93Dxx

1. INTRODUCTION

Lyapunov stability of nonlinear time-varying systems has attracted the attention of several authors (see [1, 2, 8, 18, 20] and references therein). Many applications are developped regarding the time-varying dynamical systems, in particular for complex networks, finite-time consensus criteria for Multi-agent systems with nonlinear dynamics, synchronizability of duplex networks (see [13] – [16]).

It is well known that, the asymptotic stability is more important than stability, also the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, thus the notion of practical stability is more suitable in several situations than Lyapunov stability (see [12, 17, 19] – [22]). In this case all state trajectories are bounded and approach a sufficiently small neighborhood of the origin (see [5] and references therein). One also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. To this end, the authors in [6] introduce a concept of exponential rate of convergence and for a specific class of uncertain systems they present controllers which guarantee this behavior (see [4] for the more explanation and [3]). Differential inequalities are very convenient for obtaining bounds on the solutions of nonlinear systems. The basic idea of our study is to use some new inequalities which are obtained by differentiating scalar valued Lyapunov functions along the solutions.

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In this paper, Lyapunov approach is applied to study exponential stability and stabilization of time-varying systems. Sufficient conditions for exponential stability and stabilization are obtained based on new nonlinear differential inequalities (see [9, 10]). These inequalities can be used as handy tools to research stability problems of perturbed dynamic systems. We give some new classes of perturbed systems for which the global uniform asymptotic stability of a small ball centered at the origin is obtained. The assumptions used in this work are not much more restrictive than those commonly invoked to the weaker property of global ultimate boundedness which are expressed as relation between the Lyapunov function and the existence of specific function which appear in our analysis through the solution of a scalar differential equation. As applications, based on these new established inequalities, some new results on uniform stability are obtained. Furthermore, some numerical examples are presented to illustrate the validity of the main results. As a special case, an application to control systems is given.

2. DEFINITIONS AND TOOLS

Consider the time-varying system described by the following differential equation

$$\dot{x} \quad = \quad f(t,x) \tag{1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous in (t,x) and locally Lipschitz in x on $\mathbb{R}^+ \times \mathbb{R}^n$. For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, we will denote by $x(t;t_0,x_0)$, or simply by $x(t)$, the unique solution of (1) at time t_0 starting from the point x_0.

Unless otherwise stated, we assume throughout the paper that the function encountered is sufficiently smooth. We often omit arguments of function to simplify notation, $\|.\|$ stands for the Euclidean norm vectors. We recall now some standard concepts from stability and practical stability theory; any book on Lyapunov stability can be consulted for these; particularly good references are [7, 11]: $\mathcal{K}$ is the class of functions $\mathbb{R}^+ \to \mathbb{R}^+$ which are zero at the origin, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $\mathbb{R}^+ \to \mathbb{R}^+$ which are continuous, decreasing and converging to zero as their argument tends to $+\infty$. $\mathcal{KL}$ is the class of functions $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second one. A positive definite function $\mathbb{R}^+ \to \mathbb{R}^+$ is one that is zero at the origin and positive otherwise. We define the closed ball $B_r := \Big\{x \in \mathbb{R}^n : \ \|x\| \le r\Big\}$.

2.1. Technical lemmas

To solve the problem of uniform exponential convergence to a small ball B_r of time-varying system, we introduce a technical lemma that will be crucial in establishing the main result. Let consider the scalar nonautonomous differential equation:

$$\dot{y} = -\alpha(y) + \beta(t). \tag{2}$$

Lemma 2.1. Suppose that α is a locally Lipschitz and a class $\mathcal{K}$-function defined on $[0,a]$, β is a continuous positive function on $[0,+\infty)$ verifying $\|\beta\|_\infty < \|\alpha\|_\infty$. Then, for all $(t_0,y_0) \in [0,+\infty) \times [0,a)$, the equation (2) has a unique maximal solution $y(.)$ such that $y(t_0) = y_0$. (Note that, A maximal solution to a differential equation is a solution defined on an interval I such that there is no solution with the same initial condition defined on an interval J, which properly contains I). Moreover, y is defined for all $t \geq t_0$.

P r o o f. Firstly, the equation (2) can be written as $\dot{y} = f(t, y)$ where

$$f : (t, y) \to -\alpha(y) + \beta(t).$$

Then f is continuous in (t, y) and locally Lipschitz in y. Consequently, for all $t_0 \in \mathbb{R}^+$ and $y_0 \in [0, a)$, the equation (2) has a unique maximal solution such that $y(t_0) = y_0$. Moreover, y is defined on $[t_0, T_{\max})$. In addition, we have for all $t \in [t_0, T_{\max})$

$$\dot{y}(t) = -\alpha(y(t)) + \beta(t) \leq -\alpha(y(t)) + \|\beta\|_\infty.$$

Next, we consider the equation:

$$\dot{y} = -\alpha(y) + \|\beta\|_\infty. \tag{3}$$

We denote z the maximal solution of (3) such that $z(t_0) = y_0$. We have that z is defined on $[t_0, T^*)$. We put $\xi_0 = \alpha^{-1}(\|\beta\|_\infty) \in [0, a)$. To prove that z is defined on $[t_0, +\infty)$, we distinguish three cases:

Case 1. $y_0 = \xi_0$
It is clear that the constant solution $z = \xi_0$ is the unique maximal solution of (3) verifying $z(t_0) = y_0$, and consequently $T^* = +\infty$.

Case 2. $0 \leq y_0 < \xi_0$
We prove that $0 \leq z(t) < \xi_0$ for all $t \in [t_0, T^*)$. Suppose it is not true, we can find $t_1 \in [t_0, T^*)$ such that $z(t_1) \geq \xi_0$. Hence $z(t_0) < \xi_0$, then we can find (by the Intermediate Value Theorem) $t_2 \in [t_0, t_1]$ such that $z(t_2) = \xi_0$. By the uniqueness of the solution z, we have $z = \xi_0$, which is a contradiction with $y_0 \neq \xi_0$. Therefore z remains within the compact $[0, \xi_0]$ of $[0, a)$ and thus $T^* = +\infty$.

Case 3. $\xi_0 < y_0 < a$
We obtain $z(t) \in (\xi_0, a)$ for all $t \in [t_0, T^*)$ (we apply the same reasoning). Next, we get

$$\forall t \in [t_0, T^*), \quad \dot{z}(t) = -\alpha(z(t)) + \|\beta\|_\infty < -\alpha(\xi_0) + \|\beta\|_\infty = 0,$$

therefore z is strictly decreasing, hence $\xi_0 < z(t) \leq y_0 \quad \forall t \in [t_0, T^*)$ and z remains within the compact $[\xi_0, y_0]$ of $[0, a)$ and thus $T^* = +\infty$.

We obtain in all three cases, z is defined on $[t_0, +\infty)$.

Then, by comparison, we obtain

$$0 \leq y(t) \leq z(t) \quad \forall t \in [t_0, T_{\max}).$$

Thus,

$$\begin{cases} y(t) \in [0, \xi_0] & \text{if} \quad y_0 \in [0, \xi_0] \\ y(t) \in [0, y_0] & \text{if} \quad y_0 \in (\xi_0, a). \end{cases}$$

Consequently, y remains within the compact $[0, \max(\xi_0, y_0)]$ of $[0, a)$ which implies that $T_{\max} = +\infty$ and y is defined on $[t_0, +\infty)$. □

Next, an interesting result can be done in equation (2) in the case when $\beta(t)$ tends to zero when t goes to infinity. The proof of the following lemma is based on a result of [22].

