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# ON WEIGHTED U-STATISTICS FOR STATIONARY RANDOM FIELDS

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The aim of this paper is to introduce a central limit theorem and an invariance principle for weighted U-statistics based on stationary random fields. Hsing and Wu (2004) in their paper introduced some asymptotic results for weighted U-statistics based on stationary processes. We show that it is possible also to extend their results for weighted $U$-statistics based on stationary random fields.

*Keywords:* limit theorem, U-statistics, random fields

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## 1. INTRODUCTION

The aim of this paper is to present some new results on U-statistics based on weakly dependent random fields. It is well-known that many important test statistics can be written as $U$-statistics. Therefore, it is useful to study asymptotics theorems for $U$-statistics.

The theory of $U$-statistics for i.i.d. random variables comes from Hoeffding [9] and it is well developed (see [12, 13], references therein and many others). Also, asymptotics for U-statistics generated from non-independent sample have been studied from the early fifties of the last century, therefore there also are many asymptotic results for $U$-statistics based on weakly dependent random processes. These results are mainly based on a theory of mixing conditions or on conditions on associated processes, see for example [3, 5, 6].

The other way to handle weakly dependent processes, is to use the theory of stationary processes. Recently, some important results for asymptotics of U-statistics based on stationary processes were introduced, we can mention, for example, Leucht and Neumann [14] or Hsing and Wu [10].

In a case of multi-dimensional version, best to our knowledge, there are some applications for i.i.d. random fields, and recently, a result given by Denker and Gordin was introduced, see [4]. In this paper Denker and Gordin investigated some asymptotic results for Von Mises statistics based on stationary multi-parameter processes.

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Our aim in this paper is to show that it is possible to extend the results given by Hsing and Wu [10] for stationary processes to the case of stationary random fields. We show that techniques of proofs introduced by Hsing and Wu can also be used in the case of random fields. We can apply results given by Wang and Woodroofe [16] and Volný and Wang [15] on Central limit theorem and Invariance principle for stationary random fields to obtain asymptotic results for U-statistics based on stationary random fields.

## 2. NOTATION

In this section, we introduce a basic notation. To point out that the parameter has a higher dimension, we write $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \boldsymbol{Z}^d$. We denote $\|\mathbf{n}\| := |n_1|\cdot|n_2|\cdots|n_d|$ and $\mathbf{n} \to \infty$ we write for $\min\{n_1, n_2, \ldots, n_d\} \to \infty$. Operators $(\leq,\ \geq,\ +,\ -,\ \ldots)$ are used in the coordinate-wise sense, it means that for example we write $\mathbf{n}_1 \leq \mathbf{n}_2$ if and only if $n_{1i} \leq n_{2i}$ for all $i = 1, \ldots, d$ and $|\mathbf{n}| = (|n_1|, |n_2|, \ldots, |n_d|)$.

The operator $\cdot$ is used in the coordinate-wise sense, too. More precisely, we use $\mathbf{n} \cdot \mathbf{t} = (n_1 t_1, n_2 t_2, \ldots, n_d t_d)$.

In the paper, we use $\|\cdot\|_2$ for Euclidian norm.

Through the whole paper, we deal with the same dynamical systems as Wang and Woodroofe in [16] or Volný and Wang in [15]. We consider a probability space $(\Omega, \mathcal{A}, \mu)$ equipped with a group action $T_{\mathbf{i}}$, $\mathbf{i} \in \boldsymbol{Z}^d$, of automorphisms; the $\sigma$-algebra $\mathcal{A}$ is generated by independent and identically distributed random variables $e_{\mathbf{i}} = e_{\mathbf{0}} \circ T_{\mathbf{i}}$, $\mathbf{i} \in \boldsymbol{Z}^d$ with zero mean and finite second moments. By $\mathcal{F}_{\mathbf{i}}$ we denote a $\sigma$-field: $\mathcal{F}_{\mathbf{i}} = \sigma(e_{\mathbf{j}} : \mathbf{j} \leq \mathbf{i})$.

From the construction of the filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i}\in\boldsymbol{Z}^d}$, it is easily seen, that the filtration is commuting (for more details see [11]), what is a necessary condition for our results, for more details see [15].

By $(X_{\mathbf{i}})_{\mathbf{i}\in\boldsymbol{Z}^d}$, we denote a stationary random field: $X_{\mathbf{i}} = F(e_{\mathbf{j}} : \mathbf{j} \in \boldsymbol{Z}^d)$.

To introduce projection operators, let us follow the notation given by Volný and Wang [15] and first introduce a marginal filtration $(\mathcal{F}_l^{(q)})_{l\in\boldsymbol{Z}}$:

$$\mathcal{F}_l^{(q)} = \bigvee_{\mathbf{i}\in\boldsymbol{Z}^d, i_q \leq l} \mathcal{F}_{\mathbf{i}},\ q = 1, \ldots, d,\ l \in \boldsymbol{Z}$$

and write

$$\mathcal{F}_\infty = \bigvee_{\mathbf{i}\in\boldsymbol{Z}^d} \mathcal{F}_{\mathbf{i}}.$$

We will write

$$\mathrm{E}_{\mathbf{j}}(\cdot) = \mathrm{E}(\cdot|\mathcal{F}_{\mathbf{j}}),\ \mathbf{j} \in \boldsymbol{Z}^d \text{ and } \mathrm{E}_l^{(q)}(\cdot) = \mathrm{E}(\cdot|\mathcal{F}_l^{(q)}),\ q = 1, \ldots, d,\ l \in \boldsymbol{Z}.$$

