KYBERNETIKA — VOLUME 52 (2016), NUMBER 6, PAGES 988–1002

STABILIZATION OF NONLINEAR STOCHASTIC SYSTEMS WITHOUT UNFORCED DYNAMICS VIA TIME–VARYING FEEDBACK

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In this paper we give sufficient conditions under which a nonlinear stochastic differential system without unforced dynamics is globally asymptotically stabilizable in probability via time–varying smooth feedback laws. The technique developed to design explicitly the time–varying stabilizers is based on the stochastic Lyapunov technique combined with the strategy used to construct bounded smooth stabilizing feedback laws for passive nonlinear stochastic differential systems. The interest of this work is that the class of stochastic systems considered in this paper contains a lot of systems which cannot be stabilized via time–invariant feedback laws.

Keywords: stochastic differential systems, smooth time–varying feedback law, global asymptotic stability in probability

Classification: 60H10, 93C10, 93D05, 93D15, 93E15

1. INTRODUCTION

The aim of this paper is to design explicitly time–varying feedback laws for global asymptotic stabilization in probability of nonlinear stochastic differential systems without unforced dynamics. The results obtained in this work lie on an extension to the stochastic context of the stabilization techniques via time–varying feedback laws for deterministic driftless systems developed by Lin in [15]. The impossibility to stabilize by means of time–invariant feedback laws control systems without unforced dynamics has been noticed on many occasions. For deterministic systems, Samson [17] has overcome this difficulty by stabilizing some nonholonomic robots by using time–varying feedback laws. The present paper follows this line or research for a class of stochastic systems which cannot be stabilized via time–invariant feedback laws.

Deterministic nonlinear driftless systems is an important class of nonlinear systems including mechanical systems with nonholonomic constraints studied by many authors in the past decades. The stabilization of driftless affine systems has received a great deal of attention in the literature. In [4], Brockett has given a necessary condition which shows that controllable driftless nonlinear systems may fail to be asymptotically stabilizable by time–invariant feedback laws. This fact has been noticed at many occasions when

DOI: 10.14736/kyb-2016-6-0988

studying the stabilizability of mechanical systems with nonholonomic constraints (see Samson [17] or Campion, d'Andréa–Novel and Bastin [5] for example). The problem of asymptotic stabilization for some nonholonomic systems has been solved by Samson [17] by using time–varying feedback controls. This result has lead to a fruitful line of research on the stabilization of control nonlinear systems by time–varying feedback laws. In [6], Coron has proved that driftless deterministic affine systems can be globally asymptotically stabilized by periodic state feedback laws provided the Lie algebra generated by the system coefficients has full rank. However, the question on how to design explicitly the stabilizing time–varying feedback laws has remained unsolved. This question has been solved later on by Pomet [16] and Coron and Pomet [7] by using an approach based on the Lyapunov stability theory combined with the La Salle invariance principle, similar to that developed by Jurdjevic–Quinn in [12]. The results reported in the previously cited papers show that a time–varying control strategy is a natural solution to overcome some topological obstruction which may occur in smooth state feedback stabilization. A geometric interpretation on why the topological obstruction can be weakened by using time–varying feedback has been given by Sepulchre, Campion and Wertz [18] and Sontag [19]. Note that the results obtained by Pomet in [16] has been extended to the stochastic context by Florchinger in [11].

In this work, we are concerned with the problem of global asymptotic stabilization in probability for nonlinear stochastic differential systems without unforced dynamics. By using the techniques developed in [9] to obtain a stochastic version of the Jurdjevic–Quinn theorem with the approach used by Pomet [16] and Lin [15] to design time–varying stabilizers for deterministic driftless controllable systems, we propose a constructive method to design a stabilizing time–varying feedback law provided some rank condition involving the drift coefficients is satisfied. The main tools used in this paper are the stochastic Lyapunov stability theory introduced by Khasminskii in [13] combined with the stochastic La Salle invariance principle proved by Kushner [14] and the bounded feedback design technique for passive stochastic differential systems developed in [10]. The class of systems considered in this paper cannot be incorporated in the framework handled by the works exposed in [1]–[3] on the stabilization of time–varying stochastic systems developed in the past years.

This paper is divided into four sections and is organized as follows. In section one, we recall a result on global asymptotic stabilization in probability for affine stochastic differential systems proved in [10] by using a methodology developed to design bounded smooth feedback stabilizers for passive stochastic systems. In section two, we introduce the class of nonlinear stochastic differential systems we are dealing with in this paper and we provide a global stabilization result by time–varying feedback for affine stochastic differential systems without unforced dynamics that extends Theorem 1 in [16] to the stochastic context. In section three, we pursue the idea initiated in the previous section to obtain global asymptotic stabilization in probability via time–varying feedback for the class of stochastic differential systems without unforced dynamics considered in this paper. With this aim, we design explicitly a time–varying stabilizer with a degree of freedom represented by a tuning function when assuming that the Lie algebra generated by the drift coefficients has full rank. The proof is constructive and is carried out by combining the periodic time–varying design technique for deterministic system proposed by Lin in [15] with the bounded state feedback strategy for stochastic differential systems reported in [10]. This result is illustrated with an example for which we design explicitly a time–varying stabilizer. In section four, we turn our attention to the global asymptotic stabilization in probability for the class of nonlinear stochastic differential systems considered in this paper via time–varying dynamic feedback. The result proved in this section provides a nice alternative to the global asymptotic stabilization in probability via state feedback obtained in the previous sections.

