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# INTERVAL FUZZY MATRIX EQUATIONS

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This paper deals with the solvability of interval matrix equations in fuzzy algebra. Fuzzy algebra is the algebraic structure in which the classical addition and multiplication are replaced by maximum and minimum, respectively.

The notation $\boldsymbol{A} \otimes X \otimes \boldsymbol{C} = \boldsymbol{B}$, where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are given interval matrices and $X$ is an unknown matrix, represents an interval system of matrix equations. We can define several types of solvability of interval fuzzy matrix equations. In this paper, we shall deal with four of them. We define the tolerance, weak tolerance, left-weak tolerance, and right-weak tolerance solvability and provide polynomial algorithms for checking them.

*Keywords:* fuzzy algebra, interval matrix equation, tolerance solvability, weak tolerance solvability

*Classification:* 15A06, 65G30

## 1. MOTIVATION

Fuzzy equations have found a broad area of applications in causal models which emphasize relationships between input and output variables. They are used in diagnosis models [1, 12, 16, 17] or models of nondeterministic systems [18]. Diagnostic models are particularly important because they cope with the uncertainty in many real-life situations concerning either medical diagnoses or diagnoses of technical devices. In the simplest formulation we are faced with a space of symptoms and a space of faults. Elements of faults are related with elements of symptoms by means of a fuzzy relation. Usually, the stronger the relationship between the symptom and a fault, the higher is the value of the corresponding argument. The solution of the fuzzy relational equation of the form $A \otimes x = b$, where $A$ is a matrix, $b$ and $x$ are vectors of suitable dimensions and classical addition and multiplication operations are replaced by maximum and minimum, provides a maximal set of symptoms that produce the given fault.

The solvability of the systems of fuzzy linear equations is well reviewed. In this paper, we shall deal with the solvability of fuzzy matrix equations of the form $A \otimes X \otimes C = B$, where $A$, $B$, and $C$ are given matrices of suitable sizes and $X$ is an unknown matrix. In the following example we will show one of possible applications.

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**Example 1.1.** Let us consider a situation, in which passengers from places $P_1, P_2$, $P_3$, and $P_4$ want to transfer to holiday destinations $D_1, D_2$, and $D_3$. Different transportation means provide transporting passengers from places $P_1, P_2, P_3$, and $P_4$ to airport terminals $T_1$ and $T_2$ (See Figure 1). We assume that the connection between $P_i$ and $T_l$ is possible only via one of the check points $Q_1, Q_2$, and $Q_3$. On Figure 1 there is an arrow $(P_i\,Q_j)$ if there exists the road from $P_i$ to $Q_j$ and there is an arrow $(T_l\,D_k)$ if terminal $T_l$ handles the passengers traveling to destination $D_k$ ($i = 1, 2, 3, 4$, $j = 1, 2, 3$, $k = 1, 2, 3$, $l = 1, 2$). The symbols along arrows represent the capacities of the corresponding connections.

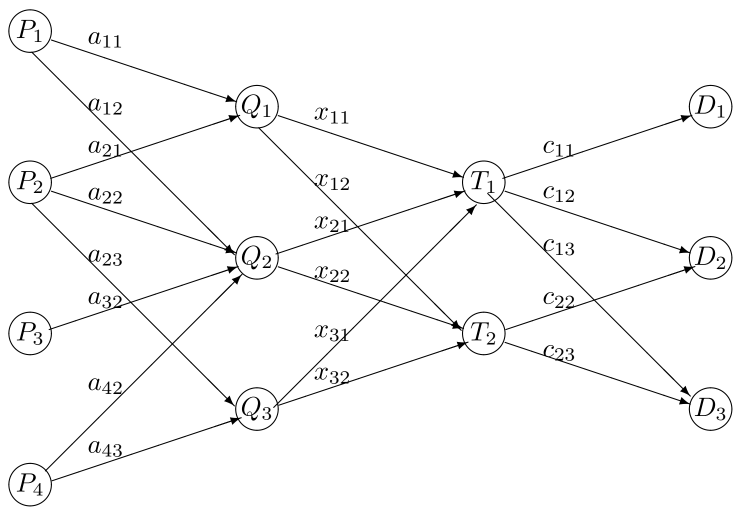


**Fig. 1.** Transportation system.

Denote by $a_{ij}$ ($c_{lk}$) the capacity of the road from $P_i$ to $Q_j$ (from $T_l$ to $D_k$). If $Q_j$ is linked with $T_l$ by a road with the capacity $x_{jl}$, then the capacity of the connection between $P_i$ and $D_k$ via $Q_j$ using terminal $T_l$ is equal to $\min\{a_{ij}, x_{jl}, c_{lk}\}$.

Suppose that the number of passengers traveling from place $P_i$ to destination $D_k$ is denoted by $b_{ik}$. To ensure the transportation for all passengers from $P_1$ to their destinations the following equations must be satisfied:

$$\max\Big\{\min\{a_{11}, x_{11}, c_{11}\}, \min\{a_{12}, x_{21}, c_{11}\}\Big\} = b_{11},$$
$$\max\Big\{\min\{a_{11}, x_{11}, c_{12}\}, \min\{a_{12}, x_{21}, c_{12}\}, \min\{a_{12}, x_{22}, c_{22}\}\Big\} = b_{12},$$
$$\max\Big\{\min\{a_{11}, x_{12}, c_{23}\}, \min\{a_{11}, x_{11}, c_{13}\}, \min\{a_{12}, x_{21}, c_{13}\}, \min\{a_{12}, x_{22}, c_{23}\}\Big\} = b_{13}.$$

Similar equalities must be satisfied to ensure the transportation for all passengers from $P_2$, $P_3$ and $P_4$ to their destinations.