Now, we can introduce the projection operators, which are defined by

$$P_l^{(q)}(f) = \mathrm{E}_l^{(q)}(f) - \mathrm{E}_{l-1}^{(q)}(f),\ q = 1, \ldots, d,\ l \in \boldsymbol{Z}$$

and

$$P_{\mathbf{j}} = \Pi_{q=1}^d P_{j_q}^q,\ \mathbf{j} \in \boldsymbol{Z}^d.$$

From Wang and Volný [15] we know that if $\{\mathcal{F}_{\mathbf{i}}\}_{\mathbf{i}\in\boldsymbol{Z}^d}$ is a commuting filtration, then $\{P_{\mathbf{j}}\}_{\mathbf{j}\in\boldsymbol{Z}^d}$ and $\{P_l^{(q)}\}_{l\in\boldsymbol{Z},\ q=1,\dots,d}$ are commuting operators and

$$P_{\mathbf{j}}(f) \in \bigcap_{q=1}^{d} \Big(L_2(\mathcal{F}^q_{j_q}) \ominus L_2(\mathcal{F}^q_{j_q-1})\Big) =: L_2^{\mathbf{j}},\ \mathbf{j}\in\boldsymbol{Z}^d.$$

Now, let us write $K$ for a symmetric measurable function from $\boldsymbol{R}^2$ to $\boldsymbol{R}$ and define the following statistic:

$$U_{\mathbf{n}} = \sum_{\mathbf{0}\le\mathbf{i},\mathbf{j}\le\mathbf{n}-\mathbf{1}} w_{\mathbf{i}-\mathbf{j}} K(X_{\mathbf{i}}, X_{\mathbf{j}}), \tag{1}$$

where $w_{\mathbf{k}}$ are coordinate-wise symmetric constant weights; it means that we suppose weights to be such that

$$w_{(k_1,k_2,\dots,k_d)} = w_{(|k_1|,|k_2|,\dots,|k_d|)} \text{ for all } \mathbf{k}\in\boldsymbol{Z}^d.$$

By the equation (1), we define so-called weighted U-statistic. It is well-known that a large class of statistics can be written in this form; for example, sample mean, sample variance and other moments, Kendall's $\tau$ can be written as U-statistics. Applications of weighted U-statistics based on random fields come from physics, geography, computer science and many other fields.

Following Hsing and Wu [10] for simplicity of notation, let us put $Y_{\mathbf{i},\mathbf{j}} = K(X_{\mathbf{i}}, X_{\mathbf{j}}) - \mathrm{E}K(X_{\mathbf{i}}, X_{\mathbf{j}})$. Hence, we can see that $(Y_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j}\in\boldsymbol{Z}^d}$ is a centered stationary process with finite second moments. The stationarity of this field is here in the sense that $(Y_{\mathbf{i},\mathbf{i}-\mathbf{k}})_{\mathbf{i}\in\boldsymbol{Z}^d}$ is a stationary random field. Moreover, due to the symmetry of the function $K$, we have $Y_{\mathbf{i},\mathbf{j}} = Y_{\mathbf{j},\mathbf{i}}$ for all $\mathbf{i},\mathbf{j}\in\boldsymbol{Z}^d$.

## 3. RESULTS

Now, we can state the first theorem – the invariance principle in case of summable weights. Let us recall that by $D[\mathbf{0},\mathbf{1}]$ we denote the space of cadlag functions on the space $[\mathbf{0},\mathbf{1}]$. For more details see [1].

**Theorem 3.1.** Let $(Y_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j}\in\boldsymbol{Z}^d}$ and $(w_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j}\in\boldsymbol{Z}^d}$ be as defined above and assume that

$$\sum_{\mathbf{k}\in\boldsymbol{Z}^d}\sum_{\mathbf{i}\in\boldsymbol{Z}^d} |w_{\mathbf{k}}| \|P_{\mathbf{0}}(Y_{\mathbf{i},\mathbf{i}-\mathbf{k}})\|_2 < \infty. \tag{2}$$

Then

$$\left(\frac{1}{\sqrt{\|\mathbf{n}\|}} \sum_{\mathbf{0}\le\mathbf{i},\mathbf{j}\le\lfloor(\mathbf{n}-\mathbf{1})\mathbf{t}\rfloor} w_{\mathbf{i}-\mathbf{j}} Y_{\mathbf{i},\mathbf{j}}\right)_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]} \xrightarrow{\mathcal{D}} (\sigma\mathcal{B}(\mathbf{t}))_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]} \tag{3}$$

in $D[\mathbf{0},\mathbf{1}]$, where $(\mathcal{B}(\mathbf{t}))_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]}$ is a standard Brownian sheet and $\sigma^2 < \infty$.

P r o o f. The proof follows the idea of the proof given by Hsing and Wu, see [10, Proof of Th 1]. Let $\ell \geq \mathbf{0}$, $\mathbf{s} \in \boldsymbol{Z}^d$, put

$$\xi_{\mathbf{s}}^{\ell} = \sum_{-\ell \leq \mathbf{k} \leq \ell} w_{\mathbf{k}} Y_{\mathbf{s},\mathbf{s}-\mathbf{k}}.$$

Then for every $\ell \in Z_+^d$, $(\xi_{\mathbf{s}}^{\ell})_{\mathbf{s}}$ forms a stationary random field.