2. STABILIZABILITY OF AFFINE STOCHASTIC SYSTEMS

In this section, we recall a stabilizability result for affine stochastic differential systems proved in [10] by using the concept of passivity for stochastic systems.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which all the processes considered in this work are defined. Consider the $\mathbb{R}^n$–valued stochastic process $(x_t)_{t\geq 0}$ solution of the stochastic differential system written in the sense of Itô,

$$x_t = x_0 + \int_0^t \left(f(x_s) + \sum_{i=1}^m u_i \overline{f}^i(x_s) \right) \mathrm{d}s + \sum_{k=1}^r \int_0^t \left(g_k(x_s) + \sum_{i=1}^m u_i \overline{g}_k^i(x_s) \right) \mathrm{d}w_s^k \quad (1)$$

where

1. x_0 is given in $\mathbb{R}^n$,
2. $(w_t)_{t\geq 0}$ is a standard Wiener process with values in $\mathbb{R}^r$,
3. u is a measurable control law with values in $\mathbb{R}^m$,
4. f, $\overline{f}^i$, $1 \leq i \leq m$, g_k, $1 \leq k \leq r$, and $\overline{g}_k^i$, $1 \leq k \leq r$, $1 \leq i \leq m$, are smooth Lipschitz functions mapping $\mathbb{R}^n$ into $\mathbb{R}^n$, vanishing in the origin and such that there exists a nonnegative constant K such that for any $x \in \mathbb{R}^n$,

$$|f(x)| + \sum_{i=1}^m |\overline{f}^i(x)| + \sum_{k=1}^r |g_k(x)| + \sum_{k=1}^r \sum_{i=1}^m |\overline{g}_k^i(x)| \leq K(1 + |x|).$$

With the stochastic differential system (1) introduce the second order differential operators Λ_i, $1 \leq i \leq m$, defined for any function φ in $C^2(\mathbb{R}^n, \mathbb{R})$ by

$$\Lambda_i \varphi(x) = \nabla\varphi(x) \overline{f}^i(x) + \sum_{k=1}^r \mathrm{Tr}\left(g_k(x) \overline{g}_k^i(x)^\tau \nabla^2 \varphi(x) \right)$$

and the first order differential operators $\mathcal{G}_k$, $1 \leq k \leq r$, defined for any function φ in $C^1(\mathbb{R}^n, \mathbb{R})$ by

$$\mathcal{G}_k \varphi(x) = \nabla\varphi(x) g_k(x).$$

Then, denoting by L_0 the infinitesimal generator of the stochastic process solution of the stochastic differential system (1) when $u = 0$, the following stabilization result has been proved in [10].

Theorem 2.1. Assume that there exists a proper smooth Lyapunov function V defined on $\mathbb{R}^n$ such that $L_0V(x) = 0$ for every $x \in \mathbb{R}^n$, the matrix

$$D(x) = Id_m + \frac{1}{2}\sum_{k=1}^{r} \overline{g}_k(x)^\tau \nabla^2 V(x) \overline{g}_k(x)$$

is invertible for every $x \in \mathbb{R}^n$ and the set

$$\begin{aligned}\Gamma = \Big\{ x \in \mathbb{R}^n \ / \ \mathcal{G}_{i_0}^{\alpha_0} L_0^{\beta_0} \ldots \mathcal{G}_{i_k}^{\alpha_k} L_0^{\beta_k} \Lambda_j V(x) = 0 \text{ and } \mathcal{G}_{i_0}^{\alpha_0} L_0^{\beta_0} \ldots \mathcal{G}_{i_k}^{\alpha_k} L_0^{\beta_k+1} V(x) = 0, \\ \forall j \in \{1, \ldots, m\}, \forall k \in \mathbb{N}, \qquad (2) \\ \forall i_0, \ldots, i_k \in \{1, \ldots, r\}, \forall \alpha_0, \beta_0, \ldots, \alpha_k, \beta_k \in \{0, \ldots, k\} \text{ s.t. } \sum_{i=0}^{k} (\alpha_i + \beta_i) = k \Big\}\end{aligned}$$

is reduced to $\{0\}$. Then, the stochastic differential system (1) is globally asymptotically stabilizable in probability by the smooth feedback law defined on $\mathbb{R}^n$ by

$$u(x) = -\beta \frac{D(x)^{-1} \left(\Lambda V(x)\right)^\tau}{1 + ||D(x)^{-1} \left(\Lambda V(x)\right)^\tau ||^2}$$

for any $\beta > 0$ where $\Lambda V(x)$ is the matrix $\Lambda V(x) = (\Lambda_1 V(x), \ldots, \Lambda_m V(x))$.

3. PROBLEM SETTING

Consider the stochastic process $(x_t)_{t \geq 0}$ with values in $\mathbb{R}^n$ solution of the stochastic differential system written in the sense of Itô,

$$\begin{aligned} x_t = x_0 + \sum_{i=0}^{m} \int_0^t u_i \overline{f}^i(x_s)\,\mathrm{d}s + \sum_{i,j=1}^{m} \int_0^t u_i \overline{F}^{ij}(x_s, u) u_j \,\mathrm{d}s + \sum_{k=1}^{r}\sum_{i=1}^{m} \int_0^t u_i \overline{g}_k^i(x_s)\,\mathrm{d}w_s^k \\ + \sum_{k=1}^{r}\sum_{i,j=1}^{m} \int_0^t u_i \overline{G}_k^{ij}(x_s, u) u_j \,\mathrm{d}w_s^k \qquad (3)\end{aligned}$$

where

1. x_0 is given in $\mathbb{R}^n$,
2. $(w_t)_{t \geq 0}$ is a standard Wiener process with values in $\mathbb{R}^r$,
3. u is a measurable control law with values in $\mathbb{R}^{m+1}$,
4. $\overline{f}^i$, $0 \leq i \leq m$, and $\overline{g}_k^i$, $1 \leq k \leq r$, $1 \leq i \leq m$, are smooth functions mapping $\mathbb{R}^n$ into $\mathbb{R}^n$, vanishing in the origin and with less than linear growth,
5. $\overline{F}^{ij}$, $1 \leq i, j \leq m$, and $\overline{G}_k^{ij}$, $1 \leq k \leq r$, $1 \leq i, j \leq m$, are smooth functions mapping $\mathbb{R}^n \times \mathbb{R}^{m+1}$ into $\mathbb{R}^n$ which do not depend on u_0, with less than linear growth and such that $\overline{F}^{ij}(x, 0) = \overline{G}_k^{ij}(x, 0) = 0$ for every $x \in \mathbb{R}^n$.

