RELATIVE CONTROLLABILITY OF NONLINEAR FRACTIONAL DELAY INTEGRODIFFERENTIAL SYSTEMS WITH MULTIPLE DELAYS IN CONTROL

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This paper describes the controllability of nonlinear fractional delay integrodifferential systems with multiple delays in control. Necessary and sufficient conditions for the controllability criteria for linear fractional delay system are established. Further sufficient conditions for the controllability of nonlinear fractional delay integrodifferential system are obtained by using fixed point arguments. Examples are provided to illustrate the results.

Keywords: fractional delay integrodifferential equation, Laplace transform, controllability, Mittag–Leffler function, Caputo fractional derivative

Classification: 34A08, 93B05

1. INTRODUCTION

The future of many processes in the world around us depends not only on the present state, but is also significantly determined by the entire prehistory. Such systems occur in automatic control, economics, medicine, biology and other areas. Mathematical description of these processes can be done with the help of differential equations, with delay, integral and integrodifferential equations. A related study on analytic solutions of linear delay differential equations has been made by Bellman and Cooke [8], Smith [36], Halaney [12], Oguztoreli [28], Smith and Hale [13]. It is interesting to study these models in fractional sense to give better results compared to me integer order case.

In some real world problems, fractional derivative provides an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rhenology etc involves derivative of fractional order. As a consequence, the subject of fractional differential equations is gaining more importance and attention. Fractional differentials and integrals provide more accurate models of systems under consideration. Many authors have demonstrated applications of fractional calculus in the frequency dependent damping behavior of viscoelastic materials [1, 2], dynamics of interfaces between nanoparticles and substrates [9], the nonlinear oscillation of earthquakes [32], bioengineering [23], continuum and statistical mechanics [24], signal processing [30], filter design, circuit theory and robotics [33].

DOI: 10.14736/kyb-2017-1-0161

Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in [14] and differential equations with fractional order have recently proved to be valuable tools in the modeling of many physical phenomena. Moreover Machoda et al. [21, 22] analyzed and designed the fractional order digital control systems and also modeled the fractional dynamics in DNA. Practical systems like manufacturing systems, chemical processes, transmission lines and rolling mill systems have delays in their dynamics. Typical time-delay systems with multiple time-varying delays include a turbojet engine, a microwave oscillator, the inferred grinding model and models of population dynamics. Satisfactory modeling of time-varying delays is also important for the synthesis of effective control systems since they show significantly different characteristics from those of fixed time delays. It is essential that system models must take into account these time delays in order to predict the true system dynamics.

The important qualitative behavior of a dynamical system is controllability. It is used to influence an object behavior so as to accomplish a desired goal. It plays a major role in both finite and infinite dimensional systems. Dauer and Gahl [10] obtained the controllability of nonlinear delay systems. Balachandran and Dauer [4] studied the controllability problems for both linear and nonlinear delay systems. The relative controllability of nonlinear fractional dynamical system with multiple delays and distributive delays in control have been discussed by Balachandran et al. [5, 6, 7]. Klamka [18, 19] established the controllability of both linear and nonlinear systems with time variable delay in control. Manzanilla et al. [25] obtained the controllability of a differential equation with delay and advanced arguments. Recently Mur et al. [27] studied the relative controllability of linear systems of fractional order with delay. Controllability of fractional control systems with control delay has been studied by Wei [37] and controllability of time delay fractional systems with and without constraints [35]. Detailed study on controllability of fractional delay dynamical systems is made in [15, 16], the Laplace transform method is used to obtain the solution representation which has been employed in this paper to obtain the relative controllability of fractional delay integrodifferential systems with multiple delay in control. After providing preliminary results in section 2, we establish in section 3, necessary and sufficient conditions for the controllability criteria for linear fractional delay systems by defining the Grammian matrix. In section 4, sufficient conditions for the controllability of nonlinear fractional delay integrodifferential systems are established using Schauder's fixed point theorem. Examples with numerical simulations are given in section 5 to illustrate the theory.

2. PRELIMINARIES

This section begins with definitions and properties of fractional operator, special functions and their Laplace transformations [13, 17, 20, 26, 31, 34].

(a) Let f be a real or complex valued function of the variable $t > 0$ and s is a real or complex parameter. The Laplace transform of f is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t, \text{ for } Re(s) > 0. \tag{1}$$

An important function occurring in electrical systems is the delayed unit step function

$$u_a(t) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases} \tag{2}$$

and its Laplace transformation is given by

$$L[u_a(t)](s) = \frac{e^{-as}}{s}, \; s > 0, \; a > 0, \; Re(s) > 0. \tag{3}$$

(b) The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, is defined as

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s,$$

where the function f has absolutely continuous derivatives upto order $n-1$. The Laplace transform of Caputo derivative is

$$L[^{C}D^{\alpha}x(t)](s) = s^{\alpha} L[x(t)](s) - \sum_{k=0}^{n-1} x^{k}(0) s^{\alpha-1-k}, \; n-1 < \alpha \leq n.$$