Lemma 2.2. Let $\alpha(.)$ be a class $\mathcal{K}$-function locally Lipschitz on $[0, a]$ and $\beta : [0, +\infty) \to [0, +\infty)$ a continuous function verifying $\|\beta\|_\infty < \|\alpha\|_\infty$ and $\lim_{+\infty} \beta(t) = 0$. Then, the maximal solution of the equation (2) with $y_0 \in [0, a)$ tends to 0 as $t \to +\infty$.

P r o o f. Let y the maximal solution of the equation (2) verifying $y_0 \in [0, a)$, by using lemma 2.1, we have

$$\forall t \geq t_0, \quad y(t) \in [0, a).$$

Moreover, y is the maximal solution of the equation

$$\dot{y} = -\hat{\alpha}\,(y) + \beta\,(t)\,,$$

verifying $y_0 \in [0, a)$, where

$$\hat{\alpha} : \mathbb{R} \to \mathbb{R}, \quad y \mapsto \begin{cases} \alpha(y) & \text{if} \quad y \in [0, a] \\ -\alpha(-y) & \text{if} \quad y \in [-a, 0] \\ \frac{\alpha(a)}{a} y & \text{if} \quad y \in [a, +\infty) \cup (-\infty, -a]. \end{cases}$$

It is clear that $\hat{\alpha}$ is a $\mathcal{K}$-function on $\mathbb{R}$. Since $\beta(t)$ tends to 0 as $t \to +\infty$, we obtain by using a result of [22] the fact that,

$$\lim_{t \to +\infty} y(t) = 0.$$

□

Lemma 2.3. Let α a class $\mathcal{K}$-function and locally Lipschitz on $[0, a]$, where a is a positive constant. $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function verifying $\|\beta\|_\infty < \|\alpha\|_\infty$ and $\lim_{t \to +\infty} \beta(t) = 0$. Then there exist a class $\mathcal{KL}$-function σ on $[0, a) \times [0, +\infty)$ and a non-negative and continuous function ε verifying

$$\lim_{t \to +\infty} \varepsilon(t) = 0$$

such that for all $t_0 \in [0, +\infty)$ and $y_0 \in [0, a)$, the equation (2) has a unique maximal solution y such that $y\,(t_0) = y_0$. Moreover y is defined for all $t \geq t_0$ and

$$0 \leq y(t) \leq \sigma(y_0, t - t_0) + \varepsilon\,(t)\,.$$

P r o o f. Under the hypothesis, we have:
for all $(t_0, y_0) \in [0, +\infty) \times [0, a)$, the equation (2) has a unique maximal solution y, which is defined on $[t_0, +\infty)$ and verifying $y(t_0) = y_0$. Let $d \in (0, a)$ such that $\alpha(d) \in (\|\beta\|_\infty, \|\alpha\|_\infty)$. We consider, for all $x \in [0, a]$

$$\overline{\alpha}(x) = \frac{1}{a} \int_0^x \alpha(u)\,\mathrm{d}u \quad \text{and} \quad \underline{\alpha}(x) = \frac{1}{a-d} \int_d^x \alpha(u)\,\mathrm{d}u.$$

Now, we put

$$\widetilde{\alpha} = \sup(\overline{\alpha}, \underline{\alpha}).$$

It is easy to verify that $\tilde{\alpha}$ is a class $\mathcal{K}$-function and convex on $[0, a]$. Moreover,

$$\tilde{\alpha}(x) \quad \leq \quad \alpha(x), \quad \forall x \in [0, a]$$

and $\|\beta\|_\infty < \|\tilde{\alpha}\|_\infty$. Now, we consider the equation:

$$\dot{y} = -\tilde{\alpha}(y) + \beta(t). \tag{4}$$

Let z the maximal solution of (4) such that $z(t_0) = y_0$ and η the maximal solution of (4) such that $\eta(t_0) = 0$. Using the fact,

- $\tilde{\alpha} \leq \alpha$
- $z(t_0) \geq \eta(t_0)$
- $y(t_0) = z(t_0)$

we can deduce by comparison, that

$$\eta(t) \leq z(t) \quad \forall t \geq t_0$$

and

$$y(t) \leq z(t) \quad \forall t \geq t_0.$$

Next, we prove that $\forall x, x' \in [0, a]$ such that $x \leq x'$

$$\tilde{\alpha}(x' - x) \leq \tilde{\alpha}(x') - \tilde{\alpha}(x). \tag{5}$$

First, the inequality (5) is trivial for $x = 0$. Now, for $x \neq 0$, we have by convexity of $\tilde{\alpha}$

$$\tilde{\alpha}(x) = \tilde{\alpha}(\frac{x}{x'}x' + (1 - \frac{x}{x'})0) \leq \frac{x}{x'}\tilde{\alpha}(x'),$$

and similarly, we get

$$\tilde{\alpha}(x' - x) \leq \frac{x' - x}{x'}\tilde{\alpha}(x'),$$

then

$$\tilde{\alpha}(x) + \tilde{\alpha}(x' - x) \leq \tilde{\alpha}(x'),$$

and finally

$$\tilde{\alpha}(x' - x) \leq \tilde{\alpha}(x') - \tilde{\alpha}(x).$$

By applying the inequality (5), we obtain

$$\dot{\widehat{(z-\eta)}}(t) = -(\tilde{\alpha}\,(z\,(t)) - \tilde{\alpha}\,(\eta\,(t))) \leq -\tilde{\alpha}(z\,(t) - \eta\,(t)).$$

Let $u = z - \eta \geq 0$ and $u\,(t_0) = z\,(t_0) - \eta\,(t_0) = y_0$, then we obtain

$$\dot{u} \leq -\tilde{\alpha}(u).$$

Now, taking the nonlinear differential equation

$$\dot{v} = -\tilde{\alpha}(v) \tag{6}$$

we denote v the maximal solution of (6) such that $v(t_0) = y_0$.
By comparison, we have

$$u(t) \leq v(t) \quad \forall t \geq t_0.$$

We consider for all $x \in (0, a)$,

$$\lambda(x) = -\int_b^x \frac{\mathrm{d}z}{\tilde{\alpha}(z)},$$

where b is an arbitrary constant in $(0, a)$. It is clear that λ is a differentiable and strictly decreasing on $(0, a)$. To prove that

$$\lim_{x \to 0^+} \lambda(x) = +\infty$$

we use the fact that $\tilde{\alpha}$ is locally Lipshitz, which implies that

$$-\tilde{\alpha}(x) = -\tilde{\alpha}(x) - (-\tilde{\alpha}(0)) = O(x - 0)[x \to 0]$$

and, consequently

$$\frac{1}{x} = O(\frac{1}{\tilde{\alpha}(x)})[x \to 0].$$

However, since $\displaystyle\int_0 \frac{\mathrm{d}x}{x} = +\infty$, then we obtain $\lambda(x) \xrightarrow[x \to 0^+]{} +\infty$.
Let

$$c := -\lim_{x \to a} \lambda(x) > 0.$$

Then the range of λ, and hence also the domain of λ^{-1}, is the open interval $(-c, +\infty)$.
If $y_0 > 0$, the function v satisfies

$$\lambda(v(t)) - \lambda(y_0) = t - t_0$$

hence

$$v(t) = \lambda^{-1}(\lambda(y_0) + t - t_0).$$

If $y_0 = 0$, then $v(t) = 0$ since $v = 0$ is an equilibrium point for the equation (6).
Define a function σ by:

$$\sigma(r, s) = \begin{cases} \lambda^{-1}(\lambda(r) + s) & r > 0 \\ 0 & r = 0 \end{cases}$$

