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PERFECT OBSERVERS FOR FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

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A perfect (exact) fractional observer of discrete-time singular linear control system of fractional order is studied. Conditions for its existence are given. The obtained results are applied to the detectability problem of the class of systems under consideration.

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1. INTRODUCTION

In various real phenomena the input variables of a given control system may be unknown, either since the control is not accessible or since inputs consider the fact of external disturbances vary due to. This is the source of one of the most important and natural problem in control theory, is the problem of the observer design. This problem places special emphasis on the state reconstruction issues, but also for example, chaotic synchronization, network communication, on system supervision or on fault diagnosis. On the other hand, singular systems, describing a large class of systems, can be met for example in electronic, economic, dynamic balances of mass and energy, see [5, 19]. Classically it is assumed that an error between the unknown state of the considered system (with some unmeasurable variables) and its estimate tends to zero in a long time period, see for example [9, 11, 21], but it can happen that this error exactly equals zero. In this case the observer is called a perfect (exact) observer. This class of observers for singular systems was studied in [12, 19, 20], where the Dai's [4] concept of observers for singular discrete-time linear systems is extended to the continuous-time case. The next step in elaborating the problem of perfect observers design has been devoted to the existence of this type of observers in fractional continuous-time case, see [13].

Now our goal is to give, basing on [4, 13], a description of a perfect observer on singular linear control discrete-time fractional systems with sampling step h. The research are motivated by the fact that in the recent years the fractional calculus is viewed as becoming more useful and effective tool in describing the behavior of real systems. Simulations and experiments seem to confirm this, see for example [6, 7, 22]. Again,

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in practice, some states variables cannot be accessible, so problem of observer design for fractional discrete-tome case appears in a natural way. The generalizations of nth order differences to their fractional forms are used. Basic properties of fractional operators were investigated firstly in [14] and next in [1, 2]. In [8] there was adopted a more general fractional h-difference Riemman–Liouville operator. The presence of h in this operator is important from both engineering and numerical points of view. On one hand h represents a sample step, on the other – when h tends to zero, the solutions of the fractional difference equation may be seen as approximations to the solutions of corresponding Riemann–Liouville fractional equations.

Classically, a linear control discrete-time fractional system can be defined by Riemann–Liouville– or by the Caputo– or by the Grünwald–Letnikov–type fractional h-difference operator. In [10] it was shown that these three types of fractional h-difference operators are related to each other. Moreover, the Grünwald–Letnikov–type fractional h-difference operator can be expressed by the Riemann–Liouville–type fractional h-difference operator. So, systems under our consideration with these types of operators are studied simultaneously.

The work was motivated by results discussed in [13, 20]. The first step, after introducing the relevant notation and facts (Section 2), based on [17] the difference linear systems of a fractional order are presented (Sections 3). Next, the perfect observer for the class of linear singular fractional order difference systems is introduced and its properties are studied (Section 4). As one of possible applications of the analysed type of observer, the problem of detectability for the considered system is studied. Extending stability results given in [17, 18] for standard discrete-time linear fractional systems onto singular systems, it is shown that for a detectable singular system one can design a fractional singular observer. This observer is no longer a perfect one.

2. PRELIMINARIES

At the beginning let us introduce a notation and some properties that will be needed further.

If $h > 0$ and $a \in \mathbb{R}$ then $(h\mathbb{N})_a := \{a, a+h, a+2h, \ldots\}$. Consider a function $x : (h\mathbb{N})_a \to \mathbb{R}$. Then the forward h-difference operator is classically defined as $(\Delta_h x)(t) = \frac{x(t+h)-x(t)}{h}$. If we put$(\Delta_h^0 x)(t) := x(t)$, then, $\Delta_h^n := \Delta_h \circ \cdots \circ \Delta_h$ is n-fold application of operator Δ_h for any $n \in \mathbb{N}_0$. We can compute that

$$(\Delta_h^n x)(t) = h^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x(t+kh)\,.$$

Let

$$\tilde{\varphi}_\mu(n) = \begin{cases} \binom{n+\mu-1}{n}, & \text{for } n \in \mathbb{N}_0 \\ 0, & \text{for } n < 0. \end{cases} \tag{1}$$

denotes the family of binomial functions on $\mathbb{Z}$ parameterized by $\mu > 0$. Let "$*$" denotes a convolution operator, i. e.