(c) The Mittag–Leffler functions of various type are defined as

$$E_{\alpha}(z) = E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \; z \in \mathbb{C}, \; Re(\alpha) > 0, \tag{4}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \; z, \beta \in \mathbb{C}, \; Re(\alpha) > 0, \tag{5}$$

$$E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\gamma)_k (-\lambda)^k}{k! \Gamma(\alpha k + \beta)} t^{\alpha k}, \tag{6}$$

where $(\gamma)_n$ is a Pochhamer symbol which is defined as $\gamma(\gamma+1)\dots(\gamma+n-1)$ and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$. The Laplace transform of Mittag–Leffler functions (4), (5) and (6) are given by

$$L[E_{\alpha,1}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha-1}}{(s^{\alpha} \pm \lambda)}, \; \mathrm{Re}(\alpha) > 0, \tag{7}$$

$$L[t^{\beta-1} E_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha-\beta}}{(s^{\alpha} \pm \lambda)}, \; \mathrm{Re}(\alpha) > 0, \quad \mathrm{Re}(\beta) > 0, \tag{8}$$

$$L[t^{\beta-1} E^{\gamma}_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} \pm \lambda)^{\gamma}}, \quad \mathrm{Re}(s) > 0, \mathrm{Re}(\beta) > 0, \; |\lambda s^{-\alpha}| < 1. \tag{9}$$

(d) If $F(s) = L[f(t)](s)$ for Re(s)>0, then [34]

$$F(s-a) = L[e^{at} f(t)](s),$$

and

$$L[u_a(t) f(t-a)](s) = e^{-as} F(s), \; a \geq 0.$$

3. LINEAR DELAY SYSTEMS

Consider the fractional delay dynamical system with multiple delays in control

$$
\begin{aligned}
{}^C D^\alpha x(t) &= Ax(t) + Bx(t-h) + \sum_{i=0}^{M} C_i u(\sigma_i(t)),\ t \in J = [0,T], \\
x(t) &= \phi(t), \quad -h < t \le 0,
\end{aligned} \tag{10}
$$

where $0 < \alpha < 1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A, B are $n \times n$ matrices and C_i are $n \times m$ matrices for $i = 0, 1, 2, \dots M$. Assume the following conditions:

(H1) The functions $\sigma_i : J \to \mathbb{R}, i = 0, 1, 2, \dots M$, are twice continuously differentiable and strictly increasing in J. Moreover

$$\sigma_i(t) \le t,\ i = 0, 1, 2 \dots M,\ \text{for } t \in J. \tag{11}$$

(H2) Introduce the time lead functions $r_i(t) : [\sigma_i(0), \sigma_i(T)] \to [0,T]$, $i = 0, 1, 2, \dots M$, such that $r_i(\sigma_i(t)) = t$ for $t \in J$. Further $\sigma_0(t) = t$ and for $t = T$. The following inequalities holds

$$
\begin{aligned}
\sigma_M(T) \le \sigma_{M-1}(T) \le \dots \sigma_{m+1}(T) \le 0 &= \sigma_m(T) < \sigma_{m-1}(T) = \dots = \sigma_1(T) \\
&= \sigma_0(T) = T.
\end{aligned} \tag{12}
$$

The following definitions of complete state of the system (10) at time t and relative controllability are assumed.

Definition 3.1. (Balachandran [3]) The set $y(t) = \{x(t), \beta(t,s)\}$, where $\beta(t,s) = u(s)$ for $s \in [\min\ h_i(t), t)$ is said to be the complete state of the system (10) at time t.

Definition 3.2. System (10) is said to be relatively controllable on $[0,T]$ if for every complete state $y(t)$ and every $x_1 \in \mathbb{R}^n$, there exists a control $u(t)$ defined on $[0,T]$, such that the solution of system (10) satisfies $x(T) = x_1$.

The solution of the system (10) is expressed as [15, 16]

$$
\begin{aligned}
x(t) = X_\alpha(t)\phi(0) \quad &+ \quad B\int_{-h}^{0} (t-s-h)^{\alpha-1} X_{\alpha,\alpha}(t-s-h)\phi(s)\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) \sum_{i=0}^{M} C_i u_i(\sigma_i(s))\mathrm{d}s.
\end{aligned} \tag{13}
$$

Using the time lead functions $r_i(t)$, the solution can be written as

$$
\begin{aligned}
x(t) &= X_\alpha(t)\phi(0) + B\int_{-h}^{0} (t-s-h)^{\alpha-1} X_{\alpha,\alpha}(t-s-h)\phi(s)\mathrm{d}s \\
&+ \sum_{i=0}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s)\mathrm{d}s.
\end{aligned} \tag{14}
$$

The solution (14) can be rewritten as

$$x(t) = x_L(t;\phi) + \sum_{i=0}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s)\mathrm{d}s,$$

where

$$x_L(t;\phi) = X_\alpha(t)\phi(0) + B\int_{-h}^{0} (t - s - h)^{\alpha-1} X_{\alpha,\alpha}(t - s - h)\phi(s)\mathrm{d}s.$$

By using the inequalities (12), we get

$$\begin{aligned} x(t) \quad = \quad & x_L(t;\phi) + \sum_{i=0}^{m} \int_{\sigma_i(0)}^{0} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s)\beta(s)\mathrm{d}s \\ & + \sum_{i=0}^{m} \int_{0}^{t} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s)\mathrm{d}s \\ & + \sum_{i=m+1}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s)\beta(s)\mathrm{d}s. \end{aligned}$$

For simplicity, let us write the solution as

$$x(t) = x_L(t;\phi) + H(t) + \sum_{i=0}^{m} \int_{0}^{t} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s)\mathrm{d}s, \qquad (15)$$

where

$$\begin{aligned} H(t) \quad = \quad & \sum_{i=0}^{m} \int_{\sigma_i(0)}^{0} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s)\beta(s)\mathrm{d}s \\ & + \sum_{i=m+1}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s)\beta(s)\mathrm{d}s. \end{aligned}$$

Now let us define the controllability Grammian matrix by

$$W = \sum_{i=0}^{m} \int_{0}^{T} (X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s))(X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s))^* \mathrm{d}s,$$

where the $*$ denotes the matrix transpose.