With the stochastic differential system (3) introduce the first order differential operators Λ_i, $0 \leq i \leq m$, defined for any function φ in $C^{0,1}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$ by

$$\Lambda_i \varphi(t,x) = \nabla_x \varphi(t,x) \overline{f}^i(x).$$

In the following, by using Theorem 2.1 we give a refinement of Theorem 3.1 in [10] for global asymptotic stabilization in probability of the stochastic differential system

$$x_t = x_0 + \sum_{i=0}^{m} \int_0^t u_i \overline{f}^i(x_s)\,\mathrm{d}s + \sum_{k=1}^{r} \sum_{i=1}^{m} \int_0^t u_i \overline{g}_k^i(x_s)\,\mathrm{d}w_s^k \tag{4}$$

by time–varying feedback law.

Theorem 3.1. Let α be the function defined on $\mathbb{R} \times \mathbb{R}^n$ by

$$\alpha(t,x) = \frac{||x||^2}{(1+||x||^2)(1+||\overline{f}^0(x)||^2)} \sin(t) \tag{5}$$

and V be a proper smooth Lyapunov function defined on $\mathbb{R} \times \mathbb{R}^n$ such that

(C1) V is 2π–periodic with respect to time

$$V(t+2\pi, x) = V(t,x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

(C2) $(V(t,x) = 0) \Leftrightarrow (x = 0)$

(C3) $(\nabla_x V(t,x) = 0) \Leftrightarrow (x = 0)$

(C4) $\nabla_t V(t,x) + \nabla_x V(t,x) \alpha(t,x) \overline{f}^0(x) = 0, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n,$

(C5) V has an infinitesimal upper limit

$$\lim_{x \to 0} \sup_{0<t} V(t,x) = 0,$$

(C6) The matrix $\Delta(t,x) = Id_m + \frac{1}{2} \sum_{k=1}^{r} \overline{g}_k(x)^\tau \nabla_x^2 V(t,x) \overline{g}_k(x)$ is invertible for every $(t,x) \in \mathbb{R} \times \mathbb{R}^n$.

Then, if

$$(\mathbf{H}) \qquad \text{rank span} \left\{ \mathrm{ad}_{\overline{f}^0}^k \overline{f}^i,\ 0 \leq i \leq m,\ k \in \mathbb{N} \right\} = n$$

the stochastic differential system (4) is globally asymptotically stabilizable in probability by the smooth time–varying feedback law

$$u(t,x) = (\alpha(t,x), 0, \ldots, 0)^\tau - \beta \frac{\overline{\Delta}(t,x)^{-1} \left(\Lambda V(t,x)\right)^\tau}{1 + ||\overline{\Delta}(t,x)^{-1} \left(\Lambda V(t,x)\right)^\tau ||^2}$$

for any $\beta > 0$ where $\overline{\Delta}(t,x)$ is the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \Delta(t,x) \end{pmatrix}$ and $\Lambda V(t,x)$ is the matrix $\Lambda V(t,x) = (\Lambda_0 V(t,x), \ldots, \Lambda_m V(t,x))$.

Proof. Since the functions α and V are 2π periodic with respect to time, setting

$$u(t,x) = (\alpha(t,x), 0, \dots, 0)^\tau + \overline{u}(t,x)$$

we deduce that the stochastic differential system (4) can be considered as the time–invariant stochastic differential system on $S^1 \times \mathbb{R}^n$, where $S^1 = \mathbb{R}/2\pi\mathbf{Z}$,

$$\mathrm{d}X_t = R(X_t)\,\mathrm{d}t + \sum_{i=0}^{m} \overline{u}_i \overline{R}^i(X_t)\,\mathrm{d}t + \sum_{k=1}^{r}\sum_{i=1}^{m} \overline{u}_i \overline{H}_k^i(X_t)\,\mathrm{d}w_t^k \tag{6}$$

with

$$X_t = \begin{pmatrix} t \\ x_t \end{pmatrix}, \quad R(X) = \begin{pmatrix} 1 \\ \alpha(t,x)\overline{f}^0(x) \end{pmatrix}, \quad \overline{R}^i(X) = \begin{pmatrix} 0 \\ \overline{f}^i(x) \end{pmatrix}, \quad 0 \le i \le m,$$

and

$$\overline{H}_k^i(X) = \begin{pmatrix} 0 \\ \overline{g}_k^i(x) \end{pmatrix}, \quad 1 \le k \le r,\ 1 \le i \le m.$$

Then, if $\mathcal{L}_0$ denotes the infinitesimal generator of the stochastic process solution of the stochastic differential system (6) when $\overline{u} = 0$, we have

$$\mathcal{L}_0 V(t,x) = \nabla_t V(t,x) + \nabla_x V(t,x)\alpha(t,x)\overline{f}^0(x)$$

and taking assumption **(C4)** into account, it yields

$$\mathcal{L}_0 V(t,x) = 0$$

for every $(t,x) \in S^1 \times \mathbb{R}^n$.

Moreover, noticing that for every $X \in S^1 \times \mathbb{R}^n$, the matrix

$$D(X) = Id_{m+1} + \frac{1}{2}\sum_{k=1}^{r} \overline{H}_k(X)^\tau \nabla^2 V(X) \overline{H}_k(X) = \begin{pmatrix} 1 & 0 \\ 0 & \Delta(t,x) \end{pmatrix}$$

we deduce from assumption **(C6)** that the matrix $D(X)$ is invertible for every $X \in S^1 \times \mathbb{R}^n$.