It is easy to verify that σ is a class $\mathcal{KL}$-function on $[0, a) \times [0, +\infty)$, moreover

$$v(t) = \sigma(y_0, t - t_0) \quad \forall t \geq t_0.$$

Since $y(t) \leq z(t)$ and $u(t) = z(t) - \eta(t)$, then we get

$$\begin{aligned} y(t) &\leq u(t) + \eta(t) \\ &\leq v(t) + \eta(t) \\ &\leq \sigma(y_0, t - t_0) + \eta(t), \quad \forall t \in [t_0, +\infty). \end{aligned}$$

Let ε the maximal solution of the equation (4) such that $\varepsilon(0) = 0$. The functions ε and η are verifying the same equation and $\eta(t_0) = 0 \leq \varepsilon(t_0)$. By comparison, we obtain

$$\eta(t) \leq \varepsilon(t), \quad \forall t \geq t_0.$$

By using Lemma 2.2, we have

$$\lim_{t \to +\infty} \varepsilon(t) = 0.$$

In conclusion,

$$y(t) \leq \sigma(y_0, t - t_0) + \varepsilon(t), \quad \forall t \geq t_0.$$

The proof is complete. □

Next, we provide some sufficient conditions to obtain a new differential inequality which will be used to show the boundedness of solutions of some classes of perturbed systems.

Proposition 2.4. Let $\alpha > \beta \geq 1$ and $\gamma \geq 0$ be such that $\frac{\beta-1}{\alpha-1} < \frac{1}{1+\gamma}$, $H(t)$ a continuous function on $[0, +\infty)$ satisfying $\lim_{r \to +\infty} \frac{H(r)}{r} = 0$. Let ψ be a non-negative absolutely continuous function on $[0, \infty)$ which fulfills, for some $K \geq 0$, $Q \geq 0$, $\varepsilon_0 > 0$ and every $\varepsilon \in (0, \varepsilon_0]$, the differential inequality:

$$\psi'(t) + \varepsilon\psi(t) \leq K\varepsilon^{\alpha} H(\psi(t)) + \varepsilon^{-\gamma} q(t),$$

where q is any nonnegative function satisfying

$$\sup_{t \geq 0} \int_t^{t+1} q(y)\, \mathrm{d}y \leq Q.$$

Then, there exists $R_0 > 0$ with the following property: for every $R \geq 0$, there is $t_R \geq 0$, such that

$$\psi(t) \leq R_0, \forall t \geq t_R,$$

whenever $\psi(0) \leq R$. Both R_0 and t_R can be explicitly computed.

P r o o f. The hypothesis on q implies that, for any $t \geq 0$,

$$\int_t^{t+\tau} q(y)\, \mathrm{d}y \leq Q(1 + \tau), \forall \tau > 0.$$

Due to the assumptions on α, β, γ, we can select $\upsilon \in (0,1)$ satisfying the inequality:

$$1-\upsilon > \max\{\beta-\alpha\upsilon, \gamma\upsilon\}.$$

Calling $\omega = 1-\gamma\upsilon > \upsilon$, we consider the function

$$\jmath(r) = -\omega r^{1-\gamma\upsilon-\upsilon} + \omega K r^{-\gamma\upsilon-\upsilon} H(r).$$

As $\lim_{r\to\infty} \jmath(r) = -\infty$, we can choose $\varrho \geq \omega Q$, such that $\varrho^{-\frac{\upsilon}{\omega}} \leq \varepsilon_0$ and

$$\jmath(r) \leq -1-2\omega Q, \forall r \geq \varrho^{\frac{1}{\omega}}.$$

Then, we introduce the auxiliary function:

$$\varphi(t) = [\psi(t)]^{\omega}.$$

We preliminarily note that, for (almost) every t such that $\varphi(t) \geq \varrho$, we have

$$\varphi^{'}(t) \leq -1-2\omega Q + \omega q(t) \qquad (*)$$

Indeed, for (almost) any fixed t, setting $\varepsilon = [\varphi(t)]^{-\frac{\upsilon}{\omega}}$(note that $\varepsilon \leq \varepsilon_0$ when $\varphi(t) \geq \varrho$), the differential inequality reads,

$$\varphi^{'}(t) \leq \jmath([\varphi(t)]^{\frac{1}{\omega}}) + \omega q(t).$$

i) If $\varphi(t) \leq \varrho$ for some $t \geq 0$, then $\varphi(t+\tau) \leq 2\varrho$, for every $\tau \geq 0$. If not, let $\tau_1 > 0$ be such that $\varphi(t+\tau_1) > 2\varrho$, and set $\tau_0 = \sup\{\tau \in [0,\tau_1] : \varphi(t+\tau) \leq \varrho\}$. Integrating $(*)$ on $[t+\tau_0, t+\tau_1]$, we obtain the contradiction

$$2\varrho < \varphi(t+\tau_1) \leq \varrho - (\tau_1-\tau_0) - 2\omega Q(\tau_1-\tau_0) + \omega Q(1+\tau_1-\tau_0) < 2\varrho.$$

ii) If $\varphi(0) > \varrho$, then $\varphi(t_*) \leq \varrho$, for some $t_* \leq \varphi(0)(1+\omega Q)^{-1}$. Indeed let $t > 0$ be such that $\varphi(\tau) > \varrho$ for all $\tau \in [0,t]$. Integrating $(*)$ on $[0,t]$, we are led to

$$\varrho < \varphi(t) \leq \varphi(0) - t - 2\omega Q t + \omega Q(1+t) \leq \varphi(0) - (1+\omega Q)t + \varrho.$$

Therefore, it must be $t < \varphi(0)(1+\omega Q)^{-1}$.

In order to come back to the original $\psi(t)$, just define

$$R_0 = (2\varrho)^{\frac{1}{\omega}} \text{ and } t_R = R^{\frac{1}{\omega}}(1+\omega Q)^{-1}.$$

By applying the results of [23], the proof follows. □

Remark 2.5. We get the same result if we suppose that

$$\lim_{r\to\infty} \frac{H(r)}{r} = l$$

where l is such that,

$$-\omega + \omega K l < 0.$$

Proposition 2.6. Let $\alpha > \beta \geq 1$ and $\gamma \geq 0$ be such that $\frac{\beta-1}{\alpha-1} < \frac{1}{1+\gamma}$, $H(t)$ a continuous function on $[0, \infty)$ satisfying $\lim_{r\to+\infty} \frac{H(r)}{r} = +\infty$. Let ψ be a non-negative absolutely continuous function on $[0, +\infty)$ which fulfills, for some $K \geq 0$, $Q \geq 0$, $\varepsilon_0 > 0$ and every $\varepsilon \in (0, \varepsilon_0]$, the differential inequality:

$$\psi^{'}(t) + \varepsilon H(\psi(t)) \leq K\varepsilon^{\alpha}[\psi(t)]^{\beta} + \varepsilon^{-\gamma} q(t),$$

where q is any non-negative function satisfying

$$\sup_{t\geq 0} \int_{t}^{t+1} q(y)\,\mathrm{d}y \leq Q.$$

Then, there exists $R_0 > 0$ with the following property: for every $R \geq 0$, there is $t_R \geq 0$, such that

$$\psi(t) \leq R_0, \forall t \geq t_R,$$

whenever $\psi(0) \leq R$. Both R_0 and t_R can be explicitly computed.