Theorem 3.3. The linear system (10) is relatively controllable on $[0,T]$ if and only if the controllability Grammian matrix

$$W = \sum_{i=0}^{m} \int_{0}^{T} (X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s))(X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s))^* \mathrm{d}s$$

is positive definite for some $T > 0$.

The proof of the theorem is similar as in [16].

4. INTEGRODIFFERENTIAL SYSTEMS

Consider the nonlinear fractional delay Integrodifferential systems with multiple delays in control of the form

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &= Ax(t) + Bx(t-h) + \sum_{i=0}^{M} C_i u(\sigma_i(t)) \\ &\quad + f(t, x(t), x(t-h), \int_0^t g(t,s,x(s-h))\mathrm{d}s, u(t)), t \in J, \end{aligned} \tag{16}$$

with $x(t) = \phi(t), -h < t \leq 0$, where $0 < \alpha < 1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and A, B are $n \times n$ matrices, C_i for $i = 0, 1, \ldots M$, are $n \times m$ matrices and $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function. Further we made the following assumptions:

(H3) The continuous function f satisfies the condition

$$\lim_{p \to \infty} \frac{|f(t,p)|}{|p|} = 0, \tag{17}$$

uniformly in $t \in J$, where $p = |x| + |y| + |u|$.

(H4) The continuous function f satisfies the condition

$$||f(t,p)|| \leq \sum_{j=1}^{q} \rho_j(t)\phi_j(p), \tag{18}$$

where $\phi_j : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ are measurable functions and $\rho_i : J \to \mathbb{R}_+$ are L_1 functions for $j = 1, 2, \ldots q$.

Let Q be the Banach space of continuous $\mathbb{R}^n \times \mathbb{R}^m$ valued functions defined on the interval J with the norm

$$||(x,u)|| = ||x|| + ||u||,$$

where $||x|| = \sup\{x(t) : t \in J\}$ and $||u|| = \sup\{u(t) : t \in J\}$. That is, $Q = C_n(J) \times C_m(J)$, where $C_n(J)$ is the Banach space of continuous $\mathbb{R}^n$ valued functions defined on the interval J with the sup norm.

(H5) $g : J \times J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the following condition

$$|\int_0^t g(t,s,x(s,h))\mathrm{d}s| \leq \sup\left(\int_0^t |a(t,s)|\mathrm{d}s\right)||x||$$

such that $\sup\left(\int_0^t |a(t,s)|\mathrm{d}s\right) \leq 1$.

Similar to the linear system, the solution of nonlinear system (17) using time lead function $r_i(t)$ is given as

$$\begin{aligned} x(t) &= x_L(t;\phi) + H(t) + \sum_{i=0}^{m} \int_0^t (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s)\mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s, x(s), x(s-1), \int_0^t g(t,s,x(s-h))\mathrm{d}s, u(s))\mathrm{d}s. \end{aligned} \tag{19}$$

Theorem 4.1. Assume that the hypotheses (H1) – (H3) and (H5) are satisfied and suppose that the linear fractional delay dynamical system (10) is relatively controllable. Then the nonlinear system (17) is relatively controllable on J.

Proof. Define $\Psi : Q \to Q$ by

$$\Psi(x, u) = (y, v)$$

where

$$\begin{aligned} v(t) &= (T - r_i(t))^{1-\alpha}(X_{\alpha,\alpha}(T - r_i(t))C_i^* \dot{r}_i(t))^* \qquad (20)\\ &\times W^{-1}\Bigg[x_1 - x_L(T;\phi) - \sum_{i=0}^{m}\int_{\sigma_i(0)}^{0}(T - r_i(s))^{\alpha-1}X_{\alpha,\alpha}(T - r_i(s))C_i\dot{r}_i(s)\beta(s)\mathrm{d}s\\ &- \sum_{i=m+1}^{M}\int_0^T (T - r_i(s))^{\alpha-1}X_{\alpha,\alpha}(T - r_i(s))C_i\dot{r}_i(s)\beta(s)\mathrm{d}s\\ &- \int_0^T (T-s)^{\alpha-1}X_{\alpha,\alpha}(T-s)f(s, x(s), x(s-h), \int_0^t g(t, s, x(s-h))\mathrm{d}s, u(s))\mathrm{d}s\Bigg], \end{aligned}$$

and

$$\begin{aligned} y(t) &= x_L(t;\phi) + \sum_{i=0}^{m}\int_{\sigma_i(0)}^{0}(t - r_i(s))^{\alpha-1}X_{\alpha,\alpha}(t - r_i(s))C_i\dot{r}_i(s)\beta(s)\mathrm{d}s \qquad (21)\\ &+ \sum_{i=0}^{m}\int_0^t (t - r_i(s))^{\alpha-1}X_{\alpha,\alpha}(t - r_i(s))C_i\dot{r}_i(s)v(s)\mathrm{d}s\\ &+ \sum_{i=m+1}^{M}\int_{\alpha_0}^{t}(t - r_i(s))^{\alpha-1}X_{\alpha,\alpha}(t - r_i(s))C_i\dot{r}_i(s)\beta(s)\mathrm{d}s\\ &+ \int_0^t (t-s)^{\alpha-1}X_{\alpha,\alpha}(t-s)f(s, x(s), x(s-1), \int_0^t g(t, s, x(s-h))\mathrm{d}s, v(s))\mathrm{d}s. \end{aligned}$$