In addition, using inductive computations as those used in the proof of Theorem 1 in [16] and following the same line of reasoning, we deduce that assumptions **(H)**, **(C2)** and **(C3)** imply that the set

$$\overline{\Gamma} = \left\{ X \in S^1 \times \mathbb{R}^n \ /\ \mathcal{L}_0^k \Lambda_i V(X) = 0,\ \forall i \in \{0,\dots,m\},\ \forall k \in \mathbb{N} \right\}$$

is reduced to $\{0\}$.

Therefore, Theorem 2.1 asserts that the stochastic differential system (6) is globally asymptotically stabilizable in probability by the smooth feedback law $\overline{u}$ defined by

$$\overline{u}(t,x) = -\beta \frac{\overline{\Delta}(t,x)^{-1} (\Lambda V(t,x))^\tau}{1 + ||\overline{\Delta}(t,x)^{-1} (\Lambda V(t,x))^\tau ||^2}$$

for any $\beta > 0$ which completes the proof of Theorem 3.1. □

Remark 3.2. The time–varying stabilizing controller proposed in the above result includes a single compensator α that can be chosen in the class of time–varying functions mapping $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}$ satisfying the following conditions,

1. α is 2π–periodic with respect to time
$$\alpha(t+2\pi, x) = \alpha(t,x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$
2. α is odd with respect to time
$$\alpha(-t, x) = -\alpha(t,x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$
3. α vanishes for $x = 0$
$$\alpha(t,0) = 0, \quad \forall t \in \mathbb{R},$$
4. There exists $K > 0$ such that
$$|\alpha(t,x)| \, ||\overline{f}^0(x)|| \leq K\left(1 + ||x||\right), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

4. TIME–VARYING STABILIZATION OF GENERAL STOCHASTIC SYSTEMS

In this section, we pursue the idea initiated in Theorem 3.1 to solve the problem of global stabilization in probability via time–varying feedback law for more general stochastic differential systems without unforced dynamics in the form (3).

After a preliminary compensation with the function α defined by (5), we apply the technique developed in [10] to design a bounded smooth stabilizer for the resulting nonlinear stochastic differential system.

Theorem 4.1. Let V be a proper smooth Lyapunov function defined on $\mathbb{R} \times \mathbb{R}^n$ which satisfies conditions **(C1)** to **(C5)** of Theorem 3.1. Then, if condition **(H)** of Theorem 3.1 is satisfied, the stochastic differential system (3) is globally asymptotically stabilizable in probability by the bounded smooth time–varying feedback law
$$u(t,x) = (\alpha(t,x), 0, \ldots, 0)^\tau - \beta_\epsilon(t,x)\frac{(\Lambda V(t,x))^\tau}{1 + ||\Lambda V(t,x)||^2}$$
where
$$\beta_\epsilon(t,x) = \frac{\epsilon/mr}{1 + \rho_\epsilon(x)^4\left(||\nabla_x V(t,x)|| + ||\nabla_x^2 V(t,x)||\right)^2}, \quad 0 < \epsilon < 1, \tag{7}$$
and $\rho_\epsilon(x)$ is a smooth function mapping $\mathbb{R}^n$ into $\mathbb{R}$ such that for every $x \in \mathbb{R}^n$,
$$\rho_\epsilon(x) \geq \max_{i,j \in \{1,\ldots,m\}} \max_{k \in \{1,\ldots,r\}} \sup_{||u|| < \epsilon} \left(||\overline{F}^{ij}(x,u)||, ||\overline{g}_k^i(x)||, ||\overline{G}_k^{ij}(x,u)||\right). \tag{8}$$

P r o o f. Setting for every $(t,x) \in \mathbb{R} \times \mathbb{R}^n$,

$$u(t,x) = (\alpha(t,x), 0, \ldots, 0)^\tau + \overline{u}(t,x)$$

the stochastic differential system (3) can be considered, since the functions α and V are 2π periodic with respect to time, as the time–invariant stochastic differential system on $S^1 \times \mathbb{R}^n$,

$$\begin{aligned}\mathrm{d}X_t = R(X_t)\,\mathrm{d}t + \sum_{i=0}^{m} \overline{u}_i \overline{R}^i(X_t)\,\mathrm{d}t + \sum_{i,j=1}^{m} \overline{u}_i \tilde{R}^{ij}(X_t,u)\overline{u}_j\,\mathrm{d}t + \sum_{k=1}^{r}\sum_{i=1}^{m} \overline{u}_i \overline{H}^i_k(X_t)\,\mathrm{d}w^t_k \\ + \sum_{k=1}^{r} \overline{u}_i \tilde{H}^{ij}_k(X_t,u)\overline{u}_j\,\mathrm{d}w^k_t \qquad (9)\end{aligned}$$

where

$$X_t = \begin{pmatrix} t \\ x_t \end{pmatrix}, \quad R(X) = \begin{pmatrix} 1 \\ \alpha(t,x)\overline{f}^0(x) \end{pmatrix}, \quad \overline{R}^i(X) = \begin{pmatrix} 0 \\ \overline{f}^i(x) \end{pmatrix}, \quad 0 \le i \le m,$$

$$\tilde{R}^{ij}(X,u) = \begin{pmatrix} 0 \\ \overline{F}^{ij}(x,u) \end{pmatrix}, \ 1 \le i,j \le m, \quad \overline{H}^i_k(X) = \begin{pmatrix} 0 \\ \overline{g}^i_k(x) \end{pmatrix}, \ 1 \le k \le r, \ 1 \le i \le m,$$

and

$$\tilde{H}^{ij}_k(X,u) = \begin{pmatrix} 0 \\ \overline{G}^{ij}_k(x,u) \end{pmatrix}, \quad 1 \le k \le r, \ 1 \le i,j \le m.$$