$$(\tilde{\varphi}_\mu * \overline{x})(n) := \sum_{s=0}^{n} \binom{n-s+\mu-1}{n-s} \overline{x}(s)\,.$$

Then for a function $x : (h\mathbb{N})_a \to \mathbb{R}$ the *fractional h-sum of order* $\alpha > 0$ is defined by

$$\left({}_a\Delta_h^{-\alpha} x\right)(t) := h^\alpha \left(\tilde{\varphi}_\alpha * \overline{x}\right)(n),$$

where $t = a + (\alpha + n)h$ for any $n \in \mathbb{N}_0$ and $\overline{x}(s) := x(a + sh)$.

The Mittag--Leffler two-parameter function on the one hand side is a generalization of the classical exponential function to fractional case, on the other hand it naturally occurs in solutions of fractional order difference equations and in the descriptions of an evaluation of fractional order difference linear control systems. So, we recall that this function is defined as follows (see [17])

$$E_{(\alpha,\beta)}(\lambda, n) := \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+\beta}(n - qk).$$

Note that, $E_{(0,1)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k$, so it is the well known geometrical series. Moreover, taking into account (1), we have the following:

$$E_{(\alpha,\alpha)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \binom{n - qk + (k+1)\alpha - 1}{n - k}$$

and

$$E_{(\alpha,1)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \binom{n - qk + k\alpha}{n - k}.$$

As the $\mathcal{Z}$-transform is the natural tool used in analysis of properties of discrete-time systems, recall that the (single-sided) $\mathcal{Z}$-transform of a sequence $\{y(n)\}_{n\in\mathbb{N}_0}$ is a complex function $Y(z)$ given by

$$Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} y(k) z^{-k},$$

where $z \in \mathbb{C}$ is a complex variable for which the series $\sum_{k=0}^{\infty} y(k)z^{-k}$ converges absolutely.

Since $\binom{k+\alpha-1}{k} = (-1)^k \binom{-\alpha}{k}$, then for $|z| > 1$ we have

$$\mathcal{Z}\left[\tilde{\varphi}_\alpha\right](z) = \sum_{k=0}^{\infty} \binom{k + \alpha - 1}{k} z^{-k} = \left(\frac{z}{z-1}\right)^\alpha. \tag{2}$$

Since, linear control discrete-time system of a fractional order can be defined by Riemann–Liouville– or by the Caputo– or by the Grünwald–Letnikov–type fractional h-difference operator, in next subsection we recall the definitions of the Caputo–, Riemann–Liouville– and Grünwald–Letnikov–type h-difference operators and some properties of the $\mathcal{Z}$-transform of the those operators.

2.1. Fractional difference operators

Let $\alpha \in (0, 1]$. The *Caputo–type h-difference operator* ${}_a\Delta_{h,*}^\alpha$ *of order* α *for a function* $x : (h\mathbb{N})_a \to \mathbb{R}$ is defined by (see [15])

$$\left({}_a\Delta_{h,*}^\alpha x\right)(t) := \left({}_a\Delta_h^{-(1-\alpha)} \left(\Delta_h x\right)\right)(t) = h^{-\alpha} \left(\tilde{\varphi}_{1-\alpha} * \Delta_{h=1}\overline{x}\right)(n)$$

for any $t = a+(1-\alpha)h+nh$ and $\overline{x}(n) = x(a+nh)$. Note that ${}_a\Delta^{\alpha}_{h,*}x : (h\mathbb{N})_{a+(1-\alpha)h} \to \mathbb{R}$. If $\alpha = 1$, then $\left({}_a\Delta^{1}_{h,*}x\right)(t) = (\Delta_h x)(t)$ for any $t \in (h\mathbb{N})_a$.

Proposition 2.1. (Mozyrska and Wyrwas [17]) Let $a \in \mathbb{R}$ and $\alpha \in (0,1]$. Let us define $y(n) := \left({}_a\Delta^{\alpha}_{h,*}x\right)(t)$ for any $t = a + (1-\alpha)h + nh$. Then

$$\mathcal{Z}\left[y\right](z) = h^{-\alpha}\left(\frac{z}{z-1}\right)^{1-\alpha}\left((z-1)X(z) - zx(a)\right),$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(n) := x(a+nh)$.