Let

$$\begin{aligned} a_i &= \sup ||X_{\alpha,\alpha}(T - r_i(s))||, b_i = \sup ||\dot{r}_i(s)||, i = 0, 1, 2, \ldots, M,\\ \nu &= \sup ||\beta(s)||, \vartheta = \sup ||X_{\alpha,\alpha}(T - s)||,\\ \mu &= \sum_{i=0}^{m} a_i b_i ||C_i||N_i + \sum_{i=m+1}^{M} a_i b_i ||C_i||M_i,\\ c_i &= 4a_i b_i ||C_i^*||||W^{-1}||\nu\alpha^{-1}T^\alpha, d_i = 4a_i b_i ||C_i^*||||W^{-1}||[|x_1| + \beta + \mu],\\ a &= \max\{b\alpha^{-1}T^\alpha ||C_i||, 1\}, b = \sum_{i=0}^{m} a_i b_i L_i, c_2 = 4\vartheta\alpha^{-1}T^\alpha, d_2 = 4[\beta + \nu\mu],\\ N_i &= \int_{\sigma_i(0)}^{0}(T - r_i(s))^{\alpha-1}\mathrm{d}s, M_i = \int_{\sigma_i(0)}^{\sigma_i(T)}(T - r_i(s))^{\alpha-1}\mathrm{d}s, \end{aligned}$$

$$\begin{aligned}
L_i &= \int_0^T (T - r_i(s))^{\alpha-1} \mathrm{d}s, c = \max\{c_i, c_2\}, d = \max\{d_i, d_2\} \\
\sup|f| &= \{\sup|f(t, x(t), x(t-1), \int_0^t g(t, s, x(s-h))\mathrm{d}s, u(t))|, t \in J\}.
\end{aligned}$$

Then

$$\begin{aligned}
|v(t)| &\leq ||C_i^*||a_i b_i||W^{-1}||[||x_1|| + \beta + \mu] + a_i b_i||C_i^*||||W^{-1}||\vartheta\alpha^{-1}T^\alpha \\
&\leq \left(\frac{d_i}{4a} + \frac{c_i}{4a}\sup|f|\right) \\
&\leq \frac{1}{4a}(d + c\sup|f|)
\end{aligned}$$

and

$$\begin{aligned}
|y(t)| &\leq \beta + \nu\mu + \left(\sum_{i=0}^m a_i b_i||C_i||L_i\alpha^{-1}T^\alpha\right) v(s) + \vartheta\alpha^{-1}T^\alpha \sup|f| \\
&\leq \frac{d}{4} + \frac{1}{4}(d + c\sup|f|) + \frac{c}{4}\sup|f| \\
&\leq \frac{d}{2} + \frac{c}{2}\sup|f|.
\end{aligned}$$

By Proposition 1 in [11], the function f satisfies the following conditions. For each pair of positive constants c and d, there exists a positive constant r such that, for $|p| \leq r$,

$$c|f(t,p)| + d \leq r \text{ for all } t \in J. \tag{22}$$

Also, for given c and d, if r is a constant such that $r < r_1$, r will also satisfy (22). Now take c and d as given above and choose r so that (22) is satisfied. Therefore $||x|| \leq \frac{r}{2}$ and $||u|| \leq \frac{r}{2}$, then $|x(s)| + |y(s)| \leq r$, for all $s \in J$. It follows that $d + c\sup|f| \leq r$. Therefore $|u(s)| \leq \frac{r}{4a}$ for all $s \in J$ and hence $||u|| \leq \frac{r}{4a}$, which gives $||x|| \leq \frac{r}{2}$. Thus

$$Q(r) = \{(x,u) \in Q : ||x|| \leq \frac{r}{2} \text{ and } ||u|| \leq \frac{r}{2}\}.$$

Then Ψ maps $Q(r)$ into itself. Our objective is to show that Ψ has a fixed point; since f is continuous, it follows that Ψ is continuous. Let Q_0 be a bounded subset of Q. Consider a sequence $\{(y_j, v_j)\}$ contained in $\Psi(Q_0)$, where we let

$$(y_j, v_j) = \Psi(x_j, u_j),$$

for some $(x_j, u_j) \in Q_0$, for $j = 1, 2, \ldots$. Since f is continuous, $|f(s, x_j(s), x_j(s - h), \int_0^t g(t, s, x_j(s-h))\mathrm{d}s, u_j(s))|$ is uniformly bounded for all $s \in J$ and $j = 1, 2, 3, \ldots$. It follows that $\{(y_j, v_j)\}$ is a bounded sequence in Q. Hence $\{v_j(t)\}$ is equicontinuous and a uniformly bounded sequence on $[0, t_1]$. Since $\{y_j(t)\}$ is a uniformly bounded and equicontinuous sequence on $[-h, t_1]$, an application of Ascoli's theorem yields a further subsequence of $\{(y_j, v_j)\}$ which converges in Q to some (y_0, v_0). It follows that $\Psi(Q_0)$ is sequentially compact, hence the closure is sequentially compact. Thus Ψ is completely continuous. Since $Q(r)$ is closed, bounded and convex, the Schauder fixed point theorem implies that Ψ has a fixed point $(x, u) \in Q(r)$, such that $(y, v) = \Psi(x, u) = (x, u)$. It follows that

$$\begin{aligned} x(t) &= x_L(t;\phi) + H(t) + \sum_{i=0}^{m} \int_0^t (t - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(t - r_i(s)) C_i \dot{r}_i(s) u(s) \mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1} X_{\alpha,\alpha}(t-s) f(s, x(s), x(s-h), u(s)) \mathrm{d}s, \end{aligned} \tag{23}$$

for $t \in J$ and $x(t) = \phi(t)$ for $t \in [-h, 0]$.