The condition (2) implies for every $\ell \in \boldsymbol{Z}^d$:

$$\sum_{\mathbf{s} \in \boldsymbol{Z}^d} \|P_{\mathbf{0}}(\xi_{\mathbf{s}}^{\ell})\|_2 \leq \sum_{\mathbf{s} \in \boldsymbol{Z}^d} \sum_{-\ell \leq \mathbf{k} \leq \ell} |w_{\mathbf{k}}| \|P_{\mathbf{0}}(Y_{\mathbf{s},\mathbf{s}-\mathbf{k}})\|_2 < \infty.$$

Therefore, we can see that the stationary random field $(\xi_{\mathbf{s}}^{\ell})_{\mathbf{s}}$ generated by i.i.d. random variables satisfies Hannan's condition. Hence, we can apply [15, Theorem 5.1] and obtain for all $\ell \in \boldsymbol{Z}_+^d$

$$\left( \frac{1}{\sqrt{\|\mathbf{n}\|}} \sum_{\mathbf{0} \leq \mathbf{k} \leq \lfloor (\mathbf{n}-\mathbf{1})\mathbf{t} \rfloor} \xi_{\mathbf{k}}^{\ell} \right)_{\mathbf{t} \in [\mathbf{0},\mathbf{1}]} \xrightarrow{\mathcal{D}} (\sigma_\ell \mathcal{B}(\mathbf{t}))_{\mathbf{t} \in [\mathbf{0},\mathbf{1}]},$$

in $D[\mathbf{0},\mathbf{1}]$, where $(\mathcal{B}(\mathbf{t}))_{\mathbf{t} \in [\mathbf{0},\mathbf{1}]}$ is a standard Brownian sheet and $\sigma_\ell^2 < \infty$. The sequence of $(\sigma_\ell^2)_\ell$ is Cauchy due to the condition (2).

Hence, we need to verify that

$$\limsup_{\ell \to \infty} \limsup_{\mathbf{n} \to \infty} \mathrm{P}\left( \frac{1}{\sqrt{\|\mathbf{n}\|}} \sup_{\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}-\mathbf{1}} \left| \sum_{\mathbf{0} \leq \mathbf{i},\mathbf{j} \leq \mathbf{k}} w_{\mathbf{i}-\mathbf{j}} Y_{\mathbf{i},\mathbf{j}} - \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{k}} \sum_{\mathbf{i}-\ell \leq \mathbf{j} \leq \mathbf{i}+\ell} w_{\mathbf{i}-\mathbf{j}} Y_{\mathbf{i},\mathbf{j}} \right| > \varepsilon \right) = 0 \tag{4}$$

for all $\varepsilon > 0$. It is easily seen that the condition (4) is equivalent to the following condition:

$$\limsup_{\ell \to \infty} \limsup_{\mathbf{n} \to \infty} \mathrm{P}\left( \frac{1}{\sqrt{\|\mathbf{n}\|}} \max_{\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}-\mathbf{1}} \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{k}} \left| \sum_{\mathbf{j} \in [\mathbf{0},\mathbf{k}] : |\mathbf{i}-\mathbf{j}| \not\leq \ell} w_{\mathbf{i}-\mathbf{j}} Y_{\mathbf{i},\mathbf{j}} \right| > \varepsilon \right) = 0. \tag{5}$$

Let us prove this relation (5). First, we can observe:

$$\mathrm{P}\left( \frac{1}{\sqrt{\|\mathbf{n}\|}} \max_{\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}-\mathbf{1}} \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{k}} \left| \sum_{\mathbf{j} \in [\mathbf{0},\mathbf{k}] : |\mathbf{i}-\mathbf{j}| \not\leq \ell} w_{\mathbf{i}-\mathbf{j}} Y_{\mathbf{i},\mathbf{j}} \right| > \varepsilon \right) \\ \leq \mathrm{P}\left( \frac{1}{\sqrt{\|\mathbf{n}\|}} \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}-\mathbf{1}} \sum_{\mathbf{j} \in [\mathbf{0},\mathbf{n}-\mathbf{1}] : |\mathbf{i}-\mathbf{j}| \not\leq \ell} |w_{\mathbf{i}-\mathbf{j}} Y_{\mathbf{i},\mathbf{j}}| > \varepsilon \right).$$

Then, we can write:

$$\left\| \sum_{-\mathbf{n}+\mathbf{1}\le\mathbf{k}\le\mathbf{n}-\mathbf{1},\ |\mathbf{k}|\not\le\ell} \sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}} |w_{\mathbf{k}} Y_{\mathbf{i},\mathbf{i}-\mathbf{k}}| \right\|_2^2$$

$$= \sum_{\mathbf{j}\in\mathbf{Z}^d} \left\| P_{\mathbf{0}} \left( \sum_{-\mathbf{n}+\mathbf{1}\le\mathbf{k}\le\mathbf{n}-\mathbf{1},\ |\mathbf{k}|\not\le\ell} \sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}} |w_{\mathbf{k}} Y_{\mathbf{i}-\mathbf{j},\mathbf{i}-\mathbf{j}-\mathbf{k}}| \right) \right\|_2^2$$