Proof. Indeed, we can keep the same proof as above, just take

$$\jmath(r) = -\omega r^{-\gamma\upsilon-\upsilon} H(r) + \omega K r^{\beta-\alpha\upsilon-\gamma\upsilon}.$$

□

We will apply in the next section the results established in this section to obtain estimates for the solutions of perturbed differential equations in finite dimensional and to get uniform boundedness and uniform convergence to a small neighborhood of the origin. The idea is to use differential inequalities which are very convenient for obtaining bounds on the solutions of nonlinear systems. The inequality is obtained by differentiating scalar valued Lyapunov function associated to the nominal system along the solutions of the system in presence of the term of perturbation.

2.2. Stability analysis

Definition 2.7. Let $V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ a Lyapunov function:

$$\begin{cases} V(t,0) = 0, & \forall t \geq 0; \\ V(t,x) > 0, & \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \setminus \{0\}. \end{cases}$$

(*i*) $V(t,x)$ is positive definite, i. e. there exists a continuous, non-decreasing scalar function $\alpha(x)$ such that $\alpha(0) = 0$ and

$$0 < \alpha(\|x\|) < V(t,x), \quad \forall x \neq 0.$$

(*ii*) $\dot{V}(t,x)$ is negative definite, that is,

$$\dot{V}(t,x) \leq -\gamma(\|x\|) < 0$$

where γ is a continuous non-decreasing scalar function such that $\gamma(0) = 0$.

(iii) $V(t,x) \leq \beta(\|x\|)$ where β is a continuous non-decreasing function and $\beta(0) = 0$, i. e. V is decrescent, i. e. the Lyapunov function is upper bounded.

(iv) V is radially unbounded, that is $\alpha(\|x\|) \to \infty$ as $\|x\| \to \infty$.

We need also the definitions given below.

Definition 2.8. (uniform stability of B_r)

1. The ball B_r is said to be uniformly stable if for all $\varepsilon > r$, there exits $\delta := \delta(\varepsilon)$ such that for all $t_0 \geq 0$
$$\|x_0\| < \delta \implies \|x(t)\| < \varepsilon, \ \forall t \geq t_0. \tag{7}$$
2. B_r is globally uniformly stable if it is uniformly stable and the solutions of system (1) are globally uniformly bounded.

Definition 2.9. (uniform attractivity of B_r) B_r is globally uniformly attractive if for all $\varepsilon > r$ and $c > 0$, there exists $T =: T(\varepsilon, c) > 0$ such that for all $t_0 \geq 0$,
$$\|x(t)\| < \varepsilon, \forall t \geq t_0 + T, \ \|x_0\| < c.$$

Sufficient condition for global uniform asymptotic stability is characterized by the existence a class $\mathcal{KL}$-function β and a constant $r > 0$ such that, given any initial state x_0, ensuing trajectory $x(t)$ satisfies
$$\|x(t)\| \quad \leq \quad \beta(\|x_0\|, t) + r, \quad \forall t \geq 0. \tag{8}$$
If the class $\mathcal{KL}$-function β on the above relation (8) is of the form $\beta(r,s) = kre^{-\lambda t}$, with $\lambda, k > 0$ we say that the ball B_r is globally uniformly exponentially stable. It is also, worth to notice that if, in the above definitions, we take $r = 0$, then one deals with the standard concept of GUAS and GUES of the origin (see [11] for more details). Moreover, in the rest of this paper, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq\| x(t) \| -r$, so that if $r = 0$ we find the classical definition of the uniform asymptotic stability of the origin viewed as an equilibrium point. For the class of systems that can be modeled by (1), the uniform exponential stability of a neighborhood of the origin can be established by requiring the existence of a Lyapunov function that satisfies certain conditions ([3]).

Theorem 2.10. Let $x = 0$ be an equilibrium point for the system:
$$\dot{x} = f(t,x) \tag{9}$$
where $f : [0, +\infty) \times D \to \mathbb{R}^n$ is continuous in (t,x) and locally Lipschitz in x and $D \subset \mathbb{R}^n$ is a domain that contains $x = 0$. We suppose that there exists a function $V : [0, +\infty) \times D \to \mathbb{R}$ continuously differentiable, such that $\forall t \geq 0$, $\forall x \in D$
$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) \tag{10}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(\|x\|) + \beta\left(t\right). \tag{11}$$

where α_1, α_2, α_3 are class $\mathcal{K}$-functions on $[0,r]$, r is a positive constant which is chosen such that $B_r \subset D$ and β is a continuous function verifying

$$\lim_{t\to+\infty} \beta(t) = 0 \quad \text{and} \quad \|\beta\|_\infty < \alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(r).$$

Then, for all $t_0 \in \mathbb{R}_+$ and $\|x_0\| < \alpha_2^{-1} \circ \alpha_1(r)$, the maximal solution x of the system (9) such that $x(t_0) = x_0$, is defined on $[t_0, +\infty)$ and satisfies,

$$\|x(t)\| \leq \sigma(\|x_0\|, t - t_0) + \varepsilon\left(t\right), \quad \forall t \geq t_0$$

where σ is a $\mathcal{KL}$-function on $[0,r] \times [0,+\infty)$ and ε is a continuous function on $[0,+\infty)$ such that

$$\lim_{t\to+\infty} \varepsilon(t) = 0.$$

P r o o f. For $r \in \mathbb{R}_+^*$, we put

$$\xi_0 = \alpha_3^{-1}(\|\beta\|_\infty).$$

Then,

$$0 \leq \|\beta\|_\infty < \alpha_3 \circ \alpha_2^{-1} \circ \alpha_1(r) \Longrightarrow \xi_0 < \alpha_2^{-1} \circ \alpha_1(r) \Longrightarrow \alpha_2(\xi_0) < \alpha_1(r).$$

On the other hand,

$$\alpha_1(r) \leq \alpha_2(r) \Longrightarrow \alpha_2^{-1} \circ \alpha_1(r) \leq r \Longrightarrow 0 \leq \xi_0 < r.$$

Let $x_0 \in D$ such that

$$\|x_0\| < \alpha_2^{-1} \circ \alpha_1(r),$$

then,

$$\|x_0\| < r.$$

Now, we consider the maximal solution x of the system (9), verifying $x(t_0) = x_0$. We prove that x is defined on $[t_0, +\infty)$ and $\|x(t)\| \leq r, \quad \forall t \geq t_0$.