$$\begin{aligned} x(T) &= x_L(T;\phi) + H(T) + \sum_{i=0}^{m} \int_0^T (T - r_i(s))^{\alpha-1} X_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i(s) \\ &\quad \times \Big\{ (T - r_i(s))^{1-\alpha} (X_{\alpha,\alpha}(T - r_i(s)) C_i \dot{r}_i)^* W^{-1} \\ &\quad \times \Big[x_1 - x_L(T;\phi) - H(T) - \int_0^T (T-s)^{\alpha-1} \\ &\quad \times X_{\alpha,\alpha}(T-s) f(s, x(s), , x(s-h), u(s)) \mathrm{d}s \Big] \Big\} \mathrm{d}s \\ &\quad + \int_0^T (T-s)^{\alpha-1} X_{\alpha,\alpha}(T-s) \\ &\quad \times f(s, x(s), x(s-h), \int_0^t g(t, s, x(s-h)) \mathrm{d}s, u(s)) \mathrm{d}s, \\ &= x_1. \end{aligned}$$

Hence the system (17) is relatively controllable on J. □

Theorem 4.2. Assume that the hypotheses (H1), (H2), (H4) and (H5) are satisfied and suppose that

$$det W(0, T) > 0. \tag{24}$$

Then the nonlinear system (17) is relatively controllable on J.

P r o o f. Now let

$$\psi_j(r) = \sup\{\phi_j(p); ||p|| \le r\}.$$

Since (H3) holds, there exists $r_0 > 0$ such that

$$r_0 - \sum_{i=1}^{q} c_j \psi_j(r_0) \ge d,$$

which implies that

$$\sum_{j=1}^{q} c_j \psi_j(r_0) + d \leq r_0.$$

Define the operator $\Psi : Q \to Q$ by

$$\Psi(x,u) = (y,v)$$

where y and v are defined as in (20) and (21). Similar to the proof of Theorem 4.1, it can be shown that Ψ has a fixed point $(z,v) \in Q(r)$, such that $\Psi(x,u) = (y,v) = (x,u)$. Hence $x(t)$ is the solution of the system (17) and easy to verify $x(T) = x_1$. Thus the control $u(t)$ steers the system (17) from the initial complete state $y(0)$ to x_1 on J. Hence the system (17) is relatively controllable on J. □

5. EXAMPLES

In this section, we apply the results obtained in the previous sections to the following fractional delay dynamical systems with multiple delays in control.

Example 5.1. Consider the linear fractional delay dynamical system with delay in control of the form

$$\begin{aligned} {}^{C}D^{\frac{1}{2}}x(t) &= x(t) + x(t-2) + u(t) + u(t-1), \\ x(t) &= 1, -2 \leq t \leq 0, \end{aligned} \tag{25}$$

where $\alpha = \frac{1}{2}$, $h = 2$, $\sigma = 1$, $\phi(t) = 1$, $A = 1, B = 1, C_0 = 1$ and $C_1 = 1$; the solution of the equation (25) by taking Laplace transform is of the form

$$\begin{aligned} x(t) &= \sum_{n=0}^{[t]} (t-n)^{\frac{1}{2}n} E_{\frac{1}{2},\frac{1}{2}n+1}((t-n)^{\frac{1}{2}}) \\ &+ \sum_{n=0}^{[t]} B^{n+1} \int_{-2}^{0} (t-s-n-2)^{\frac{1}{2}n-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}(n+1)}((t-s-n-2)^{\frac{1}{2}})\phi(s)\mathrm{d}s \\ &+ \sum_{n=0}^{[t]} \int_{0}^{t-n} (t-s-n)^{\frac{1}{2}n-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}n+\frac{1}{2}}((t-s-n)^{\frac{1}{2}})u(s)\mathrm{d}s \\ &+ \sum_{n=0}^{[t]} \int_{0}^{t-n} (t-s-n)^{\frac{1}{2}n-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}n+\frac{1}{2}}((t-s-n)^{\frac{1}{2}})u(s-1)\mathrm{d}s. \end{aligned} \tag{26}$$

Now consider the controllability on $[0,2]$, where $[t] = 0$. The solution (26) reduces to the form

$$\begin{aligned} x(t) &= E_{\frac{1}{2}}(t^{\frac{1}{2}}) + \int_{-2}^{0} (t-s-2)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}((t-s-2)^{\frac{1}{2}})\mathrm{d}s \\ &\quad + \int_{0}^{t} (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}((t-s)^{\frac{1}{2}})u(s)\mathrm{d}s + \int_{0}^{t} (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}((t-s)^{\frac{1}{2}})u(s-1)\mathrm{d}s, \end{aligned}$$

where

$$\begin{aligned} X_\alpha(t) &= E_{\frac{1}{2}}(t^{\frac{1}{2}}), \\ X_{\alpha,\alpha}(t-s-2) &= (t-s-2)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}((t-s-2)^{\frac{1}{2}}) \end{aligned}$$

and

$$X_{\alpha,\alpha}(t-s) = (t-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}((t-s)^{\frac{1}{2}}).$$