Then, if $\mathcal{L}$ denotes the infinitesimal generator of the stochastic process solution of the stochastic differential system (9) we have, for every $(t,x) \in S^1 \times \mathbb{R}^n$,

$$\begin{aligned}\mathcal{L}V(t,x) &= \nabla_t V(t,x) + \nabla_x V(t,x)\alpha(t,x)\overline{f}^0(x) + \Lambda V(t,x)\overline{u} \\ &\quad + \sum_{i,j=1}^{m} \overline{u}_i \nabla_x V(t,x)\overline{F}^{ij}(x,u)\overline{u}_j + \frac{1}{2}\overline{u}^\tau \begin{pmatrix} 0 \\ K_u V(t,x) \end{pmatrix} \overline{u}\end{aligned}$$

where

$$\begin{aligned}K_u V(t,x) = \sum_{k=1}^{r} \Big(&\overline{H}_k(X)^\tau \nabla^2_x V(t,x)\overline{H}_k(X) + 2\overline{u}^\tau \overline{H}_k(X)\nabla^2_x V(t,x)\tilde{H}_k(X,u) \\ &+ \overline{u}^\tau \tilde{H}_k(X,u)^\tau u \nabla^2_x V(t,x)\tilde{H}_k(X,u)\Big)\end{aligned}$$

and taking assumption **(C4)** into account, it yields

$$\mathcal{L}V(t,x) = \Lambda V(t,x)\overline{u} + \sum_{i,j=1}^{m} \overline{u}_i \nabla_x V(t,x)\overline{F}^{ij}(x,u)\overline{u}_j + \frac{1}{2}\overline{u}^\tau \begin{pmatrix} 0 \\ K_u V(t,x) \end{pmatrix} \overline{u}.$$

On the other hand, if for every $(t,x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\overline{u}(t,x) = -\beta_\epsilon(t,x)\frac{(\Lambda V(t,x))^\tau}{1 + ||\Lambda V(t,x)||^2} \qquad (10)$$

where β_ϵ is given by (7) we have

$$||\overline{u}|| \leq \frac{\beta_\epsilon}{2} \leq \frac{\epsilon}{2}$$

and hence, taking (8) into account, it is easy to prove that

$$\left| \sum_{i,j=1}^{m} \overline{u}_i(t,x)\nabla_x V(t,x)\overline{F}^{ij}(x,u)\overline{u}_j(t,x) \right| \leq ||\overline{u}||^2 m\rho_\epsilon(x)||\nabla_x V(t,x)||$$

and

$$||K_u V(t,x)|| \leq mr\rho_\epsilon(x)^2||\nabla_x^2 V(t,x)||.$$

Therefore, with the above estimates, we deduce that

$$\mathcal{L}V(t,x) \leq \beta_\epsilon(t,x)\frac{||\Lambda V(t,x)||^\tau}{1+||\Lambda V(t,x)||^2}\left(-1+\beta_\epsilon(t,x)mr\rho_\epsilon(x)^2\left(||\nabla_x V(t,x)||+||\nabla_x^2 V(t,x)||\right)\right)$$

and invoking (8) it yields

$$\mathcal{L}V(t,x) \leq \beta_\epsilon(t,x)\frac{||\Lambda V(t,x)||^\tau}{1+||\Lambda V(t,x)||^2}\left(-1+\frac{\epsilon}{2}\right) \leq 0. \tag{11}$$

The latter estimate implies, according with the stochastic Lyapunov theorem (Theorem 5.3.1 in [13]) that the equilibrium solution of the closed–loop system deduced from the stochastic differential system (9) with the feedback law $\overline{u}$ given by (10) is stable in probability.

Furthermore, the stochastic La Salle theorem proved by Kushner in [14] asserts that the stochastic process solution X_t of the closed–loop system deduced from the stochastic differential system (9) with the feedback law $\overline{u}$ given by (10) tends with probability one to the largest invariant set whose support is contained in the locus $\mathcal{L}V\,(t,x_t)=0$ for every $t \geq 0$.

But, if $\mathcal{L}V\,(t,x_t)=0$ for every $t\geq 0$, inequality (11) implies that $\Lambda V\,(t,x_t)=0$ for every $t\geq 0$, that is $\Lambda_i V\,(t,x_t)=0$ for every $t\geq 0$ and $i\in\{0,\dots,m\}$ and also, as a consequence, $\overline{u}\,(t,x_t)=0$ for every $t\geq 0$.

Then, if $\mathcal{L}_0$ denotes the infinitesimal generator of the stochastic process solution of the stochastic differential system (9) when $\overline{u}=0$; i.e. the first order differential operator defined for every function $\varphi\in C^{1,1}\,(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ by

$$\mathcal{L}_0\varphi(t,x) = \nabla_t\varphi(t,x) + \nabla_x\varphi(t,x)\alpha(t,x)\overline{f}^0(x)$$

we have, by application of Itô's formula to the stochastic process $\Lambda_i V\,(t,x_t)$, $i\in\{0,\dots,m\}$,

$$\mathcal{L}_0\Lambda_i V\,(t,x_t)=0$$

and, since $\mathcal{L}_0 V\,(t,x_t)=0$ by assumption **(C4)**,

$$\mathrm{ad}_{\mathcal{L}_0}\Lambda_i V\,(t,x_t)=0$$

for every $t \geq 0$ and $i \in \{0, \ldots, m\}$.

Therefore, by successive iterations of the above procedure, one can prove that if $\Lambda V(t, x_t) = 0$ for every $t \geq 0$, one has

$$\mathrm{ad}^k_{\mathcal{L}_0} \Lambda_i V(t, x_t) = 0$$

for every $t \geq 0$, $i \in \{0, \ldots, m\}$ and $k \in \mathbb{N}$.