If $\alpha = 1$ then $\mathcal{Z}\left[y\right](z) = \frac{1}{h}\left((z-1)X(z) - z\overline{x}(0)\right)$, that coincides with the transform of difference Δ_h of $\overline{x}$.

Let $\alpha \in (0,1]$. The *Riemann–Liouville–type fractional h-difference operator* ${}_a\Delta^{\alpha}_{h}$ *of order* α *for a function* $x : (h\mathbb{N})_a \to \mathbb{R}$ is defined by (see [3, 8])

$$\left({}_a\Delta^{\alpha}_{h}x\right)(t) := \left(\Delta_h\left({}_a\Delta^{-(1-\alpha)}_{h}x\right)\right)(t),$$

where $t \in (h\mathbb{N})_{a+(1-\alpha)h}$.

Proposition 2.2. (Mozyrska and Wyrwas [17]) For $a \in \mathbb{R}$, $\alpha \in (0,1]$ let us define $y(n) := \left({}_a\Delta^{\alpha}_{h}x\right)(t)$, where $t = a + (1-\alpha)h + nh$, $t \in (h\mathbb{N})_{a+(1-\alpha)h}$. Then

$$\mathcal{Z}\left[y\right](z) = z\left(\frac{hz}{z-1}\right)^{-\alpha}X(z) - zh^{-\alpha}x(a),$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(n) := x(a+nh)$.

If $\alpha = 1$, then $\mathcal{Z}\left[y\right](z) = \frac{1}{h}\left((z-1)X(z) - z\overline{x}(0)\right)$, that coincides with the transform of difference Δ_h of $\overline{x}$ similarly as in the case of the Caputo-type operator.

The third type of the operator, that we take under consideration, is the Grünwald–Letnikov–type fractional h-difference operator. If $\alpha \in \mathbb{R}$, then this operator ${}_a\widetilde{\Delta}^{\alpha}_{h}$ of order α for a function $x : (h\mathbb{N})_a \to \mathbb{R}$ is defined by (see for example [10])

$$\left({}_a\widetilde{\Delta}^{\alpha}_{h}x\right)(t) := \sum_{s=0}^{\frac{t-a}{h}} a^{(\alpha)}_s x(t-sh),$$

where $a^{(\alpha)}_s = (-1)^s\binom{\alpha}{s}\frac{1}{h^{\alpha}}$ with

$$\binom{\alpha}{s} = \begin{cases} 1 & \text{for } s = 0, \\ \frac{\alpha(\alpha-1)\cdots(\alpha-s+1)}{s!} & \text{for } s \in \mathbb{N}. \end{cases}$$

Note that if $a = (\alpha-1)h$ then (see [10])

$$\left({}_0\widetilde{\Delta}^{\alpha}_{h}y\right)(t+h) = \left({}_a\Delta^{\alpha}_{h}x\right)(t),$$

where $x(t) = y(t-a)$ for $t \in (h\mathbb{N})_a$.

Proposition 2.3. (Mozyrska and Wyrwas [17]) For $a \in \mathbb{R}$ and $\alpha \in (0,1]$ let us define $y(n) := \left({}_a\widetilde{\Delta}_h^{\alpha} x\right)(t)$, where $t = a + nh$, $t \in (h\mathbb{N})_a$, $n \in \mathbb{N}_0$. Then

$$\mathcal{Z}\left[y\right](z) = \left(\frac{hz}{z-1}\right)^{-\alpha} X(z)\,,$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(q) := x(a + nh)$.

3. DIFFERENCE LINEAR SYSTEMS OF A FRACTIONAL ORDER

Let $\alpha \in (0,1]$. Since some definitions and facts that we discuss are the same for each type of difference operators, we use the common symbol defined by its values

$$\left({}_a\Upsilon^{\alpha} x\right)(t) = \begin{cases} ({}_a\Delta_{h,*}^{\alpha} x)(t) \text{ or } ({}_a\Delta_h^{\alpha} x)(t) & \text{for } a = (\alpha - 1)h \\ ({}_0\widetilde{\Delta}_h^{\alpha} x)(t+h), & \text{for } a = 0\,. \end{cases}$$

Basing on Propositions 2.1, 2.2, 2.3, in [17] it was shown the following.