Using time lead function, the solution can be written in the form

$$\begin{aligned} x(t) &= E_{\frac{1}{2}}(t^{\frac{1}{2}}) + t^{\frac{1}{2}} E_{\frac{1}{2},\frac{3}{2}}(t^{\frac{1}{2}}) \\ &\quad + \sum_{i=0}^{1} \int_0^t (t - r_i(s))^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}((t - r_i(s))^{\frac{1}{2}}) \dot{r}_i(s) u(s) \mathrm{d}s, \end{aligned}$$

where $r_0(s) = s$ and $r_1(s) = s - 1$. The controllability Grammian matrix is defined by

$$\begin{aligned} W &= \sum_{i=0}^{1} \int_0^2 [E_{\frac{1}{2},\frac{1}{2}}((2 - r_i(s))^{\frac{1}{2}}) \dot{r}_i(s)][E_{\frac{1}{2},\frac{1}{2}}((2 - r_i(s))^{\frac{1}{2}}) \dot{r}_i(s)]^* \mathrm{d}s, \\ &= \sum_{i=0}^{1} \int_0^2 [E_{\frac{1}{2},\frac{1}{2}}((2 - r_i(s))^{\frac{1}{2}})][E_{\frac{1}{2},\frac{1}{2}}((2 - r_i(s))^{\frac{1}{2}})]^* \mathrm{d}s, \\ &= \int_0^2 (E_{\frac{1}{2},\frac{1}{2}}((2-s)^{\frac{1}{2}}))(E_{\frac{1}{2},\frac{1}{2}}((2-s)^{\frac{1}{2}}))^* \mathrm{d}s \\ &\quad + \int_0^2 (E_{\frac{1}{2},\frac{1}{2}}((2-s+1)^{\frac{1}{2}}))(E_{\frac{1}{2},\frac{1}{2}}((2-s+1)^{\frac{1}{2}}))^* \mathrm{d}s; \end{aligned}$$

on simplifying, we get

$$W = 530.780 > 0$$

is positive definite. Hence, by the Theorem 3.3, the system is controllable on $[0, 2]$. Next we give the numerical simulation of the state and control variables for the system (25) and the control

$$u(t) = [(2-t)^{\frac{1}{2}} (E_{\frac{1}{2},\frac{1}{2}}(2-t)^{\frac{1}{2}})^* + (1-t)^{\frac{1}{2}} (E_{\frac{1}{2},\frac{1}{2}}(1-t)^{\frac{1}{2}})^*] W^{-1} (10 - E_{\frac{1}{2}}(2) - E_{\frac{1}{2},\frac{3}{2}}(2)),$$

which steers $x(0) = 1$ to $x(2) = 10$.

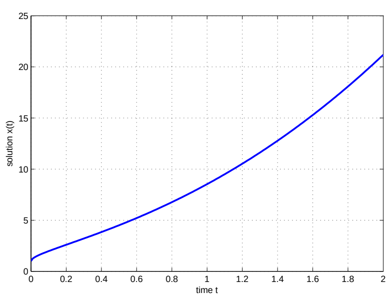

Figure 1.

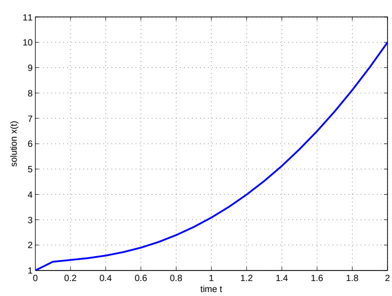

Figure 2.

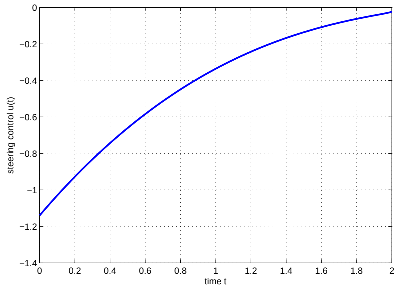

Figure 3.

Figure 1 represents the trajectory of the equation (25) without control starting from the initial point 1 and not reaching the final point 10 in $[0,2]$, Figure 2 represents the trajectory of the equation (25) with control starting from the initial point 1 and reaching the final point 10 in $[0,2]$ and Figure 3 represents the steering control.

Example 5.2. Consider the nonlinear fractional delay dynamical system with delay in control of the form

$$\begin{aligned} {}^{C}D^{\frac{1}{2}}x(t) &= x(t)+x(t-2)+C_0u(t)+C_1u(t-1)+\frac{x(t)+x(t-2)+\int_0^t \sin xe^{-\frac{1}{2}(t-s)}\mathrm{d}s}{x^2(t)+x^2(t-2)+u(t)}, \\ x(t) &= \phi(t), \end{aligned} \tag{27}$$

where α, σ, h, ϕ, A, B, C_0 and C_1 are defined as in (5.1) and the nonlinear function is

$$f(t,x(t),x(t-2),\int_0^t g(t,s,x(s))\mathrm{d}s,u(t)) = \frac{x(t)+x(t-2)+\int_0^t \sin xe^{-\frac{1}{2}(t-s)}\mathrm{d}s}{x^2(t)+x^2(t-2)+u(t)}. \tag{28}$$

Consider the controllability on $[0,2]$. Let

$$\lim_{p\to\infty}\frac{|x(t)|+|x(t-2)|+|\int_0^t \sin xe^{-\frac{1}{2}(t-s)}\mathrm{d}s|}{(|x^2(t)|+|x^2(t-2)|+|u(t)|)|p|} \le \frac{3|x(t)|}{(|x(t)|+|u(t)|)(2|x^2(t)|+|u(t)|)}$$

which tends to zero as $p\to\infty$. Since the linear delay system (25) is controllable on $[0,2]$ and the nonlinear function satisfies the hypotheses in the Theorem 4.1, the nonlinear delay system (27) is controllable on $[0,2]$.

Examples 5.1 and 5.2 describes the conditions when A, B, C_0 and C_1 are constant. Following these examples, 5.3 and 5.4 illustrate the conditions when A, B, C_0 and C_1 are matrices.