In general, suppose that there are $m$ places $P_1, P_2, \ldots, P_m$, $n$ transfer points $Q_1, Q_2, \ldots, Q_n$, $s$ terminals $T_1, T_2, \ldots, T_s$, and $r$ destinations $D_1, D_2, \ldots, D_r$. If there is no road from $P_i$ to $Q_j$ (from $T_l$ to $D_k$), we put $a_{ij} = 0$ ($c_{lk} = 0$). Our task is to choose the appropriate capacities $x_{jl}$ for any $j \in N = \{1,\ 2, \ldots,\ n\}$, and for any

$l \in S = \{1, 2, \ldots, s\}$ such that the maximum capacity of the road from $P_i$ to $D_k$ is equal to a given number $b_{ik}$ for any $i \in M = \{1, 2, \ldots, m\}$ and for any $k \in R = \{1, 2, \ldots, r\}$, i. e.,

$$\max_{j \in N,\, l \in S} \min\{a_{ij}, x_{jl}, c_{lk}\} = b_{ik}. \tag{1}$$

A certain disadvantage of any necessary and sufficient condition for the solvability of (1) stems from the fact that it only indicates the existence or non-existence of the solution but does not indicate any action to be taken to increase the degree of solvability. However, it happens quite often in modelling real situations that the obtained system turns out to be unsolvable.

One of the possible methods of restoring the solvability is to replace the exact input values by intervals of possible values. The result of the substitution is so-called interval fuzzy matrix equation. The theory of interval computations, in particular of interval systems in the classical algebra is already quite developed, see e. g. the monograph [7] or [14, 15]. Interval systems of linear equations in fuzzy algebra have been studied in [3, 4, 8, 9]. In this paper, we deal with the solvability of interval fuzzy matrix equations. We define the tolerance, right-weak tolerance, left-weak tolerance, and weak tolerance solvability and provide polynomial algorithms for checking them.

## 2. PRELIMINARIES

Fuzzy algebra is the triple $(\mathcal{I}, \oplus, \otimes)$, where $\mathcal{I} = [O, I]$ is a linear ordered set with the least element $O$, the greatest element $I$, and two binary operations $a \oplus b = \max\{a, b\}$ and $a \otimes b = \min\{a, b\}$.

Denote by $M$, $N$, $R$, and $S$ the index sets $\{1, 2, \ldots, m\}$, $\{1, 2, \ldots, n\}$, $\{1, 2, \ldots, r\}$, and $\{1, 2, \ldots, s\}$, respectively. The set of all $m \times n$ matrices over $\mathcal{I}$ is denoted by $\mathcal{I}(m, n)$ and the set of all column $n$ vectors over $\mathcal{I}$ by $\mathcal{I}(n)$.
Operations $\oplus$ and $\otimes$ are extended to matrices and vectors in the same way as the operations in the classical algebra. We will consider the *ordering* $\leq$ on the sets $\mathcal{I}(m, n)$ and $\mathcal{I}(n)$ defined as follows:

- for $A, C \in \mathcal{I}(m, n)$ : $A \leq C$ if $a_{ij} \leq c_{ij}$ for all $i \in M$, $j \in N$,
- for $x, y \in \mathcal{I}(n)$ : $x \leq y$ if $x_j \leq y_j$ for all $j \in N$.

We will use the *monotonicity of* $\otimes$, which means that for any $A, C \in \mathcal{I}(m, n)$ and for any $B$, $D \in \mathcal{I}(n, s)$ the implication

$$\text{if } A \leq C \text{ and } B \leq D \text{ then } A \otimes B \leq C \otimes D$$

holds true.

Let $A \in \mathcal{I}(m, n)$ and $b \in \mathcal{I}(m)$. In fuzzy algebra, we can write the system of equations in the matrix form

$$A \otimes x = b. \tag{2}$$

The crucial role for the solvability of system (2) in fuzzy algebra is played by the *principal* *solution* of system (2), defined by

$$x_j^*(A,b) = \min_{i\in M}\{b_i : a_{ij} > b_i\} \tag{3}$$

for any $j \in N$, where $\min\emptyset = I$.

The following theorem describes the importance of the principal solution for the solvability of (2).

**Theorem 2.1.** (Cuninghame-Green [5], Zimmermann [19]) Let $A \in \mathcal{I}(m,n)$ and $b \in \mathcal{I}(m)$ be given.

(i) If $A \otimes x = b$ for $x \in \mathcal{I}(n)$, then $x \le x^*(A,b)$.

(ii) $A \otimes x^*(A,b) \le b$.

(iii) The system $A \otimes x = b$ is solvable if and only if $x^*(A,b)$ is its solution.

The properties of a principal solution are expressed in the following assertions.

**Lemma 2.2.** (Cechlárová [3]) Let $A \in \mathcal{I}(m,n)$ and $b, d \in \mathcal{I}(m)$ be given and let $b \le d$. Then $x^*(A,b) \le x^*(A,d)$.

**Lemma 2.3.** (Myšková [8]) Let $b \in \mathcal{I}(m)$ and $C, D \in \mathcal{I}(m,n)$ be given and let $D \le C$. Then $x^*(C,b) \le x^*(D,b)$.

**Lemma 2.4.** Let $A \in \mathcal{I}(m,n)$, $b \in \mathcal{I}(m)$ and $c \in \mathcal{I}$. Then

$$\min\{x_j^*(A\otimes c, b), c\} = \min\{x_j^*(A,b), c\}$$

for any $j \in N$.

P r o o f. In the case that $x_j^*(A,b) \ge c$ we have $x_j^*(A\otimes c, b) \ge c$, according to Lemma 2.3, so both minima are equal to $c$. In the second case $x_j^*(A,b) = b_i < c$ for some $i \in M$, which follows that $a_{ij} > b_i$. Then $a_{ij} \otimes c > b_i$ and consequently $x_j^*(A\otimes c, b) \le b_i = x_j^*(A,b)$. Together with $x_j^*(A\otimes c, b) \ge x_j^*(A,b)$ we obtain the equality. □

## 3. MATRIX EQUATIONS AND TENSOR PRODUCT

Let $A \in \mathcal{I}(m,n)$, $B \in \mathcal{I}(m,r)$, $X \in \mathcal{I}(n,s)$ and $C \in \mathcal{I}(s,r)$ be given matrices. It is easy to see that $[A \otimes X \otimes C]_{ik} = \max_{j\in N,\, l\in S} \min\{a_{ij}, x_{jl}, c_{lk}\}$. Hence, we can (1) write in the form

$$A \otimes X \otimes C = B. \tag{4}$$

In the following, we shall deal with the solvability of (4). We shall use the notion of tensor product.