$$\begin{aligned}
&\le \sum_{\mathbf{j}\in\mathbf{Z}^d} \left( \sum_{-\mathbf{n}+\mathbf{1}\le\mathbf{k}\le\mathbf{n}-\mathbf{1},\ |\mathbf{k}|\not\le\ell} |w_{\mathbf{k}}| \sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}} \|P_{\mathbf{0}}(Y_{\mathbf{i}-\mathbf{j},\mathbf{i}-\mathbf{j}-\mathbf{k}})\|_2 \right)^2 \\
&\le C \sum_{\mathbf{j}\in\mathbf{Z}^d} \sum_{-\mathbf{n}+\mathbf{1}\le\mathbf{k}\le\mathbf{n}-\mathbf{1},\ |\mathbf{k}|\not\le\ell} |w_{\mathbf{k}}| \sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}} \|P_{\mathbf{0}}\left(Y_{\mathbf{i}-\mathbf{j},\mathbf{i}-\mathbf{j}-\mathbf{k}}\right)\|_2 \\
&\le C\|\mathbf{n}\| \sum_{\mathbf{j}\in\mathbf{Z}^d} \sum_{-\mathbf{n}+\mathbf{1}\le\mathbf{k}\le\mathbf{n}-\mathbf{1},\ |\mathbf{k}|\not\le\ell} |w_{\mathbf{k}}| \, \|P_{\mathbf{0}}\left(Y_{\mathbf{j},\mathbf{j}-\mathbf{k}}\right)\|_2 .
\end{aligned}$$

We could use the estimation by the constant $C$ due to the condition (2). Hence, we derived that the left-hand side of (5) is less than or equal to

$$\limsup_{|\ell|\to\infty} \limsup_{\mathbf{n}\to\infty} \frac{C\|\mathbf{n}\| \sum_{\mathbf{j}\in\mathbf{Z}^d} \sum_{\mathbf{k}\in\mathbf{Z}^d,\ |\mathbf{k}|\not\le\ell} |w_{\mathbf{k}}| \, \|P_{\mathbf{0}}(Y_{\mathbf{j},\mathbf{j}-\mathbf{k}})\|_2}{\|\mathbf{n}\|\varepsilon^2}.$$

From (2) we derive that this limit is equal to 0 for all $\varepsilon > 0$ and the proof is finished.
□

Now, let us consider a more general case – the case of non-summable weights. To prove a result in this case, we will follow the idea given by Hsing and Wu [10]again. This idea of the proof is close to the idea which is used by Wang and Woodroofe, see [16], to prove their invariance principle – to use the approximation by m-dependent processes.

We show that we can approximate the sum of $w_{\mathbf{k}} Y_{\mathbf{i},\mathbf{i}-\mathbf{k}}$ by the sum of $w_{\mathbf{k}} \hat{Y}_{\mathbf{i},\mathbf{i}-\mathbf{k}}$, where $\hat{Y}_{\mathbf{i},\mathbf{j}}$ is some type of projection of $Y_{\mathbf{i},\mathbf{j}}$.

Let us denote

$$\hat{X}_{\mathbf{i}}^{\,m} := \mathrm{E}(X_{\mathbf{i}} | \sigma(e_{\mathbf{j}} : \mathbf{i}-\mathbf{m}+\mathbf{1} \le \mathbf{j} \le \mathbf{i}+\mathbf{m})),$$

where $\mathbf{m} = (m, m, \ldots, m)$. Then, by $\hat{Y}_{\mathbf{i},\mathbf{j}}^m$ we denote $K(\hat{X}_{\mathbf{i}}^m, \hat{X}_{\mathbf{j}}^m) - \mathrm{E}K(\hat{X}_{\mathbf{i}}^m, \hat{X}_{\mathbf{j}}^m)$.

For simplicity of notation, let us denote:

$$W_{\mathbf{n}}(\mathbf{i}) = \sum_{\mathbf{0}\le\mathbf{j}\le\mathbf{n}-\mathbf{1}} w_{\mathbf{i}-\mathbf{j}} \quad \text{and} \quad W_{\mathbf{n}} = \left( \sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}} W_{\mathbf{n}}^2(\mathbf{i}) / \|\mathbf{n}\| \right)^{1/2}.$$

Now, we can formulate a theorem, which is a multi-dimensional version of Hsing–Wu's theorem [10, Theorem 2].

**Theorem 3.2.** Assume that $\lim_{m\to\infty}\sup_{\mathbf{j}\in\mathbf{Z}^d}\|Y_{\mathbf{0},\mathbf{j}}-\hat{Y}^m_{\mathbf{0},\mathbf{j}}\|_2=0$,

$$\liminf_{\mathbf{n}\to\infty}\frac{W_{\mathbf{n}}}{\sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}}|w_{\mathbf{i}}|}>0$$

and

$$\lim_{\epsilon\to 0}\sup_{\mathbf{k}\in\mathbf{Z}^d}\sum_{\mathbf{i}\in\mathbf{Z}^d}\min\left(\sup_{m\ge 1}\|P_{\mathbf{0}}\hat{Y}^m_{\mathbf{i},\mathbf{i}-\mathbf{k}}\|_2,\epsilon\right)=0.$$

Then

$$\lim_{m\to\infty}\limsup_{\mathbf{n}\to\infty}\frac{1}{\|\mathbf{n}\|W^2_{\mathbf{n}}}\left\|\sum_{\mathbf{0}\le\mathbf{i},\mathbf{j}\le\mathbf{n}-\mathbf{1}}\left(w_{\mathbf{i}-\mathbf{j}}Y_{\mathbf{i},\mathbf{j}}-w_{\mathbf{i}-\mathbf{j}}\hat{Y}^m_{\mathbf{i},\mathbf{j}}\right)\right\|^2_2=0.$$