Moreover, following the same line of reasoning as in the proof of Theorem 1 in [16] straightforward inductive computations show that assumption **(H)** implies that for every $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ such that $\alpha(t, x) \neq 0$,

$$\text{rank span} \left\{ \mathrm{ad}^k_{\mathcal{L}_0} \Lambda_i, \ 0 \leq i \leq m, \ k \in \mathbb{N} \right\} = n.$$

This, in turn, implies that the equations

$$\mathrm{ad}^k_{\mathcal{L}_0} \Lambda_i V(t, x_t) = 0$$

for every $t \geq 0$, $i \in \{0, \ldots, m\}$ and $k \in \mathbb{N}$ have, under assumptions **(C2)** and **(C3)**, a unique solution $x_t = 0$.

Thus, the stochastic La Salle theorem asserts that the equilibrium solution of the closed–loop system deduced from the stochastic differential system (9) with the feedback law $\overline{u}$ given by (10) is globally asymptotically stable in probability which completes the proof of Theorem 4.1. □

Remark 4.2. In general it is not easy to find a Lyapunov function V satisfying assumptions **(C1)** to **(C3)** in Theorem 4.1. However, as already noticed by Pomet in [16], when $\overline{f}^0 = (1, 0, \ldots, 0)^\tau$ a possible choice for V is

$$V(t, x) = \frac{1}{2} \left(\left(x_1 + \left(x_2^2 + \cdots + x_n^2 \right) \cos t \right)^2 + x_2^2 + \cdots + x_n^2 \right)$$

using

$$\alpha(t, x) = \left(x_2^2 + \cdots + x_n^2 \right) \sin t$$

instead of the function defined in (5).

Example 4.3. Let x_0 be given in $\mathbb{R}^3$ and denote by $(x_t)_{t \geq 0}$ the stochastic process with values in $\mathbb{R}^3$ solution of the stochastic differential system

$$\mathrm{d}x_t = \begin{pmatrix} u_0 \\ \mathrm{e}^{x_{1,t} u_1} - 1 \\ u_1 \end{pmatrix} \mathrm{d}t + u_1 \begin{pmatrix} x_{1,t} \\ x_{2,t}^2 \\ \mathrm{e}^{x_{3,t}} \end{pmatrix} \mathrm{d}w_t \tag{12}$$

where $(w_t)_{t \geq 0}$ is a standard real–valued Wiener process and u is a measurable control law with values in $\mathbb{R}^2$.

Obviously, the stochastic differential system (12) can be rewritten as

$$\mathrm{d}x_t = u_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}t + u_1 \begin{pmatrix} 0 \\ x_{1,t} \\ 1 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ \mathrm{e}^{x_{1,t} u_1} - 1 - x_{1,t} u_1 \\ 0 \end{pmatrix} \mathrm{d}t + u_1 \begin{pmatrix} x_{1,t} \\ x_{2,t}^2 \\ \mathrm{e}^{x_{3,t}} \end{pmatrix} \mathrm{d}w_t$$

and hypothesis **(H)** is satisfied with the coefficients $\overline{f}^0(x) = (1,0,0)^\tau$ and $\overline{f}^1(x) = (0, x_1, 1)^\tau$.

Furthermore, if we take

$$V(t,x) = \frac{1}{2}\left(\left(x_1 + \left(x_2^2 + x_3^2\right)\cos t\right)^2 + x_2^2 + x_3^2\right)$$

and

$$\alpha(t,x) = \left(x_2^2 + x_3^2\right)\sin t$$

hypothesis **(C1)** to **(C5)** in Theorem 3.1 are satisfied. As a consequence, we obtain the stabilizing time–varying feedback law

$$u_0(t,x) = \left(x_2^2 + x_3^2\right)\sin t - \beta_\epsilon(t,x)\frac{x_1 + \left(x_2^2 + x_3^2\right)\cos t}{1 + ||\Lambda V(t,x)||^2}$$

and

$$u_1(t,x) = -\beta_\epsilon(t,x)\frac{(x_3 + x_1x_2)\left(1 + 2\left(x_1 + \left(x_2^2 + x_3^2\right)\cos t\right)\cos t\right)}{1 + ||\Lambda V(t,x)||^2}$$

where

$$||\Lambda V(t,x)||^2 = \left(x_1 + \left(x_2^2 + x_3^2\right)\cos t\right)^2 + (x_3 + x_1x_2)^2\left(1 + 2\left(x_1 + \left(x_2^2 + x_3^2\right)\cos t\right)\cos t\right)^2$$

and

$$\beta_\epsilon(t,x) = \frac{\epsilon}{2}\left(1 + \rho_\epsilon(x)^4\left(||\nabla_x V(t,x)|| + ||\nabla_x^2 V(t,x)||\right)^2\right)^{-1}, \quad 0 < \epsilon < 1$$

with

$$\rho_\epsilon(x) = \left(x_1^2 + x_2^4 + \mathrm{e}^{2x_3}\right)^{1/2} + \mathrm{e}^{1+x_1^2}.$$

5. STABILIZATION VIA TIME–VARYING DYNAMIC FEEDBACK

The stabilizability via dynamic state feedback law has been introduced for deterministic nonlinear systems by Sontag and Sussmann in [20] and later on extended to nonlinear stochastic differential systems by Florchinger in [8].

In this section, we focus our attention to the global asymptotic stabilization in probability for the class of nonlinear stochastic differential systems studied in this paper via time–varying dynamic feedback. With this aim, we use a methodology in the spirit of the previous section which relies on the stochastic Lyapunov second method and the stochastic La Salle invariance principle.

First note that if $\mathcal{L}_u$ denotes the infinitesimal generator of the stochastic process solution of the stochastic differential system (3) it is obvious that for any function φ in $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$ the quantity $\mathcal{L}_u\varphi(t,x) - \nabla_t\varphi(t,x)$ is linear in u and consequently, it can be expressed as

$$\mathcal{L}_u\varphi(t,x) - \nabla_t\varphi(t,x) = T_u\varphi(t,x)u. \tag{13}$$

Then, with the previous notation, the following result holds.