**Definition 3.1.** Let $A = (a_{ij})$ be an $m \times n$ matrix and let $B = (b_{ij})$ be an $r \times s$ matrix. The *tensor product* of $A$ and $B$ is the following $mr \times ns$ matrix:

$$A \boxtimes B = \begin{pmatrix} A \otimes b_{11} & A \otimes b_{12} & \dots & A \otimes b_{1s} \\ A \otimes b_{21} & A \otimes b_{22} & \dots & A \otimes b_{2s} \\ \dots & \dots & \dots & \dots \\ A \otimes b_{r1} & A \otimes b_{r2} & \dots & A \otimes b_{rs} \end{pmatrix}.$$

Let $X \in B(n,s)$. Denote by $\mathrm{vec}\,(X)$ the vector $(X_1, X_2, \dots, X_s)^\top$, where $X_l$ is $l$th column of matrix $X$. Similarly we define $\mathrm{vec}\,(B)$.

**Theorem 3.2.** Matrix equation

$$(A_1 \otimes X \otimes C_1) \oplus (A_2 \otimes X \otimes C_2) \oplus \dots \oplus (A_r \otimes X \otimes C_r) = B, \tag{5}$$

where $A_i$, $C_i$, and $B$ are matrices of compatible sizes, is equivalent to the vector-matrix system

$$(A_1 \boxtimes C_1^\top \oplus A_2 \boxtimes C_2^\top \oplus \dots A_r \boxtimes C_r^\top) \otimes \mathrm{vec}\,(X) = \mathrm{vec}\,(B). \tag{6}$$

P r o o f. The proof is equivalent to the similar one in the max-plus algebra, which is given in [2]. □

For $r = 1$, the matrix equation in the form (5) takes the form $A \otimes X \otimes C = B$.

Denote by $X^*(A,B,C) = \big(x^*_{jl}(A,B,C)\big)$ the matrix defined as follows

$$x^*_{jl}(A,B,C) = \min_{k \in R}\{x^*_j(A \otimes c_{lk}, B_k)\}. \tag{7}$$

We shall call the matrix $X^*(A,B,C)$ a *principal matrix solution* of (4). The following theorem expresses the properties of $X^*(A,B,C)$ and gives the necessary and sufficient condition for the solvability of (4).

**Theorem 3.3.** Let $A \in \mathcal{I}(m,n)$, $B \in \mathcal{I}(m,r)$ and $C \in \mathcal{I}(m,n)$.

(i) If $A \otimes X \otimes C = B$ for $X \in \mathcal{I}(n,s)$, then $X \le X^*(A,B,C)$.

(ii) $A \otimes X^*(A,B,C) \otimes C \le B$.

(iii) The matrix equation $A \otimes X \otimes C = B$ is solvable if and only if $X^*(A,B,C)$ is its solution.

P r o o f. The consequence of Theorem 3.2 is that interval fuzzy matrix equation (4) is solvable if and only if the vector-matrix equation

$$(A \boxtimes C^\top) \otimes \mathrm{vec}\,(X) = \mathrm{vec}\,(B) \tag{8}$$

is solvable. By Theorem 2.1 (iii) the solvability of (8) is equivalent to the equality

$$(A \boxtimes C^\top) \otimes x^*(A \boxtimes C^\top, \mathrm{vec}\,(B)) = \mathrm{vec}\,(B).$$

We will prove that $x^*(A \boxtimes C^\top, \text{vec}(B)) = \text{vec}(X^*(A,B,C))$. For this reason, we rewrite (8) into the following form:

$$\begin{pmatrix} A\otimes c_{11} & A\otimes c_{21} & \dots & A\otimes c_{s1} \\ A\otimes c_{12} & A\otimes c_{22} & \dots & A\otimes c_{s2} \\ \dots & \dots & \dots & \dots \\ A\otimes c_{1r} & A\otimes c_{2r} & \dots & A\otimes c_{sr} \end{pmatrix} \otimes \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_s \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_r. \end{pmatrix}$$

Using (3) we have

$$x^*_{jl} = \min\Big\{\min_{i\in M}\{b_{i1} : a_{ij}\otimes c_{l1} > b_{i1}\}, \min_{i\in M}\{b_{i2} : a_{ij}\otimes c_{l2} > b_{i2}\}, \dots,$$

$$\dots \min_{i\in M}\{b_{ir} : a_{ij}\otimes c_{lr} > b_{ir}\}\Big\}$$

$$= \min\Big\{x^*_j(A\otimes c_{l1}, B_1), x^*_j(A\otimes c_{l2}, B_2), \dots, x^*_j(A\otimes c_{lr}, B_r)\Big\} = \min_{k\in R} x^*_j(A\otimes c_{lk}, B_k).$$

Hence the proof of parts (i), (ii) and (iii) follows directly from Theorem 2.1. □

**Remark 3.4.** Equality (7) can be written in the form

$$X^*(A,B,C) = (X^*_1(A,B,C), X^*_2(A,B,C), \dots, X^*_s(A,B,C)),$$

where

$$X^*_l(A,B,C) = \min_{k\in R} x^*(A\otimes c_{lk}, B_k). \tag{9}$$

**Lemma 3.5.** Let $A$, $A^{(1)}$, $A^{(2)} \in \mathcal{I}(m,n)$, $B$, $B^{(1)}$, $B^{(2)} \in \mathcal{I}(m,r)$ and $C$, $C^{(1)}$, $C^{(2)} \in \mathcal{I}(s,r)$.

(i) If $A^{(2)} \le A^{(1)}$ then $X^*(A^{(1)}B,C) \le X^*(A^{(2)}B,C)$.

(ii) If $B^{(1)} \le B^{(2)}$ then $X^*(A,B^{(1)},C) \le X^*(A,B^{(2)},C)$.