Proof. We follow again the proof given by Hsing and Wu, see [10, Proof of Theorem 2]. By Cauchy inequality and the triangle inequality we have:

$$\begin{aligned}\|P_{\mathbf{0}}(w_{\mathbf{k}}Y_{\mathbf{i},\mathbf{i}-\mathbf{k}}-w_{\mathbf{k}}\hat{Y}^m_{\mathbf{i},\mathbf{i}-\mathbf{k}})\|_2 &\le |w_{\mathbf{k}}|\sup_{\mathbf{j}\in\mathbf{Z}^d}\|Y_{\mathbf{0},\mathbf{j}}-\hat{Y}^m_{\mathbf{0},\mathbf{j}}\|_2,\\ \|P_{\mathbf{0}}(w_{\mathbf{k}}Y_{\mathbf{i},\mathbf{i}-\mathbf{k}}-w_{\mathbf{k}}\hat{Y}^m_{\mathbf{i},\mathbf{i}-\mathbf{k}})\|_2 &\le C|w_{\mathbf{k}}|\sup_{m\ge 1}\|P_{\mathbf{0}}\hat{Y}^m_{\mathbf{i},\mathbf{i}-\mathbf{k}}\|_2.\end{aligned}$$

Thus, there exists a $C>0$ such that for all $\mathbf{i},\mathbf{j},m$:

$$\|P_{\mathbf{0}}(w_{\mathbf{i}-\mathbf{j}}Y_{\mathbf{i},\mathbf{j}}-w_{\mathbf{i}-\mathbf{j}}\hat{Y}^m_{\mathbf{i},\mathbf{j}})\|_2\le C|w_{\mathbf{i}-\mathbf{j}}|\min\left(\sup_{\mathbf{j}\in\mathbf{Z}^d}\|Y_{\mathbf{0},\mathbf{j}}-\hat{Y}^m_{\mathbf{0},\mathbf{j}}\|_2,\sup_{m\ge 1}\|P_{\mathbf{0}}\hat{Y}^m_{\mathbf{i},\mathbf{j}}\|_2\right).$$

For short notation, we put

$$\begin{aligned}\hat{\theta}_{\mathbf{i},\mathbf{j}} &= \sup_{m\ge 1}\|P_{\mathbf{0}}\hat{Y}^m_{\mathbf{i},\mathbf{j}}\|_2,\\ \delta_m &= \sup_{\mathbf{j}\in\mathbf{Z}^d}\|Y_{\mathbf{0},\mathbf{j}}-\hat{Y}^m_{\mathbf{0},\mathbf{j}}\|_2.\end{aligned}$$

In the following, the constant $C$ depends on $d$ and it can vary from line to line. Thus, we can write

$$\lim_{m\to\infty}\limsup_{\mathbf{n}\to\infty}\frac{1}{\|\mathbf{n}\|W^2_{\mathbf{n}}}\left\|\sum_{\mathbf{0}\le\mathbf{i},\mathbf{j}\le\mathbf{n}-\mathbf{1}}\left(w_{\mathbf{i}-\mathbf{j}}Y_{\mathbf{i},\mathbf{j}}-w_{\mathbf{i}-\mathbf{j}}\hat{Y}^m_{\mathbf{i},\mathbf{j}}\right)\right\|^2_2 \tag{6}$$

$$= \lim_{m\to\infty}\limsup_{\mathbf{n}\to\infty}\frac{1}{\|\mathbf{n}\|W^2_{\mathbf{n}}}\sum_{\mathbf{s}\in\mathbf{Z}^d}\left\|\sum_{\mathbf{0}\le\mathbf{i},\mathbf{j}\le\mathbf{n}-\mathbf{1}}P_{\mathbf{s}}\left(w_{\mathbf{i}-\mathbf{j}}Y_{\mathbf{i},\mathbf{j}}-w_{\mathbf{i}-\mathbf{j}}\hat{Y}^m_{\mathbf{i},\mathbf{j}}\right)\right\|^2_2$$

$$\le \lim_{m\to\infty}\limsup_{\mathbf{n}\to\infty}\frac{C}{\|\mathbf{n}\|W^2_{\mathbf{n}}}\sum_{\mathbf{s}\in\mathbf{Z}^d}\left(\sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}}\ \sum_{-\mathbf{n}+\mathbf{1}+\mathbf{i}\le\mathbf{k}\le\mathbf{i}}|w_{\mathbf{k}}|\min(\hat{\theta}_{\mathbf{i}-\mathbf{s},\mathbf{i}-\mathbf{k}-\mathbf{s}},\delta_m)\right)^2$$

$$\le \lim_{m\to\infty}\limsup_{\mathbf{n}\to\infty}\frac{C}{\|\mathbf{n}\|W^2_{\mathbf{n}}}\sup_{\ell\in\mathbf{Z}^d}\sum_{\mathbf{s}\in\mathbf{Z}^d}\left(\sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}}\min(\hat{\theta}_{\mathbf{i}-\mathbf{s},\mathbf{i}-\ell-\mathbf{s}},\delta_m)\sum_{-\mathbf{n}+\mathbf{1}+\mathbf{i}\le\mathbf{k}\le\mathbf{i}}|w_{\mathbf{k}}|\right)^2.$$