We distinguish two cases:

Case 1. $\|x_0\| \leq \xi_0$
Suppose that we can find $t' \geq t_0$ such that $\|x(t')\| > r$. We denote

$$t_1 = \min\left\{t \geq t_0, \|x(t)\| = r\right\}$$

and

$$t_2 = \max\left\{t \in [t_0, t_1], \|x(t)\| = \xi_0\right\}.$$

It is clear that $t_2 < t_1$ and

$$\forall t \in [t_2, t_1], \quad \xi_0 \leq \|x(t)\| \leq r.$$

Since,

$$\frac{d(V(s,x(s))}{ds}\mid_{s=t}\leq -\alpha_3(\|x(t)\|)+\|\beta\|_\infty\leq 0,$$

the function $t\mapsto V(t,x(t))$ is decreasing on $[t_2,t_1]$ which implies that

$$V\left(t_1,x(t_1)\right)\leq V\left(t_2,x(t_2)\right)$$

and

$$V\left(t_2,x(t_2)\right)\leq \alpha_2(\|x(t_2)\|)=\alpha_2(\xi_0)$$

$$V\left(t_1,x(t_1)\right)\geq \alpha_1(\|x(t_1)\|)=\alpha_1(r)$$

which is in contradiction with

$$\alpha_2(\xi_0)<\alpha_1(r).$$

Consequently $\|x(t)\|\leq r,\quad \forall t\in[t_0,T^*)$, which means that x remains within the compact B_r of D and thus

$$T^*=+\infty.$$

Case 2. $\|x_0\|>\xi_0$
We suppose that there exists $T\geq t_0$ such that $\|x(T)\|>r$ and we consider

$$t_1=\min\left\{t\geq t_0,\|x(t)\|=r\right\}.$$

• If $\exists s\in[t_0,t_1]$, $\|x(s)\|\leq\xi_0$. Let

$$t_2=\max\left\{t\in[t_0,t_1],\|x(t)\|=\xi_0\right\}.$$

Moreover, $\forall t\in[t_2,t_1]\quad \xi_0\leq\|x(t)\|\leq r$. Then $t\mapsto V(t,x(t))$ is a decreasing function and thus $V(t_1,x(t_1))\leq V(t_2,x(t_2))$, however

$$V\left(t_2,x(t_2)\right)\leq \alpha_2(\|x(t_2)\|)=\alpha_2(\xi_0)$$

$$V\left(t_1,x(t_1)\right)\geq \alpha_1(\|x(t_1)\|)=\alpha_1(r)$$

which is in contradiction with

$$\alpha_2(\xi_0)<\alpha_1(r).$$

• If $\forall t\in[t_0,t_1]$, $\|x(t)\|\ >\xi_0$. Using the fact $\|x_0\|<r$, we obtain

$$\xi_0<\|x(t)\|\leq r,\quad \forall t\in[t_0,t_1].$$

Then, $t\mapsto V(t,x(t))$ is a decreasing function on $[t_0,t_1]$, and consequently

$$V(t_1,x(t_1))\leq V(t_0,x(t_0)).$$

However

$$V\left(t_0,x(t_0)\right)\leq \alpha_2(\|x(t_0)\|)$$

and

$$\|x(t_0)\|<\alpha_2^{-1}(\alpha_1(r))$$

then $V(t_0, x(t_0)) < \alpha_1(r)$.

Since

$$V(t_1, x(t_1)) \geq \alpha_1(\|x(t_1)\|) = \alpha_1(r),$$

which is impossible. Consequently, we conclude that

$$\forall t \in [t_0, T^*), \quad \|x(t)\| \leq r,$$

which means that x remains within the compact B_r of D and thus

$$T^* = +\infty.$$

Next, the functions V and $\dot{V}$ verify

$$\forall t \geq t_0, \quad V(t, x(t)) \leq \alpha_2(\|x(t)\|)$$

and

$$\dot{V}(t, x(t)) \leq -\alpha_3(\|x(t)\|) + \beta(t).$$

Consequently

$$\dot{V} \leq -\alpha(V) + \beta(t)$$

where $\alpha = \alpha_3 \circ \alpha_2^{-1}$ is a class $\mathcal{K}$-function on $[0, \alpha_2(r)]$. Let $d > 0$ such that

$$\alpha(\alpha_2(r) - d) = \frac{\|\beta\|_\infty + \alpha_3(r)}{2},$$

and we define γ by

$$\gamma(u) = \frac{1}{d} \int_{\max(u-d,0)}^{u} \alpha(s)\,\mathrm{d}s \qquad \forall u \in [0, \alpha_2(r)].$$

It is easy to verify that γ is a class $\mathcal{K}$-function and locally Lipschitz on $[0, \alpha_2(r)]$ and satisfies

$$\dot{V} \leq -\gamma(V) + \beta(t).$$

Let y the maximal solution of the equation:

$$\dot{y} = -\gamma(y) + |\beta(t)|$$

and

$$y(t_0) = V(t_0, x_0).$$

It is obvious that $y(t_0) \in [0, \alpha_2(r))$ and $\|\beta\|_\infty < \gamma(\alpha_2(r))$. By using Lemma 2.2, we see that there is a class $\mathcal{KL}$-function $\sigma(.,.)$ defined on $[0, \alpha_2(r)) \times [0, +\infty)$ and verifies

$$y(t) \leq \sigma(y(t_0),\ t - t_0) + \delta(t), \quad \forall t \geq t_0$$

where δ is a continuous function on $[0, +\infty)$ such that $\lim_{+\infty} \delta = 0$. By comparison

$$\forall t \geq t_0, \quad V(t, x(t)) \leq y(t).$$

Now, we consider a function $\widetilde{\alpha}_1$ which is $\mathcal{K}_\infty$ and satisfies

$$\widetilde{\alpha}_1(u) = \alpha_1(u) \quad \forall u \in [0, r]$$

Then

$$\begin{aligned} \|x(t)\| &\leq \widetilde{\alpha}_1^{-1}(V(t, x(t))) \\ &\leq \widetilde{\alpha}_1^{-1}(\sigma(V(t_0, x_0), t - t_0) + \delta(t)) \\ &\leq \widetilde{\alpha}_1^{-1}(\sigma(\alpha_2(\|x_0\|), t - t_0) + \delta(t)) \\ &\leq \widetilde{\alpha}_1^{-1}(2\sigma(\alpha_2(\|x_0\|), t - t_0)) + \widetilde{\alpha}_1^{-1}(2\delta(t)). \end{aligned}$$

We put $\forall (a, b) \in [0, r) \times [0, +\infty)$.

$$\phi(a, b) = \widetilde{\alpha}_1^{-1}(2\sigma(\alpha_2(a), b))$$

and

$$\theta(b) = \widetilde{\alpha}_1^{-1}(2\delta(b)).$$

ϕ is a class $\mathcal{KL}$-function on $[0, r) \times [0, +\infty)$ and θ is a continuous function on $[0, +\infty)$ such that $\lim_{t \to +\infty} \theta = 0$. Finally, we obtain

$$\|x(t)\| \leq \phi(\|x_0\|, t - t_0) + \theta(t) \quad \forall t \geq t_0.$$

□

Now, the following corollary is an immediate consequence of Theorem 2.10.