(iii) If $C^{(2)} \le C^{(1)}$ then $X^*(A,B,C^{(1)}) \le X^*(A,B,C^{(2)})$.

P r o o f.

(i) Since $A^{(2)} \otimes c_{lk} \le A^{(1)} \otimes c_{lk}$, by Lemma 2.3 we obtain the inequality $x^*_j(A^{(1)}\otimes c_{lk}, B_k) \le x^*_j(A^{(2)}\otimes c_{lk}, B_k)$ for any $k\in R$ which implies $x^*_{jl}(A^{(1)},B,C) \le x^*_{jl}(A^{(2)},B,C)$ for any $j\in N$, $l\in S$.

(ii) By Lemma 2.2 we have $x^*_j(A\otimes c_{lk}, B^{(1)}_k) \le x^*_j(A\otimes c_{lk}, B^{(2)}_k)$ for any $k\in R$ which implies $x^*_{jl}(A,B^{(1)},C) \le x^*_{jl}(A,B^{(2)},C)$ for any $j\in N$, $l\in S$.

(iii) By Lemma 2.3 we have $x^*_j(A\otimes c^{(1)}_{lk}, B_k) \le x^*_j(A\otimes c^{(2)}_{lk}, B_k)$ for any $k\in R$ which implies $x^*_{jl}(A,B,C^{(1)}) \le x^*_{jl}(A,B,C^{(2)})$ for any $j\in N$, $l\in S$.

□

**Lemma 3.6.** Let $A^{(1)}$, $A^{(2)} \in \mathcal{I}(m,n)$, $B^{(1)}$, $B^{(2)} \in \mathcal{I}(m,r)$, and $C^{(1)}$, $C^{(2)} \in \mathcal{I}(s,r)$. The system of matrix inequalities of the form

$$A^{(1)} \otimes X \otimes C^{(1)} \leq B^{(1)}, \tag{10}$$

$$A^{(2)} \otimes X \otimes C^{(2)} \geq B^{(2)} \tag{11}$$

is solvable if and only if

$$A^{(2)} \otimes X^*(A^{(1)}, B^{(1)}, C^{(1)}) \otimes C^{(2)} \geq B^{(2)}. \tag{12}$$

Proof. According to Theorem 3.3 (ii) the matrix $X^*(A^{(1)}, B^{(1)}, C^{(1)})$ satisfies inequality (10). If (12) is satisfied, then the matrix $X^*(A^{(1)}, B^{(1)}, C^{(1)})$ satisfies the inequality (11), too, so the system of inequalities (10), (11) is solvable with solution $X^*(A^{(1)}, B^{(1)}, C^{(1)})$.

For the converse implication suppose that the system of inequalities (10), (11) is solvable and a matrix $Y$ is its solution. It follows from $A^{(1)} \otimes Y \otimes C^{(1)} \leq B^{(1)}$ that there exists a matrix $D \in \mathcal{I}(m,r)$ such that $A^{(1)} \otimes Y \otimes C^{(1)} = D \leq B^{(1)}$. According to Theorem 3.3 (i) we have $Y \leq X^*(A^{(1)}, D, C^{(1)}) \leq X^*(A^{(1)}, B^{(1)}, C^{(1)})$, where the last inequality follows from Lemma 3.5 (ii). We obtain

$$A^{(2)} \otimes X^*(A^{(1)}, B^{(1)}, C^{(1)}) \otimes C^{(2)} \geq A^{(2)} \otimes Y \otimes C^{(2)} \geq B^{(2)}.$$

Hence inequality (12) is satisfied. □

## 4. INTERVAL MATRIX EQUATIONS

Similarly to [8, 9, 13], we define *interval matrices* $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ as follows:

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \left\{ A \in \mathcal{I}(m,n);\ \underline{A} \leq A \leq \overline{A} \right\},$$

$$\boldsymbol{B} = [\underline{B}, \overline{B}] = \left\{ B \in \mathcal{I}(m,r);\ \underline{B} \leq B \leq \overline{B} \right\},$$

$$\boldsymbol{C} = [\underline{C}, \overline{C}] = \left\{ C \in \mathcal{I}(s,r);\ \underline{C} \leq C \leq \overline{C} \right\}.$$

Denote by

$$\boldsymbol{A} \otimes X \otimes \boldsymbol{C} = \boldsymbol{B} \tag{13}$$

the set of all matrix equations of the form (4) such that $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, and $C \in \boldsymbol{C}$. We call (13) an *interval fuzzy matrix equation.*

We shall think over the solvability of interval fuzzy matrix equation on the ground of the solvability of matrix equations of the form (4) such that $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$, and $C \in \boldsymbol{C}$. We can define several types of solvability of an interval fuzzy matrix equation.

Let us return to Example 1.1. Suppose that we do not know exactly capacities of connections from places $P_i$ to check points $Q_j$ and the capacities of the flights from $T_l$ to $D_k$. We only know that they are from the given intervals of possible values. We want to observe transportations capacities from $Q_j$ to $T_l$ such that in each case all capacities of connection from $P_i$ to $D_k$ will be in the given intervals of possible values. The existence of such transportation times is called the *tolerance solvability.*

### 4.1. Tolerance Solvability

**Definition 4.1.** A matrix $X$ is called a *tolerance solution* of interval fuzzy matrix equation of the form (13) if for any $A \in \boldsymbol{A}$ and for any $C \in \boldsymbol{C}$ is $A \otimes X \otimes C \in \boldsymbol{B}$.