Due to the condition on symmetric weights, we can see that for $\mathbf{i} \in [\mathbf{0}, \mathbf{n}-\mathbf{1}]$:

$$\sum_{-\mathbf{n}+\mathbf{1}+\mathbf{i} \le \mathbf{k} \le \mathbf{i}} |w_{\mathbf{k}}| \le C \sum_{\mathbf{0} \le \mathbf{k} \le \mathbf{n}-\mathbf{1}} |w_{\mathbf{k}}|,$$

where the constant $C$ depends only on $d$ again. Therefore, left-hand side of (6) is less than or equal to

$$\lim_{m\to\infty} \limsup_{\mathbf{n}\to\infty} \frac{C}{\|\mathbf{n}\|} \sup_{\ell \in \boldsymbol{Z}^d} \left[ \left( \sum_{\mathbf{i}\in \boldsymbol{Z}^d} \min(\hat{\theta}_{\mathbf{i},\mathbf{i}-\ell}, \delta_m) \right) \sum_{\mathbf{s}\in \boldsymbol{Z}^d} \left( \sum_{\mathbf{0}\le \mathbf{i} \le \mathbf{n}-\mathbf{1}} \min(\hat{\theta}_{\mathbf{i}-\mathbf{s},\mathbf{i}-\ell-\mathbf{s}}, \delta_m) \right) \right]$$

$$\le \lim_{m\to\infty} C \sup_{\ell\in \boldsymbol{Z}^d} \left( \sum_{\mathbf{s}\in \boldsymbol{Z}^d} \min(\hat{\theta}_{\mathbf{s},\mathbf{s}-\ell}, \delta_m) \right)^2 \to 0.$$

□

**Theorem 3.3.** Assume that $\sum_{\mathbf{i}\in \boldsymbol{Z}^d} |w_{\mathbf{i}}| = \infty$ and $\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}-\mathbf{1}} \|\mathbf{n}-\mathbf{k}\| w_{\mathbf{k}}^2 = o(\|\mathbf{n}\| W_{\mathbf{n}}^2)$. Then under the conditions of Theorem 3.2,

$$\frac{1}{\sqrt{\|\mathbf{n}\| W_{\mathbf{n}}^2}} \sum_{\mathbf{0}\le \mathbf{i},\mathbf{j} \le \mathbf{n}-\mathbf{1}} w_{\mathbf{i}-\mathbf{j}} Y_{\mathbf{i},\mathbf{j}} \xrightarrow{\mathcal{D}} \boldsymbol{N}(\mathbf{0}, \sigma^2),$$

for some $\sigma$ positive.

P r o o f. To prove this theorem, we can again follow the proof given by Hsing and Wu see [10, Proof of Th 3] and we adapt it for the multi-dimensional case.

Following the proof given by Hsing and Wu, we observe that it is enough to prove:

$$\frac{1}{\sqrt{\|\mathbf{n}\| W_{\mathbf{n}}^2}} \sum_{\mathbf{0}\le \mathbf{i},\mathbf{j} \le \mathbf{n}-\mathbf{1}} w_{\mathbf{i}-\mathbf{j}} \hat{Y}^l_{\mathbf{i},\mathbf{j}} \xrightarrow{\mathcal{D}} \boldsymbol{N}(\mathbf{0}, (\hat{\sigma}^l)^2), \tag{7}$$

for all $l \in \boldsymbol{N}$ as $\mathbf{n} \to \infty$. It follows from Theorem 3.2 that $(\hat{\sigma}^l)^2$ is Cauchy in $l$, thus the sequence converges to a finite constant $\sigma^2$ as $l \to \infty$.

Since $(\hat{Y}_{\mathbf{i},\mathbf{j}})$ is a stationary random field with finite second moments, from the construction of this process, we can observe that:

$$\left\| \sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}} \sum_{\mathbf{0}\le\mathbf{j}\le\mathbf{n}-\mathbf{1}:|\mathbf{i}-\mathbf{j}|\le 2\ell} w_{\mathbf{i}-\mathbf{j}} \hat{Y}^l_{\mathbf{i},\mathbf{j}} \right\|_2 = O\left( \sqrt{\|\mathbf{n}\|} \sum_{\mathbf{0}\le\mathbf{m}\le 2\ell} |w_{\mathbf{m}}| \right).$$

Hence, from the condition on non-summable weights, we derive that

$$\left\| \sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}} \sum_{\mathbf{0}\le\mathbf{j}\le\mathbf{n}-\mathbf{1}:|\mathbf{i}-\mathbf{j}|\le 2\ell} w_{\mathbf{i}-\mathbf{j}} \hat{Y}^l_{\mathbf{i},\mathbf{j}} \right\|_2 = o\left( (\|\mathbf{n}\| W_{\mathbf{n}}^2)^{1/2} \right).$$

We suppose $W_{\mathbf{n}} \to \infty$, thus to finish the proof of the theorem it suffices to show that

$$\frac{1}{\sqrt{\|\mathbf{n}\| W_{\mathbf{n}}^2}} \sum_{\mathbf{0}\le\mathbf{i}\le\mathbf{n}-\mathbf{1}} \sum_{\mathbf{0}\le\mathbf{j}\le\mathbf{n}-\mathbf{1},\ \max_k |i_k-j_k|>2l} w_{\mathbf{i}-\mathbf{j}} \hat{Y}^l_{\mathbf{i},\mathbf{j}} \xrightarrow{\mathcal{D}} \boldsymbol{N}(\mathbf{0}, (\hat{\sigma}^l)^2) \text{ as } m \to \infty,$$

for some finite $(\hat{\sigma}^l)^2$.