Corollary 2.11. Let $x = 0$ be an equilibrium point for the system:

$$\dot{x} = f(t, x) \tag{12}$$

where $f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (t, x) and locally Lipschitz in x. We suppose that there exist a function $V : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ continuously differentiable such that for all $t \geq 0$ and all $x \in \mathbb{R}^n$

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{13}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x\|) + \beta(t), \tag{14}$$

where α_1, α_2, α_3 are class $\mathcal{K}_\infty$-functions, and β is a continuous function verifying

$$\lim_{t \to +\infty} \beta(t) = 0.$$

Then, for all $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$, the maximal solution x of the system (12) such that $x(t_0) = x_0$, is defined on $[t_0, +\infty)$ and satisfies:

$$\|x(t)\| \leq \sigma(\|x_0\|, t - t_0) + \varepsilon(t), \quad \forall t \geq t_0, \tag{15}$$

where σ is a class $\mathcal{KL}$-function on $[0, +\infty) \times [0, +\infty)$ and ε is a continuous function on $[0, +\infty)$ such that

$$\lim_{t \to +\infty} \varepsilon(t) = 0.$$

The estimation (15) implies that the system (12) is globally uniformly practically asymptotically stable in the sense that the ball $B_{\|\varepsilon\|_\infty}$ is globally uniformly asymptotically stable.

2.3. Example 1

We consider the following scalar equation which can be viewed as a perturbed systems:

$$\dot{x} = -x\left(1 - \frac{2\sin^2 t}{1+(tx)^2}\right), \quad x(t_0) = x_0 \tag{16}$$

with $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}$. It is known that the maximal solution x of (16) is defined on $[t_0, +\infty)$. We want to provide estimates of $|x(t)|$. To do this, we consider

$$f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}, \quad (t,x) \longmapsto -x\left(1 - \frac{2\sin^2 t}{1+(tx)^2}\right)$$

and

$$V(t,x) = \frac{1}{2}x^2, \quad \forall (t,x) \in [0,+\infty) \times \mathbb{R}.$$

The derivative of V along the trajectories of the system (16) is given by

$$\begin{aligned} \dot{V}(t,x) &= -x^2\left(1 - \frac{2\sin^2 t}{1+(tx)^2}\right) \\ &\leq -x^2 + \frac{2(tx)^2}{1+(tx)^2}\frac{\sin^2(t)}{t^2} \\ &\leq -x^2 + 2\left(\frac{\sin t}{t}\right)^2. \end{aligned}$$

We see that α_1, α_2 and α_3 can be taken as

$$\alpha_1(x) = \alpha_2(x) = \frac{1}{2}x^2,$$

$$\alpha_3(x) = x^2, \quad \forall x \in \mathbb{R}$$

and

$$\beta(t) = 2\left(\frac{\sin t}{t}\right)^2, \quad \forall t \geq 0.$$

It is clear that

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$$

and

$$\dot{V}(t,x) \leq -\alpha_3(|x|) + \beta(t).$$

Using Corollary 2.11, we obtain the following estimation of $|x(t)|$:

$$\forall t \geq t_0, \quad |x(t)| \leq \sigma(|x_0|, t-t_0) + \varepsilon(t)$$

where σ is a class $\mathcal{KL}$-function on $[0,+\infty) \times [0,+\infty)$ and ε is a continuous function on $[0,+\infty)$ such that $\lim_{t\to+\infty} \varepsilon(t) = 0$. To make explicit the functions σ and ε, we write:

$$\begin{aligned} \dot{x} &= -x + \frac{2x\sin^2(t)}{1+(tx)^2} \\ &= -x + \frac{2tx\sin t}{1+(tx)^2}\frac{\sin t}{t} \\ &\leq -x + \left|\frac{\sin t}{t}\right|. \end{aligned}$$

Consider the equation

$$\dot{y} = -y + \mid \frac{\sin t}{t} \mid, \quad y(t_0) = x_0.$$

By using the variation of constants formula, we can write

$$y(t) = y(t_0)e^{-(t-t_0)} + \left(\int_{t_0}^{t} e^u \mid \frac{\sin u}{u} \mid \mathrm{d}u \right) e^{-t},$$

by comparison, we obtain

$$x(t) \leq y(t), \quad \forall t \geq t_0$$

which implies that

$$\forall t \geq t_0, \quad |x(t)| \leq \sigma(|x_0|, t - t_0) \ + \varepsilon(t)$$

where

$$\sigma(a, b) = ae^{-b}, \quad \forall a, b \geq 0$$

and

$$\varepsilon(t) = \left(\int_{0}^{t} e^u \mid \frac{\sin u}{u} \mid \mathrm{d}u \right) e^{-t}, \quad \forall t \geq 0.$$

It is clear that σ is a class $\mathcal{KL}$-function on $[0, +\infty) \times [0, +\infty)$. to prove that

$$\lim_{t \to +\infty} \varepsilon(t) = 0,$$

we write

$$\varepsilon(t) = \frac{N(t)}{D(t)}$$

where $N(t) = \int_0^t e^s \mid \frac{\sin s}{s} \mid \mathrm{d}s$ and $D(t) = e^t$. Since $D'(t) \neq 0$, $\lim_{t \to +\infty} D(t) = +\infty$ and

$$\frac{N'(t)}{D'(t)} = \mid \frac{\sin t}{t} \mid \xrightarrow[t \to +\infty]{} 0$$

we obtain, by using Hospital's-Rule:

$$\lim_{t \to +\infty} \varepsilon(t) = 0.$$

Moreover, we have

$$\mid \varepsilon(t) \mid \leq 1, \quad \forall t \geq 0.$$

We can obtain an estimation on the solutions as in (15), by application of corollary 2.11, the system (16) is globally uniformly "practically" asymptotically stable in the sense that the ball of radius 1, B_1 is globally uniformly asymptotically stable. Moreover, we see that $\lim_{t \to +\infty} x(t) = 0$ which implies that the origin $x = 0$ is an attractive equilibrium point for the system (16), see Figure 1.

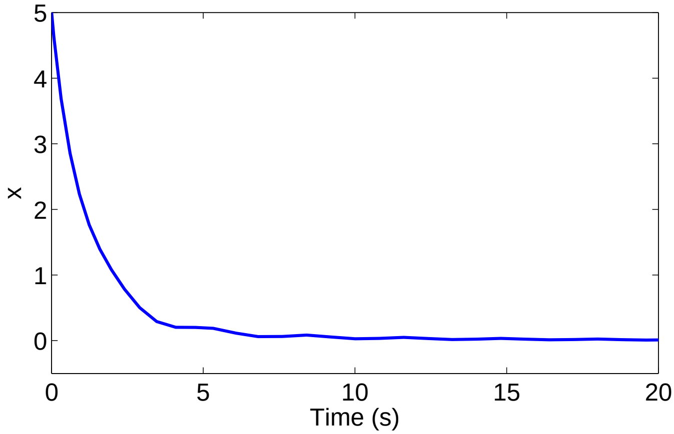

Fig. 1. Time evolution of the state $x(s)$ of system (16).

2.4. Example 2

We consider the nonlinear perturbed system:

$$\dot{x} = f(t,x) + g(t,x)$$

where $f(t,x)=f(t,(x_1,x_2))=\begin{pmatrix} -x_1 \\ -x_2. \end{pmatrix}$

We consider $V(t,x) = x_1^2 + x_2^2$ as a Lyapunov function candidate which satisfies by considering the derivative along the trajectories:

$$\dot{V}(t,x) \leq -2\|x\|^2,\ \|\frac{\partial V}{\partial x}(t,x)\| \leq 2\|x\|.$$

If we take

$$g(t,x) = \begin{cases} \varepsilon^3 x_1^2 + q \\ \\ \frac{1}{2}(1-\varepsilon)\|x\| \end{cases}$$

where $\varepsilon \in]0,1[$, q is a a function chosen as in Proposition 2.6, g satisfies

$$\|g(t,x)\| \leq \frac{1}{2}(1-\varepsilon)\|x\| + \varepsilon^{\alpha}\|x\|^2 + q.$$

Hence, by using Proposition 2.6, the solutions are uniformly bounded.