**Theorem 4.2.** A matrix $X$ is a tolerance solution of (13) if and only if it satisfies the system of inequalities

$$\overline{A} \otimes X \otimes \overline{C} \leq \overline{B}, \tag{14}$$

$$\underline{A} \otimes X \otimes \underline{C} \geq \underline{B}. \tag{15}$$

Proof. A matrix $X$ is a tolerance solution of (13) if for any $A \in \boldsymbol{A}$ and for any $C \in \boldsymbol{C}$ the product $A \otimes X \otimes C$ lies in $\boldsymbol{B}$. This leads to the requirement for the validity of the system of matrix inequalities $\underline{B} \leq A \otimes X \otimes C \leq \overline{B}$ for any $A \in \boldsymbol{A}$ and for any $C \in \boldsymbol{C}$. The left inequality is satisfied for any $A \in \boldsymbol{A}$ and for any $C \in \boldsymbol{C}$ if and only if $\underline{A} \otimes X \otimes \underline{C} \geq \underline{B}$, i.e., inequality (15) holds. The right inequality is satisfied for any $A \in \boldsymbol{A}$ and for any $C \in \boldsymbol{C}$ if and only if it holds for the matrices $\overline{A}$ and $\overline{C}$, so (14) holds. □

**Definition 4.3.** Interval fuzzy matrix equation of the form (13) is called *tolerance solvable* if there exists $X \in \mathcal{I}(n, s)$ such that $X$ is a tolerance solution of (13).

**Theorem 4.4.** Interval fuzzy matrix equation of the form (13) is tolerance solvable if and only if

$$\underline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \underline{C} \geq \underline{B}. \tag{16}$$

Proof. The tolerance solvability of (13) means that there exists a vector $X \in \mathcal{I}(n, s)$ such that $X$ is a tolerance solution. According to Theorem 4.2, it is equivalent to the solvability of the system of inequalities (14), (15). Using Lemma 3.6 we obtain (16). □

The following theorem deals with the complexity of checking the tolerance solvability of an interval fuzzy matrix equation. For the sake of simplicity, in the next theorem we will suppose that $m = r = s = n$.

**Theorem 4.5.** There is an algorithm which decides whether the given interval fuzzy matrix equation is tolerance solvable in $O(n^4)$ steps.

Proof. Checking the tolerance solvability is based on the verifying of inequality (16). Since computing $x^*(\overline{A} \otimes \overline{c}_{lk}, \overline{B}_k)$ requires $O(n^2)$ arithmetic operations, computing $X_l^*(\overline{A}, \overline{B}, \overline{C})$ by (9) for fixed $l$ requires $n \cdot O(n^2) = O(n^3)$ arithmetic operation. Hence, computing the matrix $X^*(\overline{A}, \overline{B}, \overline{C})$ requires $n \cdot O(n^3) = O(n^4)$ operations. Matrix multiplications need $O(n^3)$ arithmetic operations and checking matrix inequality (16) requires $O(n^2)$ arithmetic operations.

Hence the total complexity of the algorithm for checking the tolerance solvability of (13) is $O(n^4) + O(n^3) + O(n^2) = O(n^4)$. □

**Example 4.6.** Let $\mathcal{I} = [0, 10]$ and let

$$\boldsymbol{A} = \begin{pmatrix} [1,3] & [5,8] & [3,5] \\ [1,2] & [4,6] & [3,4] \\ [2,7] & [2,3] & [4,6] \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} [3,5] & [2,5] \\ [3,5] & [4,5] \\ [4,6] & [2,6] \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} [4,6] & [6,7] \\ [3,3] & [3,4] \end{pmatrix}.$$

We check whether the interval fuzzy matrix equation $\boldsymbol{A} \otimes X \otimes \boldsymbol{C} = \boldsymbol{B}$ is tolerance solvable.

*Solution:*
We have

$$\overline{A} \otimes \overline{c}_{11} = \begin{pmatrix} 3 & 6 & 5 \\ 2 & 6 & 4 \\ 6 & 3 & 6 \end{pmatrix}, \ \overline{A} \otimes \overline{c}_{12} = \begin{pmatrix} 3 & 7 & 5 \\ 2 & 6 & 4 \\ 7 & 3 & 6 \end{pmatrix},$$

$$\overline{A} \otimes \overline{c}_{21} = \begin{pmatrix} 3 & 3 & 3 \\ 2 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}, \ \overline{A} \otimes \overline{c}_{22} = \begin{pmatrix} 3 & 4 & 4 \\ 2 & 4 & 4 \\ 4 & 3 & 4 \end{pmatrix}.$$

We compute the principal matrix solution by (9):

$$X^*(\overline{A}, \overline{B}, \overline{C}) = \left( \min \left\{ \begin{pmatrix} 10 \\ 5 \\ 10 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \\ 10 \end{pmatrix} \right\}, \min \left\{ \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix}, \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix} \right\} \right) = \begin{pmatrix} 6 & 10 \\ 5 & 10 \\ 10 & 10 \end{pmatrix}.$$

We have to check inequality (16):

$$\underline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \underline{C} = \begin{pmatrix} 4 & 5 \\ 4 & 4 \\ 4 & 4 \end{pmatrix} \geq \underline{B}.$$

According to Theorem 4.4 the given interval fuzzy matrix equation is tolerance solvable.

### 4.2. One-side weak tolerance solvability

We define two types of one-side weak tolerance solvability.

**Definition 4.7.** Interval fuzzy matrix equation of the form (13) is called

(i) *right-weakly tolerance solvable* if for any $C \in \boldsymbol{C}$ there exists $X \in \mathcal{I}(n, s)$ such that for any $A \in \boldsymbol{A}$ is $A \otimes X \otimes C \in \boldsymbol{B}$,

(ii) *left-weakly tolerance solvable* if for any $A \in \boldsymbol{A}$ there exists $X \in \mathcal{I}(n, s)$ such that for any $C \in \boldsymbol{C}$ is $A \otimes X \otimes C \in \boldsymbol{B}$.