Example 5.3. Consider the linear fractional delay dynamical systems with delay in control of the form

$$\begin{aligned} {}^{C}D^{\frac{3}{4}}x(t) &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x(t-2) \\ &\quad + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t-1), \end{aligned} \tag{29}$$

where $\alpha = \frac{3}{4}$, $h = 2$, $\sigma = 1$ $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, $C_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $C_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $x(t) = \phi(t) \in \mathbb{R}^2$ with initial condition $x(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and finial condition $x(2) = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$. Here $x(t)$ is not the state variable but it is the pseudo state variable. The solution of the equation (29), by using Laplace transform, is of the form

$$\begin{aligned}
x(t) &= \sum_{n=0}^{[t]} B^n (t-n)^{\frac{3}{4}n} E_{\frac{3}{4},\frac{3}{4}n+1}(A(t-n)^{\frac{3}{4}}) \\
&+ B\sum_{n=0}^{[t]} B^n \int_{-1}^{0} (t-s-n-2)^{\frac{3}{4}n+\frac{3}{4}-1} E_{\frac{3}{4},\frac{3}{4}(n+1)}(A(t-s-n-2)^{\frac{3}{4}})\phi(s)\mathrm{d}s \\
&+ \sum_{n=0}^{[t]}\sum_{i=0}^{1} B^n \int_{0}^{t-n} (t-r_i(s)-n)^{\frac{3}{4}n-\frac{1}{4}} (A(t-r_i(s)-n))^{\frac{3}{4}} C_i \dot{r}_i(s) u(s)\mathrm{d}s.
\end{aligned}$$

Now considering the controllability on $[0, 2]$, where [t]=0, we have

$$\begin{aligned}
x(t) &= E_{\frac{3}{4},1}(At^{\frac{3}{4}})x(0) + B\int_{-2}^{0} (t-s-2)^{-\frac{1}{4}} E_{\frac{3}{4},\frac{3}{4}}(A(t-s-2)^{\frac{3}{4}})x(s)\mathrm{d}s \\
&+ \sum_{i=0}^{1} \int_{0}^{t} (t-r_i(s))^{-\frac{1}{4}} E_{\frac{3}{4},\frac{3}{4}}(A(t-r_i(s))^{\frac{3}{4}}) C_i \dot{r}_i(s) u(s)\mathrm{d}s;
\end{aligned}$$

on further solving, we get

$$\begin{aligned}
x(t) &= E_{\frac{3}{4}}(At^{\frac{3}{4}})x(0) + Bt^{\frac{3}{4}} E_{\frac{3}{4},\frac{7}{4}}(A(t^{\frac{3}{4}}))x(0) \\
&+ \sum_{i=0}^{1} \int_{0}^{t} (t-r_i(s))^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(A(t-r_i(s))^{\frac{1}{2}}) C_i \dot{r}_i(s) u(s)\mathrm{d}s,
\end{aligned}$$

where

$$E_{\frac{3}{4},1}(At^{\frac{3}{4}}) = \begin{pmatrix} E_{\frac{3}{2},1}(2t^{\frac{3}{2}}) & t^{\frac{3}{2}} E_{\frac{3}{2},\frac{7}{4}}(2t^{\frac{3}{2}}) \\ -2t^{\frac{3}{4}} E_{\frac{3}{2},\frac{7}{4}}(2t^{\frac{3}{2}}) & E_{\frac{3}{2},1}(2t^{\frac{3}{2}}) \end{pmatrix}, \ E_{\frac{3}{4},\frac{3}{4}}(A(2-s)^{\frac{3}{4}}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where

$$\begin{aligned}
a_{11} &= (2-s)^{-\frac{1}{4}} E_{\frac{3}{2},\frac{3}{4}}(2(2-s)^{\frac{3}{4}}), \\
a_{12} &= (2-s)^{\frac{1}{2}} E_{\frac{3}{2},\frac{3}{2}}(2(2-s)^{\frac{3}{2}}), \\
a_{21} &= -2(2-s)^{\frac{1}{2}} E_{\frac{3}{2},\frac{3}{2}}(2(2-s)^{\frac{3}{2}}), \\
a_{22} &= (2-s)^{-\frac{1}{4}} E_{\frac{3}{2},\frac{3}{4}}(2(2-s)^{\frac{3}{4}}),
\end{aligned}$$

and

$$E_{\frac{3}{4},\frac{3}{4}}(A(2-s+1)^{\frac{3}{4}}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$
\begin{aligned}
b_{11} &= (2-s+1)^{-\frac{1}{4}} E_{\frac{3}{2},\frac{3}{4}}(2(2-s+1)^{\frac{3}{4}}), \\
b_{12} &= (2-s+1)^{\frac{1}{2}} E_{\frac{3}{2},\frac{3}{2}}(2(2-s+1)^{\frac{3}{2}}), \\
b_{21} &= -2(2-s+1)^{\frac{1}{2}} E_{\frac{3}{2},\frac{3}{2}}(2(2-s+1)^{\frac{3}{2}}), \\
b_{22} &= (2-s+1)^{-\frac{1}{4}} E_{\frac{3}{2},\frac{3}{4}}(2(2-s+1)^{\frac{3}{4}}).
\end{aligned}
$$