Proposition 3.1. (Mozyrska and Wyrwas [17]) For $a \in \mathbb{R}$, $\alpha \in (0,1]$ let us define $y(n) := \left({}_a\Upsilon^{\alpha} x\right)(t)$, where $t \in (h\mathbb{N})_{a+(1-\alpha)h}$ and $t = a + (1-\alpha)h + nh$. Then

$$\mathcal{Z}\left[\left({}_a\Upsilon^{\alpha} x\right)\right](z) = z\left(\frac{hz}{z-1}\right)^{-\alpha}\left(X(z) - \left(\frac{z}{z-1}\right)^{\beta} x(a)\right),$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$, $\overline{x}(n) := x(a + nh)$ and $\beta = \alpha$ for the Riemann–Liouville– or Grünwald–Letnikov–type operators and $\beta = 1$ for the Caputo–type operator, and $a = \alpha - 1$ for the Riemann–Liouville– or Caputo–type operators and $a = 0$ for the Grünwald–Letnikov–type operator.

Let us consider the following common form of control systems

$$E\left({}_a\Upsilon^{\alpha} x\right)(nh) = Ax(nh + a) + Bu(nh)\,, \tag{3a}$$

$$y(nh) = Cx(nh + a) + Du(nh)\,, \tag{3b}$$

$$x(a) = x_0 \tag{3c}$$

where $n \in \mathbb{N}_0$, $x : (h\mathbb{N})_a \to \mathbb{R}^p$ denotes a state vector, $y : (h\mathbb{N})_0 \to \mathbb{R}^r$ is an output vector, $u : (h\mathbb{N})_0 \to \mathbb{R}^m$ is a control sequence and $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times m}$, $C \in \mathbb{R}^{r\times p}$, $D \in \mathbb{R}^{r\times m}$, $E \in \mathbb{R}^{p\times p}$ are real matrices with constant coefficients. If $\det E = 0$ then system (3) is called the *singular* system. If $\det E \neq 0$, then using pre-multiplication of the equation (3a) by E^{-1} we obtain the standard linear system of the fractional order. By Kroncecker-Capelli's Theorem the fractional order dynamics (3a) with initial condition (3c) has the unique solution if and only if $\operatorname{rank}\left[Eh^{-\alpha} z\left(\frac{z}{z-1}\right)^{-\alpha} - A\right] = \operatorname{rank}\left[Eh^{-\alpha}\left(\frac{z}{z-1}\right)^{1-\alpha} x_0\right]$.

Proposition 3.2. (Mozyrska et al. [16]) If $\det E \neq 0$, then the fractional order dynamics (3a) with initial condition (3c), $\alpha \in (0,1]$, $a = (\alpha - 1)h$ and a fixed control u has the unique solution given by

$$x(nh+a) = E_{(\alpha,\beta)}(E^{-1}Ah^{\alpha}, n)x_0 + \left(E^{\rho}_{(\alpha,\alpha)}(E^{-1}Ah^{\alpha}, \cdot) * B\overline{u}\right)(n),$$

where $\overline{u}(n) = h^{\alpha}u(nh)$ and $\beta = 1$ for the Caputo–type operator, $\beta = \alpha$ for the Riemman–Liouville– and Grünwald–Letnikov–type operators, $E^{\rho}_{(\alpha,\alpha)}(Ah^{\alpha}, n) := E_{(\alpha,\alpha)}(Ah^{\alpha}, n-1)$.

The standard definition of observability says that a system is observable in finite number of steps if from the knowledge of the output of a given system we can uniquely reconstruct the initial state.

Proposition 3.3. (Mozyrska et al. [16]) If $\det E \neq 0$ then system (3) is observable (in q steps) if and only if one of the following conditions is satisfied

(i) columns of the matrix $E_{(\alpha,\beta)}(Ah^{\alpha}, q)$ are linearly independent

(ii) $\operatorname{rank}\begin{bmatrix} C \\ CE_{(\alpha,\beta)}(Ah^{\alpha},1) \\ \vdots \\ CE_{(\alpha,\beta)}(Ah^{\alpha},q-1) \end{bmatrix} = p.$

4. PERFECT OBSERVERS

The aim of this section is to present conditions for existing of full order perfect observer of system (3).