**Lemma 4.8.** Interval fuzzy matrix equation of the form (13) is

(i) right-weakly tolerance solvable if and only if for any $C \in \boldsymbol{C}$ holds the inequality

$$\underline{A} \otimes X^*(\overline{A}, \overline{B}, C) \otimes C \geq \underline{B}, \tag{17}$$

(ii) *left-weakly tolerance solvable* left-weakly tolerance solvable if and only if for any $A \in \boldsymbol{A}$ holds the inequality

$$A \otimes X^*(A, \overline{B}, \overline{C}) \otimes \underline{C} \geq \underline{B}. \tag{18}$$

P r o o f. (i) Let $C \in \boldsymbol{C}$ be arbitrary but fixed. The existence of $X \in \mathcal{I}(n, s)$ such that $A \otimes X \otimes C \in [\underline{B}, \overline{B}]$ for any $A \in \boldsymbol{A}$ is equivalent to the tolerance solvability of the fuzzy matrix equation with constant matrix $C = \underline{C} = \overline{C}$, which is, according to Theorem 4.4, equivalent to (17). Therefore, interval fuzzy matrix equation (13) is right-weakly tolerance solvable if and only if inequality (17) is fulfilled for any matrix $C \in \boldsymbol{C}$.

(ii) For a given matrix $A \in \boldsymbol{A}$ the existence of $X \in \mathcal{I}(n, s)$ such that $A \otimes X \otimes C \in \boldsymbol{B}$ for any $C \in \boldsymbol{C}$ is equivalent to the tolerance solvability of the fuzzy matrix equation with constant matrix $A = \underline{A} = \overline{A}$, which is equivalent to (18). To ensure the left-weak tolerance solvability, inequality (18) has to be satisfied for any matrix $A \in \boldsymbol{A}$. □

Lemma 4.8 does not give an algorithm for checking the one-side weak tolerance solvability. It follows from the definitions that the tolerance solvability implies the right-weak and left-weak tolerance solvability. The converse implications may not be valid. In the following we will prove that both types of the one-side weak tolerance solvability are equivalent to the tolerance solvability in fuzzy algebra.

**Theorem 4.9.** The following assertions are equivalent:

(i) Interval fuzzy matrix equation of the form (13) is tolerance solvable.

(ii) Interval fuzzy matrix equation of the form (13) is right-weakly tolerance solvable.

(iii) Interval fuzzy matrix equation of the form (13) is left-weakly tolerance solvable.

P r o o f. As mentioned above the implications (i) ⇒ (ii) and (i) ⇒ (iii) are trivial. It is sufficient to prove the implications (ii) ⇒ (i) and (iii) ⇒ (i).

I. (ii)⇒ (i): Suppose that an interval fuzzy matrix equation is not tolerance solvable. It means that there exist $i \in M$, and $p \in R$ such that $\left[\underline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \underline{C}\right]_{ip} < \underline{b}_{ip}$.

Denote by $C^{(p)}$ the matrix with the following entries

$$c_{lk}^{(p)} = \begin{cases} \underline{c}_{lk} & \text{for } k = p,\ l \in S, \\ \overline{c}_{lk} & \text{for } k \in R,\ k \neq p,\ l \in S. \end{cases} \tag{19}$$

We will prove that

$$\left[\underline{A} \otimes X^*(\overline{A}, \overline{B}, C^{(p)}) \otimes C^{(p)}\right]_{ip} = \left[\underline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \underline{C}\right]_{ip}. \tag{20}$$

We can rewrite the both sides of (20) as

$$\left[\underline{A} \otimes X^*(\overline{A}, \overline{B}, C^{(p)}) \otimes C^{(p)}\right]_{ip} = \max_{j \in N,\, l \in S} \min\{\underline{a}_{ij}, x_{jl}^*(\overline{A}, \overline{B}, C^{(p)}), c_{lp}^{(p)}\}$$

and

$$\left[\underline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \underline{C}\right]_{ip} = \max_{j \in N,\, l \in S} \min\{\underline{a}_{ij}, x^*_{jl}(\overline{A}, \overline{B}, \overline{C}), \underline{c}_{lp}\}$$

We prove that

$$\min\{\underline{a}_{ij}, x^*_{jl}(\overline{A}, \overline{B}, C^{(p)}), c^{(p)}_{lp}\} = \min\{\underline{a}_{ij}, x^*_{jl}(\overline{A}, \overline{B}, \overline{C}), \underline{c}_{lp}\} \tag{21}$$

for any $j \in N$, $l \in S$. The left-hand side of (21) is equal to

$$\min\{\underline{a}_{ij}, x^*_{jl}(\overline{A}, \overline{B}, C^{(p)}), c^{(p)}_{lp}\} = \min\left\{\underline{a}_{ij}, \min_{k \neq p} x^*_j(\overline{A} \otimes \overline{c}_{lk}, \overline{B}_k), x^*_j(\overline{A} \otimes \underline{c}_{lp}, \overline{B}_p), \underline{c}_{lp}\right\}$$

and the right-hand side is equal to

$$\min\{\underline{a}_{ij}, x^*_{jl}(\overline{A}, \overline{B}, \overline{C}), \underline{c}_{lp}\} = \min\left\{\underline{a}_{ij}, \min_{k \neq p} x^*_j(\overline{A} \otimes \overline{c}_{lk}, \overline{B}_k), x^*_j(\overline{A} \otimes \overline{c}_{lp}, \overline{B}_p), \underline{c}_{lp}\right\}.$$

We shall prove that

$$\min\{x^*_j(\overline{A} \otimes \underline{c}_{lp}, \overline{B}_p), \underline{c}_{lp}\} = \min\{x^*_j(\overline{A} \otimes \overline{c}_{lp}, \overline{B}_p), \underline{c}_{lp}\}. \tag{22}$$

According to Lemma 2.4 we obtain

$$\min\{x^*_j(\overline{A} \otimes \underline{c}_{lp}, \overline{B}_p), \underline{c}_{lp}\} = \min\{x^*_j(\overline{A}, \overline{B}_p), \underline{c}_{lp}\}$$

and

$$\min\{x^*_j(\overline{A} \otimes \overline{c}_{lp}, \overline{B}_p), \underline{c}_{lp}\} = \min\{x^*_j(\overline{A} \otimes \overline{c}_{lp}, \overline{B}_p), \underline{c}_{lp}, \overline{c}_{lp}\} = \min\{x^*_j(\overline{A}, \overline{B}_p), \underline{c}_{lp}\}.$$

From the assumption and (20) we obtain $\left[\underline{A} \otimes X^*(\overline{A}, \overline{B}, C^{(p)}) \otimes C^{(p)}\right]_{ip} < \underline{b}_{ip}$. Hence, inequality (17) is not satisfied for the matrix $C^{(p)}$ and according to Lemma 4.8 (i) an interval fuzzy matrix equation is not right-weakly tolerance solvable.