The Grammian matrix is defined by

$$W = \sum_{i=0}^{1} \int_0^2 [E_{\frac{3}{4},\frac{3}{4}}(A(2-r_i(s))^{\frac{3}{4}})C_i\dot{r}_i(s)][E_{\frac{3}{4},\frac{3}{4}}(A(2-r_i(s))^{\frac{3}{4}})C_i\dot{r}_i(s)]^* \mathrm{d}s,$$

where $r_i(s)$ is a time lead function which is defined by $r_0(s) = s$ and $r_1(s) = s-1$. Then the Grammian matrix can be written as

$$
\begin{aligned}
W &= \int_0^2 [E_{\frac{3}{4},\frac{3}{4}}(A(2-s))^{\frac{3}{4}})C_0][E_{\frac{3}{4},\frac{3}{4}}(A(2-s))^{\frac{3}{4}})C_0]^* \mathrm{d}s \\
&\quad + \int_0^2 [E_{\frac{3}{4},\frac{3}{4}}(A(2-s+1))^{\frac{3}{4}})C_1][E_{\frac{3}{4},\frac{3}{4}}(A(2-s+1))^{\frac{3}{4}})C_1]^* \mathrm{d}s,
\end{aligned}
$$

on further calculation, we get

$$W = \begin{pmatrix} 0.0003 \times 10^4 & 0 \\ 0 & 5.7995 \times 10^4 \end{pmatrix} > 0$$

is positive definite. Then, by the Theorem 3.3, the system is controllable on $[0, 2]$. Next we give the numerical simulation of the state and control variables for the system (29) and the control

$$
\begin{aligned}
u(t) &= [(2-t)^{\frac{1}{2}}(E_{\frac{1}{2},\frac{1}{2}}(A(2-t)^{\frac{1}{2}})C_0)^* + (1-t)^{\frac{1}{2}}(E_{\frac{1}{2},\frac{1}{2}}(A(1-t)^{\frac{1}{2}})C_1)^*] \\
&\quad \times W^{-1}(x(2) - E_{\frac{1}{2}}(2)x(0) - E_{\frac{1}{2},\frac{3}{2}}(2)x(0)),
\end{aligned}
$$

which steers $x(0) = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$ to $x(2) = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$.

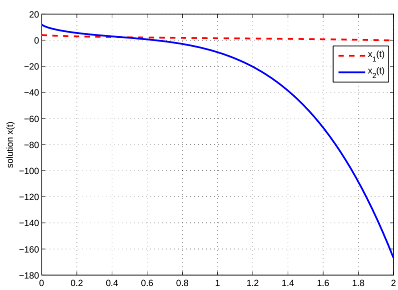

Figure 4.

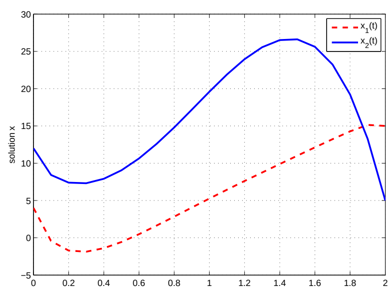

Figure 5.

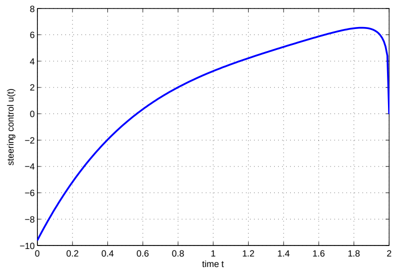

Figure 6.

Figure 4 represents the trajectory of the equation (29) without control starting from the initial vector $x(0) = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$ and not reaching the final point $x(2) = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$ in [0, 2], Figure 5 represents the trajectory of the equation (29) with control starting from the initial point $x(0) = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$ and reaching the final point $x(2) = \begin{pmatrix} 15 \\ 5 \end{pmatrix}$ in [0, 2] and Figure 6 represents the steering control.

Example 5.4. Consider the nonlinear fractional delay dynamical system with delay in control of the form

$$\begin{aligned} {}^{C}D^{\frac{3}{4}}x(t) &= Ax(t) + Bx(t-2) + C_0u(t) + C_1u(t-1) \\ &\quad + f(t, x(t), x(t-2), \int_0^t g(t,s,x(s))\mathrm{d}s, u(t)), \end{aligned} \tag{30}$$

where $\alpha, h, \sigma, \phi(t)$, A, B, C_0 and C_1 are defined as in (5.3). Consider the controllability on the $[0, 2]$. The nonlinear function f is defined by

$$f(t, x(t), x(t-2), u(t)) = \begin{pmatrix} 0 \\ \frac{x_1(t)\sin t}{x_1^2(t)+x_2^2(t)} + \frac{+\int_0^t x_1(s)e^{-\sin s}\mathrm{d}s}{x_1^2(t-2)+u(t)} \end{pmatrix}. \tag{31}$$

Since the linear delay system (29) is controllable and nonlinear function (31) satisfies the hypotheses in the Theorem 4.1 similarly as in example 5.2, we say that the nonlinear delay system (30) is controllable on interval $[0, 2]$. Let us take the nonlinear function as

$$f(t, x(t), x(t-2), u(t)) = \begin{pmatrix} 1 \\ \frac{x_1(t)}{x_1(t)+x_2(t)} + \frac{\int_0^t x_1(s)e^{-s}\mathrm{d}s}{1+x_1(t)+u(t)} \end{pmatrix}. \tag{32}$$

It does not satisfy the hypotheses in the Theorem 4.1, because it does not tend to zero as $p \to \infty$ but satisfies the hypotheses in the Theorem 4.2. Hence the nonlinear system (30) is controllable on $[0, 2]$.

6. CONCLUSION

This paper deals with the controllability of linear and nonlinear fractional delay integrodifferential systems with multiple delays in control. It should be noted that the solution representation has been established by using Laplace transform technique and Mittag–Leffler function. Necessary and sufficient conditions for the controllability of linear delay systems are derived. Consequently sufficient conditions for nonlinear delay integrodifferential systems are established by using Schauder's fixed point theorem. To show the effectiveness of the theory, examples are provided to illustrate the results.

ACKNOWLEDGMENT

The first author is thankful to the University Grants Commission (UGC), New Delhi for providing MANF (Maulana Azad National Fellowship) to carryout the research work.

(Received December 8, 2015)

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