Following the proof given by Hsing and Wu, let us denote
$\hat{J}^{l,\mathbf{i}}_{\mathbf{i},\mathbf{j}} = \mathrm{E}(\hat{Y}^l_{\mathbf{i},\mathbf{j}} | (e_{\mathbf{k}} : \mathbf{i} - \ell + \mathbf{1} \le \mathbf{k} \le \mathbf{i} + \ell))$, where $\ell \in \boldsymbol{Z}^d$ is such that $\ell = (l, l, \dots, l)$; similarly
$\hat{J}^{l,\mathbf{j}}_{\mathbf{i},\mathbf{j}} = \mathrm{E}(\hat{Y}^l_{\mathbf{i},\mathbf{j}} | (e_{\mathbf{k}} : \mathbf{j} - \ell + \mathbf{1} \le \mathbf{k} \le \mathbf{j} + \ell))$ and put

$$\hat{R}^l_{\mathbf{i},\mathbf{j}} = w_{\mathbf{i}-\mathbf{j}} \left( \hat{Y}^l_{\mathbf{i},\mathbf{j}} - \hat{J}^{l,\mathbf{i}}_{\mathbf{i},\mathbf{j}} - \hat{J}^{l,\mathbf{j}}_{\mathbf{i},\mathbf{j}} \right).$$

Let $\mathbf{i}_1 = \mathbf{r}_1 + \mathbf{q}_1 \cdot 2l$ and $\mathbf{i}_2 = \mathbf{r}_2 + \mathbf{q}_2 \cdot 2l$, where $-\ell + \mathbf{1} \le \mathbf{r}_1, \mathbf{r}_2 \le \ell$ and $-\mathbf{q} = -\lfloor \mathbf{n}/2l \rfloor \le \mathbf{q}_1, \mathbf{q}_2 \le \mathbf{q} = \lfloor \mathbf{n}/2l \rfloor$, $\mathbf{q}_1 \ne \mathbf{q}_2$. (We write $\lfloor \mathbf{n}/2l \rfloor$ for $(\lfloor n_1/2l \rfloor, \lfloor n_2/2l \rfloor, \dots, \lfloor n_d/2l \rfloor)$.

Let us remark, that $R^l_{\mathbf{i}_1,\mathbf{j}_1}$ and $R^l_{\mathbf{i}_2,\mathbf{j}_2}$ are uncorrelated whenever $|\mathbf{i}_1 - \mathbf{i}_2| \not\le 2\ell$ and $|\mathbf{j}_1 - \mathbf{j}_2| \not\le 2\ell$:

$$\left\| \sum_{\mathbf{q}_1,\mathbf{q}_2 : -\mathbf{q}\le\mathbf{q}_1,\mathbf{q}_2\le\mathbf{q} : \mathbf{q}_1\ne\mathbf{q}_2} \hat{R}^l_{\mathbf{r}_1+\mathbf{q}_1\cdot 2l, \mathbf{r}_2+\mathbf{q}_2\cdot 2l} \right\|_2^2 \le C \sum_{\mathbf{q}_1,\mathbf{q}_2 : -\mathbf{q}\le\mathbf{q}_1,\mathbf{q}_2\le\mathbf{q} : \mathbf{q}_1\ne\mathbf{q}_2} w^2_{\mathbf{r}_1+\mathbf{q}_1\cdot 2l - \mathbf{r}_2 - \mathbf{q}_2\cdot 2l}.$$

Using the same arguments as Hsing and Wu, we obtain (we suppose symmetric weights and the constant $C$ can vary from line to line)

$$\begin{aligned} \left\| \sum_{\mathbf{0}\le\mathbf{i}_1,\mathbf{i}_2\le\mathbf{n}-\mathbf{1}, \max_k |i_{1k}-i_{2k}|>2l} \hat{R}^l_{\mathbf{i}_1,\mathbf{i}_2} \right\|_2^2 &\le C \sum_{-\ell+\mathbf{1}\le\mathbf{r}_1,\mathbf{r}_2\le\ell} \sum_{-\mathbf{q}\le\mathbf{q}_1,\mathbf{q}_2\le\mathbf{q} : \mathbf{q}_1\ne\mathbf{q}_2} w^2_{\mathbf{r}_1+\mathbf{q}_1\cdot 2l-\mathbf{r}_2-\mathbf{q}_2\cdot 2l} \\ &\le C \sum_{\mathbf{0}\le\mathbf{i}_1,\mathbf{i}_2\le\mathbf{n}-\mathbf{1}} w^2_{\mathbf{i}_1-\mathbf{i}_2} \\ &\le C \sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}-\mathbf{1}} \|\mathbf{n}-\mathbf{k}\| w^2_{\mathbf{k}} = o(\|\mathbf{n}\| W^2_{\mathbf{n}}). \end{aligned}$$

So (7) follows from

$$\frac{1}{\sqrt{\|\mathbf{n}\|} W_{\mathbf{n}}} \sum_{\mathbf{0}\le\mathbf{i},\mathbf{j}\le\mathbf{n}-\mathbf{1}, \max_k |i_k-j_k|>2l} w_{\mathbf{i}-\mathbf{j}} (\hat{J}^{l,\mathbf{i}}_{\mathbf{i},\mathbf{j}} + \hat{J}^{l,\mathbf{j}}_{\mathbf{i},\mathbf{j}}) \xrightarrow{\mathcal{D}} \boldsymbol{N}(0, (\hat{\sigma}^l)^2), \tag{8}$$

for some finite $(\hat{\sigma}^l)^2$. To show (8), we can apply a central limit theorem for m-dependent random fields given by Heinrich, see [8, Th 2]. (It is also possible to find it in [7].)