II. (iii)$\Rightarrow$ (i): To prove the converse implication let us suppose that there are $q \in M$ and $t \in R$ such that $\left[\underline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \underline{C}\right]_{qt} < \underline{b}_{qt}$. Denote by $A^{(q)}$ the matrix with the entries

$$a^{(q)}_{ij} = \begin{cases} \underline{a}_{ij} & \text{for } i = q,\ j \in N, \\ \overline{a}_{ij} & \text{for } i \in M,\ i \neq q,\ j \in N. \end{cases} \tag{23}$$

We will prove that

$$\left[A^{(q)} \otimes X^*(A^{(q)}, \overline{B}, \overline{C}) \otimes \underline{C}\right]_{qt} = \left[\underline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \underline{C}\right]_{qt}. \tag{24}$$

The both sides of (24) can be rewritten as

$$\left[A^{(q)} \otimes X^*(A^{(q)}, \overline{B}, \overline{C}) \otimes \underline{C}\right]_{qt} = \max_{j \in N,\, l \in S} \min\{\underline{a}_{qj}, x^*_{jl}(A^{(q)}, \overline{B}, \overline{C}), \underline{c}_{lt}\}$$

and

$$\left[\underline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \underline{C}\right]_{qt} = \max_{j \in N,\, l \in S} \min\{\underline{a}_{qj}, x^*_{jl}(\overline{A}, \overline{B}, \overline{C}), \underline{c}_{lt}\}.$$

We prove that

$$\min\{\underline{a}_{qj}, x^*_{jl}(A^{(q)}, \overline{B}, \overline{C}), \underline{c}_{lt}\} = \min\{\underline{a}_{qj}, x^*_{jl}(\overline{A}, \overline{B}, \overline{C}), \underline{c}_{lt}\}$$

for any $j \in N$, $l \in S$. Let us remember that

$$x^*_{jl}(A^{(q)}, \overline{B}, \overline{C}) = \min\left\{ \min_{k \in R, i \neq q}\{\overline{b}_{ik} : \overline{a}_{ij} \otimes \overline{c}_{lk} > \overline{b}_{ik}\}, \min_{k \in R}\{\overline{b}_{qk} : \underline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}\right\}$$

and

$$x^*_{jl}(\overline{A}, \overline{B}, \overline{C}) = \min\left\{ \min_{k \in R, i \neq q}\{\overline{b}_{ik} : \overline{a}_{ij} \otimes \overline{c}_{lk} > \overline{b}_{ik}\}, \min_{k \in R}\{\overline{b}_{qk} : \overline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}\right\}.$$

To prove (24), we show that

$$\min\left\{ \min_{k \in R}\{\overline{b}_{qk} : \underline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}, \underline{a}_{qj}\right\} = \min\left\{ \min_{k \in R}\{\overline{b}_{qk} : \overline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}, \underline{a}_{qj}\right\}. \quad (25)$$

There are two possibilities: either $\{\overline{b}_{qk} : \underline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\} = \{\overline{b}_{qk} : \overline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}$ or $\{\overline{b}_{qk} : \underline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\} \subsetneq \{\overline{b}_{qk} : \overline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}$.

In the first case equality (25) trivially holds. In the second case denote by $R^*$ the set $R^* = \{k \in R : \underline{a}_{qj} \otimes \overline{c}_{lk} \leq \overline{b}_{qk} \wedge \overline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}$. Since for any $k \in R^*$ the inequality $\underline{a}_{qj} \leq \overline{b}_{qk}$ holds, we obtain $\min\{\min_{k \in R^*} \overline{b}_{qk}, \underline{a}_{qj}\} = \underline{a}_{qj}$. Hence, the right-hand side of (25) is equal to

$$\min\left\{ \min_{k \in R}\{\overline{b}_{qk} : \overline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}, \underline{a}_{qj}\right\} = \min\left\{ \min_{k \in R}\{\overline{b}_{qk} : \underline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}, \underline{a}_{qj}, \min_{k \in R^*} \overline{b}_{qk}\right\} =$$

$$\min\left\{ \min_{k \in R}\{\overline{b}_{qk} : \underline{a}_{qj} \otimes \overline{c}_{lk} > \overline{b}_{qk}\}, \underline{a}_{qj}\right\},$$

so equality (25) is satisfied. Since (24) is satisfied, from the assumption we obtain $\left[A^{(q)} \otimes X^*(A^{(q)}, \overline{B}, \overline{C}) \otimes \underline{C}\right]_{qt} < \underline{b}_{qt}$. According to Lemma 4.8 (ii), an interval fuzzy matrix equation is not left-weakly tolerance solvable. □

### 4.3. Weak tolerance solvability

**Definition 4.10.** Interval fuzzy matrix equation of the form (13) is called *weakly tolerance solvable* if for any $A \in \boldsymbol{A}$ and for any $C \in \boldsymbol{C}$ there exist $X \in \mathcal{I}(n,s)$ such that $A \otimes X \otimes C \in \boldsymbol{B}$.

**Theorem 4.11.** Interval fuzzy matrix equation of the form (13) is weakly tolerance solvable if and only if it is tolerance solvable.