If we take

$$g(t,x) = \begin{cases} \varepsilon q \arctan\|x\| \\ \\ \frac{1}{2}(1-\varepsilon)\|x\| \end{cases}$$

$\varepsilon \in]0,1[$, which satisfies

$$\|g(t,x)\| \leq \frac{1}{2}(1-\varepsilon)\|x\| + \varepsilon q H(\|x\|),$$

then by using Proposition 2.6, the solutions are uniformly bounded.

Next, we will study the practical exponential stabilization problem of a class of nonlinear control systems by means of a state feecback law. It is worth to notice that the origin is not required to be an equilibrium point for this system, this may be in many situations meaningful from a practical point of view specially, when stability for control systems is investigated.

2.5. Application to control nonlinear systems

An interesting field where the previous theorems can be applied is uncertain systems which can considered as perturbed systems. Consider an uncertain system described by

$$\begin{aligned} \dot{x} &= G(t,x) + F(t,x).\Delta\phi\,(u_1,u_2,...,u_m,x,t) \\ x(t_0) &= x_0 \end{aligned} \tag{17}$$

where $x \in \mathbb{R}^n$ is the state vector, $u = (u_1,u_2,...,u_m) \in \mathbb{R}^m$ is the input vector and x_0 is a given initial state. The functions $\Delta\phi : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{R}^m$, $F : \mathbb{R}. \times \mathbb{R}_+ \longrightarrow \mathbb{R}^{n\times m}$, and $G : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ are assumed to be continuous and locally Lipschitz. Before presenting our main result, we make some assumptions as follows.

(A_1) The components of the control vector are physically limited by

$$|u_i| \ \leq \ k_i, \quad \forall i = 1,2,...m \tag{18}$$

with $k_i > 0$, for all i.

(A_2) There exists a sufficiently smooth function $V(t,x)$ and positive constants $\lambda_1,\ \lambda_2,\ \lambda_3,\ p$ and c, such that for all $x \in \mathbb{R}^n, t \geq t_0$

$$\lambda_1\|x\|^p \ \leq \ V(t,x) \ \leq \ \lambda_2\|x\|^p \tag{19a}$$

$$\nabla_t V + \nabla_x^\top V G(t,x) \ \leq \ -\lambda_3 V(t,x) + c. \tag{19b}$$

(A_3) There exist positive continuous functions $f_1(t,x),\ f_2(t,x)$ and $f_3(t,x)$ satisfying, for all $y \in \mathbb{R}^m, x \in \mathbb{R}^n, t \geq t_0$

$$y^\top.\Delta\phi_1\,(y,x,t) \ \geq \ -f_1(t,x)\|y\| + f_2(t,x)\|y\|^2 - f_3(t,x)\|y\|^3 \tag{20a}$$

$$f_2^2(t,x) \ \geq \ 4f_1(t,x)f_3(t,x) \tag{20b}$$

$$\Delta\phi_1\,(y,x,t) \ =: \ \Delta\phi\left(\frac{2k_1}{\pi}\arctan(y_1),\cdots,\frac{2k_m}{\pi}\arctan(y_m),x,t\right) \tag{20c}$$

Remark 2.12. Note that, by using assumption (A_2) and because of the presence of the constant c, the solutions of the nominal system $\dot{x}(t) = G(x,t)$ converge to a small ball centered at the origin.

Now, we show that system (17) can be globally asymptotically stabilized by a state feedback law, in the sense that the tragectories of the closed-loop system converge to a small ball centered at the origin. We have the following theorem.

Theorem 2.13. Under assumptions $(A_1), (A_2)$ and (A_3), system (17) is globally exponentially "practically" stabilizable, under the control

$$u_i(t) = \frac{2k_i}{\pi}\arctan(y_i(t)), \quad \forall i = 1, \cdots, m. \tag{21a}$$

$$\big(y_1(t), \quad \cdots, \quad y_m(t)\big)^\top = -\rho(x,t)^\top K(x,t) \tag{21b}$$

$$\rho(x,t) =: \frac{f_2(x,t)}{2f_3(x,t)\left(\|\nabla_x^\top V(x,t)F(x,t)\| + \eta f_1^{-1}(x,t)\right)} \tag{21c}$$

where $K(x,t) = F^\top(x,t)\nabla_x V(x,t)$ and η is positive constant.

P r o o f. Inserting the control u into equations (17), yields

$$\begin{aligned} \dot{x} &= G(t,x) + F(t,x).\Delta\phi\,(u_1, u_2, ..., u_m, x, t) \\ &= G(t,x) + F(t,x).\Delta\phi\left(\tfrac{2k_1}{\pi}\arctan(y_1), \cdots, \tfrac{2k_m}{\pi}\arctan(y_m), x, t\right) \\ &= G(t,x) + F(t,x).\Delta\phi_1\,(y,x,t)\,. \end{aligned} \tag{22}$$

The time derivative of V along the trajectories of the closed-loop system is bounded by

$$\begin{aligned} \dot{V} &= \nabla_t V + \nabla_x^\top V\,(G(t,x) + F(t,x).\Delta\phi_1\,(y,x,t)) \\ &\leq -\lambda_3\,(V(t,x) + c) + \nabla_x^\top V F.\Delta\phi_1. \end{aligned} \tag{23}$$

From (20a) and (21b), we have

$$-\rho K^\top.\Delta\phi_1 \quad \geq \quad -f_1\rho\|K\| + \rho^2 f_2\|K\|^2 - \rho^3 f_3\|K\|^3 \tag{24}$$

which implies, since ρ does not vanish everywhere,

$$K^\top.\Delta\phi_1 \quad \leq \quad f_1\|K\| - \rho f_2\|K\|^2 + \rho^2 f_3\|K\|^3. \tag{25}$$

We obtain

$$\begin{aligned} \dot{V} &= \nabla_t V + \nabla_x^\top V\,(G(t,x) + F(t,x).\Delta\phi_1\,(y,x,t)) \\ &\leq -\lambda_3 V(t,x) + c + f_1\|K\| - \rho f_2\|K\|^2 + \rho^2 f_3\|K\|^3 \\ &\leq -\lambda_3 V(t,x) + c + f_1\|K\| - \tfrac{f_2^2\|K\|^2}{2f_3\left(\|K\|+\eta f_1^{-1}\right)} + \tfrac{f_2^2\|K\|^3}{4f_3\left(\|K\|+\eta f_1^{-1}\right)^2}. \end{aligned} \tag{26}$$