Theorem 5.1. Let V be a proper smooth Lyapunov function defined on $\mathbb{R}\times\mathbb{R}^n$ which satisfies conditions **(C1)** to **(C5)** of Theorem 3.1. Then, if condition **(H)** of Theorem 3.1 is satisfied, the stochastic differential system (3) is globally asymptotically stabilizable in probability by the time–varying dynamic feedback

$$u(t,x,\zeta) = (\alpha(t,x),0,\dots,0)^\tau + \zeta_t \tag{14}$$

$$\dot{\zeta}_t = -\zeta_t - T_{\zeta_t}V(t,x)^\tau. \tag{15}$$

P r o o f. First note that the function $\overline{V}$ defined for any $(t,x,\zeta)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$ by

$$\overline{V}(t,x,\zeta) = V(t,x) + \frac{1}{2}||\zeta||^2$$

is a proper smooth Lyapunov function which has an infinitesimal upper limit.

Then, denoting by $\mathcal{K}$ the infinitesimal generator of the stochastic process solution (x_t,ζ_t) of the closed–loop system deduced from the stochastic differential system (3) when u is given by (14) with (15), we have for every $(t,x,\zeta)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^m$,

$$\mathcal{K}\overline{V}(t,x,\zeta) = \nabla_x V(t,x)\alpha(t,x)\overline{f}^0(x) + \mathcal{L}_\zeta V(t,x) + \dot{\zeta}^\tau\zeta.$$

Taking into account hypothesis **(C5)** and (13), we get

$$\mathcal{K}\overline{V}(t,x,\zeta) = T_\zeta V(t,x)\zeta + \dot{\zeta}^\tau\zeta$$

which implies, according with (15), that

$$\mathcal{K}\overline{V}(t,x,\zeta) = T_\zeta V(t,x)\zeta - (\zeta + T_\zeta V(t,x)^\tau)^\tau\,\zeta = -||\zeta||^2 \le 0. \tag{16}$$

Therefore, the stochastic Lyapunov theorem asserts that the equilibrium solution of the closed–loop system deduced from the stochastic differential system (3) when u is given by (14) with (15) is stable in probability.

Furthermore, the stochastic La Salle theorem implies that the stochastic process solution (x_t,ζ_t) of the closed–loop system deduced from the stochastic differential system (3) when u is given by (14) with (15) tends with probability one to the largest invariant set whose support is contained in the locus $\mathcal{K}\overline{V}\,(t,x_t,\zeta_t)=0$ for every $t\ge 0$.

But, if $\mathcal{K}\overline{V}\,(t,x_t,\zeta_t)=0$ for every $t\ge 0$, we deduce from inequality (16) that $\zeta_t=0$ for every $t\ge 0$ which implies that $\dot{\zeta}_t=0$ for every $t\ge 0$ and hence, from (15), we deduce that $T_0V\,(t,x_t)=0$ for every $t\ge 0$.

Then, noticing that for every $(t,x)\in\mathbb{R}\times\mathbb{R}^n$,

$$T_0V(t,x) = \Lambda V(t,x)$$

we obtain that if $\mathcal{K}\overline{V}\,(t,x_t,\zeta_t)=0$ for every $t\ge 0$ it yields $\Lambda V\,(t,x_t)=0$ for every $t\ge 0$.

Now, arguying as in the proof of Theorem 4.1, we deduce by successive iterations of Itô's formula that under assumptions **(C2)**, **(C3)** and **(H)**, we have $x_t=0$ for every $t\ge 0$.

Thus, the stochastic La Salle theorem asserts that the equilibrium solution of the closed–loop system deduced from the stochastic differential system (3) when u is given by (14) with (15) is globally asymptotically stable in probability which completes the proof of Theorem 5.1. □

Example 5.2. Let x_0 be given in $\mathbb{R}^3$ and denote by $(x_t)_{t\geq 0}$ the stochastic process with values in $\mathbb{R}^3$ solution of the stochastic differential system

$$\mathrm{d}x_t = \begin{pmatrix} u_0 \\ \mathrm{e}^{x_{1,t}u_1} - 1 \\ u_1 \end{pmatrix} \mathrm{d}t + u_1 \begin{pmatrix} x_{1,t} \\ 0 \\ 0 \end{pmatrix} \mathrm{d}w_t \tag{17}$$

where $(w_t)_{t\geq 0}$ is a standard real–valued Wiener process and u is a measurable control law with values in $\mathbb{R}^2$.

Obviously, the stochastic differential system (17) can be rewritten as

$$\mathrm{d}x_t = u_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}t + u_1 \begin{pmatrix} 0 \\ x_{1,t} \\ 1 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ \mathrm{e}^{x_{1,t}u_1} - 1 - x_{1,t}u_1 \\ 0 \end{pmatrix} \mathrm{d}t + u_1 \begin{pmatrix} x_{1,t} \\ 0 \\ 0 \end{pmatrix} \mathrm{d}w_t$$

and hypothesis **(H)** is satisfied with the coefficients $\overline{f}^0(x) = (1,0,0)^\tau$ and $\overline{f}^1(x) = (0, x_1, 1)^\tau$.