Proof. Suppose that (13) is weakly tolerance solvable. We obtain the following sequence of implications

$$(\forall A \in \boldsymbol{A})(\forall C \in \boldsymbol{C})(\exists X \in \mathcal{I}(n,s)) A \otimes X \otimes C \in \boldsymbol{B} \overset{\text{Th 4.9}}{\Longrightarrow}$$

$$(\forall A \in \boldsymbol{A})(\exists X \in \mathcal{I}(n,s))(\forall C \in \boldsymbol{C}) A \otimes X \otimes C \in \boldsymbol{B} \overset{\text{Th 4.9}}{\Longrightarrow}$$

$$(\exists X \in \mathcal{I}(n,s))(\forall A \in \boldsymbol{A})(\forall C \in \boldsymbol{C})\, A \otimes X \otimes C \in \boldsymbol{B},$$

hence (13) is tolerance solvable. The converse implication trivially holds. □

**Remark 4.12.** Interval matrix equations in max-plus algebra have been studied in [11]. There is a similar condition for the tolerance solvability in max-plus and fuzzy algebra. A significant difference is in the left-weak and right-weak tolerance solvability, which are equivalent to the tolerance solvability in the fuzzy algebra, but not in the max-plus algebra.

**Remark 4.13.** Suppose that interval matrix $\boldsymbol{B}$ is not a closed interval. If $\boldsymbol{B}$ is a left open interval, then inequality (16) turns into inequality $\underline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \underline{C} > \underline{B}$. If $\boldsymbol{B}$ is a right open interval, then we have to add the inequality $\overline{A} \otimes X^*(\overline{A}, \overline{B}, \overline{C}) \otimes \overline{C} < \overline{B}$. In case that $\boldsymbol{B}$ is an open interval, we have to make both changes in Theorem 4.4. The cases that the entries of interval matrices matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ are not closed intervals have to be studied separately.

## 5. CONCLUSION

In this paper, we dealt with the solvability of matrix equations in fuzzy algebra. Fuzzy algebra is a useful tool for describing real situation in the economy and industry. In Example 1.1, the values $a_{ij}, x_{jl}$, and $c_{lk}$ represent the capacities of corresponding connections. In economics, those values may represent for example the financial costs for the production or transporting of some products. In another possibility, $a_{ij}$ represents a measure of the preference of the property $P_i$ of some object before the property $Q_j$, similarly $x_{jl}$ ($c_{lk}$) represent a measure of the preference of the property $Q_j$ before the property $T_l$ (the property $T_l$ before the property $D_k$).

In practice, the values $a_{ij}$ and $c_{lk}$ may depend on external conditions, so they are from intervals of possible values. Due to this fact, it is significant to deal with fuzzy matrix equations with interval data. We have studied four types of the solvability of interval fuzzy matrix equation. We intend to deal with another solvability concepts in further research.

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## REFERENCES

[1] A. Asse, P. Mangin, and D. Witlaeys: Assisted diagnosis using fuzzy information. NAFIPS 2 Congress, Schenectudy 1983. DOI:10.1109/ifsa-nafips.2013.6608528

[2] P. Butkovič and M. Fiedler: Tropical tensor product and beyond. `http://web.mat.bham.ac.uk/P.Butkovic/Pubs.html`.

[3] K. Cechlárová: Solutions of interval systems in fuzzy algebra. In: Proc. SOR 2001 (V. Rupnik, L.Zadnik-Stirn, and S. Drobne, eds.), Preddvor, pp. 321–326.

[4] K.Cechlárová and R. A. Cuninghame-Green: Interval systems of max-separable linear equations. Lin. Algebra Appl. *340* (2002), 215–224. DOI:10.1016/s0024-3795(01)00405-0

[5] R. A. Cuninghame-Green: Minimax Algebra. Lecture Notes in Economics and Mathematical Systems 1966, Springer, Berlin 1979. DOI:10.1007/978-3-642-48708-8

[6] M. Gavalec and J. Plavka: Monotone interval eigenproblem in fuzzy algebra. Kybernetika *46* (2010), 3, 387–396.

[7] J. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl: Computational Complexity of Feasibility of Data Processing and Interval Computations. Kluwer, Dordrecht 1998. DOI:10.1007/978-1-4757-2793-7

[8] H. Myšková: Interval systems of max-separable linear equations. Lin. Algebra Appl. *403* (2005), 263–272. DOI:10.1016/j.laa.2005.02.011

[9] H. Myšková: Control solvability of interval systems of max-separable linear equations. Lin. Algebra Appl. *416* (2006), 215–223. DOI:10.1016/j.laa.2005.11.008

[10] H. Myšková: On an algorithm for testing T4 solvability of fuzzy interval systems. Kybernetika *48* (2012), 5, 924–938.

[11] H. Myšková: Interval max-plus matrix equations. Lin. Algebra Appl. *492* (2016), 111–127. DOI:10.1016/j.laa.2015.10.031

[12] A. Di Nola, S. Salvatore, W. Pedrycz, and E. Sanchez: Fuzzy Relation Equations and Their Applications to Knowledge Engineering. Kluwer Academic Publishers, Dordrecht 1989. DOI:10.1007/978-94-017-1650-5

[13] J. Plavka: On the $O(n^3)$ algorithm for checking the strong robustness of interval fuzzy matrices. Discrete Appl. Math. *160* (2012), 640–647. DOI:10.1016/j.dam.2011.11.010

[14] J. Rohn: Systems of Interval Linear Equations and Inequalities (Rectangular Case). Technical Report 875, Institute of Computer Science, Academy of Sciences of the Czech Republic, Praha 2002.

[15] J. Rohn: Complexity of some linear problems with interval data. Reliable Computing *3* (1997), 315–323. DOI:10.1023/a:1009987227018

[16] E. Sanchez: Medical diagnosis and composite relations. In: Advances in Fuzzy Set Theory and Applications (M. M. Gupta, R. K. Ragade, R. R. Yager, eds.), North-Holland, Amsterdam–New York 1979, pp. 437–444.

[17] T. Terano and Y. Tsukamoto: Failure diagnosis by using fuzzy logic. In: Proc. IEEE Conference on Decision Control, New Orleans 1977, pp. 1390–1395. DOI:10.1109/cdc.1977.271521

[18] L. A. Zadeh: Toward a theory of fuzzy systems. In: Aspects of Network and Systems Theory (R. E. Kalman and N. De Claris, eds.), Hold, Rinehart and Winston, New York 1971, pp. 209–245.

[19] K. Zimmermann: Extremální algebra. Ekonomicko-matematická laboratoř Ekonomického ústavu ČSAV, Praha 1976.



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