After adding and subtracting η to righthand side of the expression above, we obtain

$$\begin{aligned} l\dot{V} &\leq -\lambda_3 V(t,x)+c+\eta+\frac{f_2^2\|K\|^3-2f_2^2\|K\|^2\left(\|K\|+\eta f_1^{-1}\right)+4f_1f_3\|K\|\left(\|K\|+\eta f_1^{-1}\right)^2-4\eta f_3\left(\|K\|+\eta f_1^{-1}\right)^2}{4f_3\left(\|K\|+\eta f_1^{-1}\right)^2} \\ &\leq -\lambda_3 V(t,x)+c+\eta+\frac{-\|K\|^3\left(f_2^2-4f_1f_3\right)-2\eta f_1^{-1}\|K\|^2\left(f_2^2-4f_1f_3\right)+4f_3\eta\left(\eta\|K\|f_1^{-1}-\left(\|K\|+\eta f_1^{-1}\right)^2\right)}{4f_3\left(\|K\|+\eta f_1^{-1}\right)^2} \end{aligned} \tag{27}$$

By keeping in mind that the term $f_2^2-4f_1f_3$ is positive, it yields by completing the squares

$$\begin{aligned} \dot{V} &\leq -\lambda_3 V(t,x)+c+\eta+\frac{\eta\left(\eta\|K\|f_1^{-1}-\left(\|K\|+\eta f_1^{-1}\right)^2\right)}{\left(\|K\|+\eta f_1^{-1}\right)^2} \\ &\leq -\lambda_3 V(t,x)+c+\eta-\frac{\eta\left(\left(\|K\|+\frac{1}{2}\eta f_1^{-1}\right)^2-\frac{3}{4}\|K\|^2\right)}{\left(\|K\|+\eta f_1^{-1}\right)^2} \\ &\leq -\lambda_3 V(t,x)+c+\eta. \end{aligned} \tag{28}$$

By applying Theorem 2.13, we have

i. if $p \geq 1$, the ball B_{α_1}, where $\alpha_1 = \left(\frac{c+\eta}{\lambda_1\lambda_3}\right)^{\frac{1}{p}}$, is globally uniformly exponentially stable.

ii. if $p < 1$, the ball B_{α_1}, where $\alpha_1 = 2^{\frac{1}{p}-1}\left(\frac{c+\eta}{\lambda_1\lambda_3}\right)^{\frac{1}{c}}$, is globally uniformly exponentially stable.

This completes the proof. □

Note that, if we take c and η depending on t, $c =: c(t)$ and $\eta =: \eta(t)$ where the functions $c(t)$ and $\eta(t)$ tend to zero when t tends to $+\infty$, one gets α_1 tends to zero and the solutions of the system converge to the origin when t tends to $+\infty$ in the two cases $p \leq 1$ and $p > 1$.

2.6. Example 3

We present now an application to control system that implement the previous theorem. Consider the following uncertain nonlinear system:

$$\dot{x} = G(t,x) + F(t,x)\triangle\phi(t,x,u) \tag{29}$$

where

$$\begin{aligned} G(t,x) &= \begin{pmatrix} -x_1 + \frac{x_1}{1+x_1^2}\exp(-x_1^2) \\ -x_2 + \exp(-x_2) \end{pmatrix} \\ F(t,x) &= \begin{pmatrix} -x_2 \\ x_1^2 \end{pmatrix} \\ \triangle\phi(t,x,u) &= \left(a(t)x_1^2 + b(t)u + c(t)u^2 + (d(t)|x_1| + e(t))\tan(u)\right) \end{aligned} \tag{30}$$

where $u \in \mathbb{R}, x = (x_1, x_2)^\top \in \mathbb{R}^2$,

$$a(t) \in [-2,2],\ b(t),\ c(t) \in [-1,1]$$

and

$$d(t) \in [3,4],\ e(t) \in [5,7]$$

for all $t \geq t_0$. The control u is limited by

$$a(t) \in [-\frac{\pi}{2},\ \frac{\pi}{2}].$$

Choosing the quadratic Lyapunov function

$$V(t,x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2.$$

The time derivative of V along the trajectories of the nominal system $\dot{x} = F(x,t)$ is bounded by

$$\dot{V}(t,x) \quad \leq \quad -2V(t,x) + 1 + \tfrac{1}{e}. \tag{31}$$

Then (19) is satisfied with $\lambda_1 = \lambda_2 = \frac{1}{2}$, $\lambda_3 = 2$, $p = 2$ and $c = 1 + \frac{1}{e}$.

From (20c), we have

$$\begin{aligned} \Delta\phi_1(y,x,t) &= \Delta\phi(\arctan(y),x,t) \\ &= a(t)x_1^2 + b(t)\arctan(y) + c(t)\arctan^2(y) + (d(t)|x_1| + e(t))y. \end{aligned} \tag{32}$$

It yields

$$y^\top.\Delta\phi_1(y,x,t) \geq -\left(2x_1^2 + \tfrac{\pi}{2} + \tfrac{\pi^2}{4}\right)|y|^2 + (3|x_1| + 5)\,|y|^2. \tag{33}$$

This suggests to take

$$f_1(x,t) = 2x_1^2 + \frac{\pi}{2} + \frac{\pi^2}{4},\ f_2(x,t) = 3|x_1| + 5,\ f_3(x,t) = \frac{1}{2}. \tag{34}$$

It is clear that $f_2^2 - 4f_1f_3$ is positive. Pick $\eta = \frac{1}{2}$. Easy computation shows that

$$K(x,t) = -x_1x_2 - x_2x_1^2$$

and

$$\rho(x,t) = \frac{2\,(3|x_1| + 5)\left(2x_1^2 + \frac{\pi}{2} + \frac{\pi^2}{4}\right)}{\left(2\left(2x_1^2 + \frac{\pi}{2} + \frac{\pi^2}{4}\right)|x_1x_2 + x_2x_1^2| + 1\right)}. \tag{35}$$

Finally, one has

$$u(t) = \arctan\left(-\frac{2\left|x_1x_2 + x_2x_1^2\right|(3|x_1| + 5)\left(2x_1^2 + \frac{\pi}{2} + \frac{\pi^2}{4}\right)}{\left(2\left(2x_1^2 + \frac{\pi}{2} + \frac{\pi^2}{4}\right)|x_1x_2 + x_2x_1^2| + 1\right)}\right). \tag{36}$$

According to Theorem 2.13, we conclude that the solutions of system (29) in closed-loop with control (36) converge exponentially to the closed ball B_α where we put $\alpha = \left(\frac{3e+1}{2}\right)^{\frac{1}{2}}$. Simulations results are performed with the parameters $a(t) = 2$, $b(t) = c(t) = 1$, $d(t) = 3$, $e(t) = 5$, and $x_1(0) = -2; x_2(0) = 0$; the state trajectories of the feedback controlled system are bounded for t large enough.

3. CONCLUSION

In this paper some sufficient conditions are given to ensure the global practical uniform asymptotic stability of certain time-varying systems. These conditions are expressed as relation between the Lyapunov function and the existence of a class $\mathcal{KL}$-function which appear in our analysis through the solution of a time varying perturbed scalar differential equation, where the second term goes to infinity as t goes to infinity. We have proven that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. The idea is to use differential inequalities which are very convenient for obtaining bounds on the solutions of some nonlinear perturbed systems by differentiating scalar valued Lyapunov functions along the solutions. Some examples and simulations results are given to illustrate the applicability of the main results. Finally, the current obtained results can be extended to some others dynamical models such the case of multi-layer time-varying complex networks. The problem of uniform stability of certain dynamical network can be therefore investigated. It will be the future prospects and interesting issues.

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