Moreover, with the stochastic differential system (17), for every $(t,x,u) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2$ and $\varphi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$, we have

$$T_u\varphi(t,x) = \begin{pmatrix} \dfrac{\partial \varphi}{\partial x_1}(t,x) \\ x_1 \dfrac{\partial \varphi}{\partial x_2}(t,x) + \dfrac{\partial \varphi}{\partial x_3}(t,x) + \dfrac{\mathrm{e}^{x_1u_1} - 1 - x_1u_1}{u_1} \dfrac{\partial \varphi}{\partial x_2}(t,x) + u_1 x_1^2 \dfrac{\partial^2 \varphi}{\partial x_1^2}(t,x) \end{pmatrix}.$$

Therefore, if we take

$$V(t,x) = \frac{1}{2}\left(\left(x_1 + \left(x_2^2 + x_3^2\right)\cos t\right)^2 + x_2^2 + x_3^2\right)$$

and

$$\alpha(t,x) = \left(x_2^2 + x_3^2\right)\sin t$$

hypothesis **(C1)** to **(C5)** in Theorem 3.1 are satisfied. Thus, one concludes from Theorem 5.1 that the time–varying dynamic feedback law

$$u(t,x,\zeta) = (\alpha(t,x), 0, \dots, 0)^\tau + \zeta_t$$

with

$$\begin{aligned} \dot{\zeta}_{0,t} &= -\zeta_{0,t} - \left(x_{1,t} + \left(x_{2,t}^2 + x_{3,t}^2\right)\cos t\right) \\ \dot{\zeta}_{1,t} &= -\zeta_{1,t} - \left(x_{1,t}x_{2,t} + x_{3,t} + \frac{\mathrm{e}^{x_{1,t}\zeta_{1,t}} - 1 - x_{1,t}\zeta_{1,t}}{\zeta_{1,t}} x_{2,t}\right) . \\ &\qquad \left(1 + 2\left(x_{1,t} + \left(x_{2,t}^2 + x_{3,t}^2\right)\cos t\right)\cos t\right) + \zeta_{1,t}x_{1,t}^2 \end{aligned}$$

renders the stochastic differential system (17) globally asymptotically stable in probability.

(Received May 27, 2016)

REFERENCES

[1] F. Abedi, M. A. Hassan, and N. Arifin: Control Lyapunov function for feedback stabilization of affine in the control stochastic time–varying systems. Int. J. Math. Anal. *5* (2011), 175–188.

[2] F. Abedi, W. J. Leong, and S. S. Chaharborj: On the aymptotic and practical stability of stochastic control systems. Math. Problems in Engineering *2013* (2013), 560647. DOI:10.1155/2013/560647

[3] F. Abedi, W. J. Leong, and M. Abedi: Lyapunov characterization of the stability of stochastic control systems. Math. Problems in Engineering *2015* (2015), 584935. DOI:10.1155/2015/584935

[4] R. Brockett: Asymptotic stability and feedback stabilization. In: Differential Geometric Control Theory (R. Brockett, R. Millman and H. Sussmann, eds.), Birkhäuser, Basel, Boston 1983, pp. 181–191.

[5] G. Campion, B. d'Andréa–Novel, and G. Bastin: Controllability and state feedback stabilization of nonholonomic mechanical systems. In: International Workshop in Adaptive and Nonlinear Control: Issues in Robotics, Springer–Verlag 1990. DOI:10.1007/bfb0039268

[6] J. M. Coron: Global asymptotic stabilization for controllable systems without drift. Math. Control Signal Systems *5* (1992), 295–312. DOI:10.1007/bf01211563

[7] J. M. Coron and J. B. Pomet: A remark on the design of time–varying stabilization feedback laws for controllable systems without drift. In: Proc. IFAC NOLCOS, Bordeaux 1992, pp. 413–417.

[8] P. Florchinger: Lyapunov–like techniques for stochastic stability. SIAM J. Control Optim. *33* (1995), 4, 1151–1169. DOI:10.1137/s0363012993252309

[9] P. Florchinger: A stochastic Jurdjevic–Quinn theorem. SIAM J. Control Optim. *41* (2002), 83–88. DOI:10.1137/s0363012900370788

[10] P. Florchinger: Global asymptotic stabilization in probability of nonlinear stochastic systems via passivity. Int. J. Control *89* (2016), 1406–1415. DOI:10.1080/00207179.2015.1132009

[11] P. Florchinger: Time–varying stabiliziers for stochastic systems with no unforced dynamics. Preprint 2016.

[12] V. Jurdjevic and J. P. Quinn: Controllability and stability. J. Differential Equations *28* (1978), 381–389. DOI:10.1016/0022-0396(78)90135-3

[13] R. Z. Khasminskii: Stochastic Stability of Differential Equations. Sijthoff and Noordhoff, Alphen aan den Rijn 1980.

[14] H. J. Kushner: Stochastic stability. In: Stability of Stochastic Dynamical Systems (R. Curtain, ed.), Lecture Notes in Mathematics 294, Springer Verlag, Berlin, Heidelberg, New York 1972, pp. 97–124. DOI:10.1007/bfb0064937

[15] W. Lin: Time–varying feedback control of nonaffine nonlinear systems without drift. Systems Control Lett. *29* (1996), 101–110. DOI:10.1016/s0167-6911(96)00050-3

[16] J. B. Pomet: Explicit design of time–varying stabilizing control law for a class of controllable systems without drift. Systems Control Lett. *18* (1992), 147–158. DOI:10.1016/0167-6911(92)90019-o

[17] C. Samson: Time–varying Stabilization of a Nonholonomic Car–like Mobile Robot. Repport de recherche 1515, INRIA Sophia–Antipolis 1991.

[18] R. Sepulchre, G. Campion, and V. Wertz: Some remarks about periodic feedback stabilization. In: Proc. IFAC NOLCOS, Bordeaux 1992, pp. 418–423.

[19] E. Sontag: Feedback stabilization of nonlinear systems. In: Robust Control of Linear Systems and Nonlinear Control (M. K. Kaashoek et al., eds.), Birkhäuser, Boston 1990, pp. 61–81. DOI:10.1007/978-1-4612-4484-4_4

[20] E. D.Sontag and H. J. Sussmann: Remarks on continuous feedback. In: Proc. 19th IEEE Conference on Decision and Control, Albuquerque 1980, pp. 916–921. DOI:10.1109/cdc.1980.271934

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