Let us recall this theorem.

**Theorem (Heinrich (1988)]** Let $(\Gamma_n)_{n\geq 1}$ be a sequence of finite subsets of $Z^d$ with $|\Gamma_n| \to \infty$ as $n \to \infty$ and let $(m_n)_{n\geq 1}$ be a sequence of positive integers. For each $n \geq 1$, let $(U_n(\mathbf{j}), \mathbf{j} \in \boldsymbol{Z}^d)$ be an $m_n$-dependent random field with $\mathrm{E}U_n(\mathbf{j}) = 0$ for all $\mathbf{j} \in \boldsymbol{Z}^d$. Assume that $\mathrm{E}\left(\sum_{\mathbf{j}\in\Gamma_n} U_n(\mathbf{j})\right)^2 \to \sigma^2$ as $n \to \infty$ with $\sigma^2 < \infty$.

Then $\sum_{\mathbf{j}\in\Gamma_n} U_n(\mathbf{j})$ converges in distribution to a Gaussian random variable with mean zero and variance $\sigma^2$ if there exists a finite constant $c > 0$ such that for any $n \geq 1$,

$$\sum_{\mathbf{j}\in\Gamma_n} \mathrm{E}U_n^2(\mathbf{j}) \leq c$$

and for any $\varepsilon > 0$ it holds that

$$\lim_{n\to\infty} L_n(\varepsilon) = \lim_{n\to\infty} m_n^{2d} \sum_{\mathbf{j}\in\Gamma_n} \mathrm{E}(U_n^2(\mathbf{j})\boldsymbol{I}_{\{|U_n(\mathbf{j})|\geq \varepsilon m_n^{-2d}\}}) = 0.$$

To apply Heindrich's theorem, we can put

$$U_n^l(\mathbf{j}) = \frac{1}{\sqrt{\|\mathbf{n}\|}W_{\mathbf{n}}} \sum_{\mathbf{i}:\ \mathbf{0}\leq\mathbf{i}\leq\mathbf{n}-\mathbf{1}, \max_k |i_k - j_k| > 2l} w_{\mathbf{i}-\mathbf{j}}(\hat{J}_{\mathbf{i},\mathbf{j}}^{l,\mathbf{i}} + \hat{J}_{\mathbf{i},\mathbf{j}}^{l,\mathbf{j}}).$$

Then conditions of Heinrich's theorem are fulfilled and the proof of Theorem 3.3 is finished. □

**Example (Linear random field):** Let $(\xi_{\mathbf{i}})_{\mathbf{i}\in\boldsymbol{Z}^d}$ be a random field of independent, identically distributed random variables with zero mean and finite second moment. Let us define a linear random field:

$$X_{\mathbf{k}} = \sum_{\mathbf{j}\in\boldsymbol{Z}^d} a_{\mathbf{j}}\xi_{\mathbf{k}-\mathbf{j}}.$$

Let us assume that $\sum_{\mathbf{j}\in\boldsymbol{Z}^d} |a_{\mathbf{j}}| < \infty$, in such a case we have so-called short-memory linear random field.

Now, let us define some common test statistics for this random field which can be expressed as U-statistics and discuss whether they satisfy our limit theorems.

First, the sample mean. Let $K(X_{\mathbf{i}}, X_{\mathbf{j}}) = X_{\mathbf{i}} + X_{\mathbf{j}}$ and $w_{\mathbf{i}-\mathbf{j}} = \frac{1}{2}\boldsymbol{I}_{\{\mathbf{i}-\mathbf{j}=\mathbf{0}\}}$. Then Theorem 1 is satisfied. More generally, to fulfill conditions of Theorem 1, it is enough to suppose that the weights $(w_{\mathbf{k}})_{\mathbf{k}\in\boldsymbol{Z}^d}$ are absolutely summable: $\sum_{\mathbf{k}\in\boldsymbol{Z}^d} |w_{\mathbf{k}}| < \infty$.

Second, let us put $K(X_{\mathbf{i}}, X_{\mathbf{j}}) = X_{\mathbf{i}} \cdot X_{\mathbf{j}}$, choose a $\mathbf{k} \geq \mathbf{0}$ and define weights: $w_{\mathbf{i}-\mathbf{j}} = \boldsymbol{I}_{\{|\mathbf{i}-\mathbf{j}|=\mathbf{k}\}}$. In such a case our U-statistic is a sample covariance and Theorem 1 takes place, too.

Last, if we put $K(X_{\mathbf{i}}, X_{\mathbf{j}}) = \boldsymbol{I}_{\{X_{\mathbf{i}}+X_{\mathbf{j}}>0\}}$ and $w_{\mathbf{i}-\mathbf{j}} = \boldsymbol{I}_{\{\mathbf{i}-\mathbf{j}\neq\mathbf{0}\}}$ then we get a 1-sample Wilcoxon statistic and it is easy to verify that conditions of Theorem 3 are satisfied.

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