Definition 4.1. Let $E, F \in \mathbb{R}^{p\times p}$, $G \in \mathbb{R}^{p\times m}$ and $H \in \mathbb{R}^{p\times r}$. A linear control systems of the form:

$$E\left({}_{a}\Upsilon^{\beta}\hat{x}\right)(nh) = F\hat{x}_j(nh+a) + Gu(nh) + Hy(nh), \tag{4}$$

where $\hat{x} : \mathbb{N}_0 \to \mathbb{R}$ is the estimate of the unknown state vector x, is called

i. an *observer of order* β of system (3) if and only if

$$\lim_{n\to\infty} \hat{x}(nh+a) = x(nh+a),$$

ii. a *perfect observer of order* β of system (3) if and only if

$$\hat{x}(nh+a) = x(nh+a) \quad \text{for any} \quad n \in \mathbb{N}_0.$$

In practice often there is an access to control u and output y. So, this leads the following problem: given a system (3) having a knowledge of $u(nh)$ and $y(nh)$ find $\hat{x}(nh)$ – an estimate of the unknown state vector x such that $\hat{x}(nh+a) = x(nh+a)$.

Let us recall that that a matrix $N \in \mathbb{R}^{p\times p}$ is called nilpotent if there exists a natural number ν such that $N^{\nu} = 0$ but $N^{\nu-1} \neq 0$. Such number ν is called the *nilpotent index* of matrix N. Additionally, if N is nilpotent and I is identity matrix of respective dimension, then matrices $I - N$ and $I + N$ are invertible.

Proposition 4.2. Let us consider a linear system

$$N\left({}_a\Upsilon^{\alpha}x\right)(nh) = \Lambda x(nh+a), \quad x(a) = x_0 \tag{5}$$

of fractional order $\alpha \in (0,1]$ where $N \in \mathbb{R}^{p\times p}$ is a nilpotent matrix of the nilpotency order q, $q \geq 2$, and $\Lambda \in \mathbb{R}^{p\times p}$ is a diagonal matrix with nonzero elements. Then the system (5) has a unique solution given by

$$x(nh+a) = \mathcal{Z}^{-1}\left[\left(\frac{z}{z-1}\right)^{\beta}\left[N - z^{-1}h^{\alpha}\left(\frac{z}{z-1}\right)^{\alpha}\Lambda\right]^{-1} Nx_0\right](n)$$

if and only if $\Lambda \neq N\frac{h^{\alpha}}{z}\mathcal{Z}[\tilde{\varphi}_{-\alpha}](z)$ where $\tilde{\varphi}_{-\alpha}$ is given by (1).

P r o o f. Taking the $\mathcal{Z}$-transform of both sides of (5), from Proposition 3.1 we obtain that

$$zh^{-\alpha}N\left(\frac{z}{z-1}\right)^{-\alpha}X(z) - zh^{-\alpha}N\left(\frac{z}{z-1}\right)^{-\alpha+\beta}x_0 = \Lambda X(z)$$

where $\alpha = \beta$ for Riemann–Liouville or Grünwald–Letnikov-type operators and $\beta = 1$ for Caputo-type operator (see also Propositions 2.1, 2.2, 2.3). Then

$$\left[N - z^{-1}h^{\alpha}\left(\frac{z}{z-1}\right)^{\alpha}\Lambda\right]X(z) = \left(\frac{z}{z-1}\right)^{\beta}Nx_0.$$

From properties of nilpotent matrices, it follows that matrix on the left hand side of (4) is invertible. So

$$X(z) = \left(\frac{z}{z-1}\right)^{\beta}\left[N - z^{-1}h^{\alpha}\left(\frac{z}{z-1}\right)^{\alpha}\Lambda\right]^{-1}Nx_0$$

if and only if $N \neq z^{-1}h^{\alpha}\left(\frac{z}{z-1}\right)^{\alpha}\Lambda$, i. e., from (2), if and only if $\Lambda \neq N\frac{h^{\alpha}}{z}\mathcal{Z}[\tilde{\varphi}_{-\alpha}](z)$. Using the inverse of $\mathcal{Z}$-transform to (4) we find the unique solution to equation (5).

□

Let $e(nh) := x(nh) - \hat{x}(nh)$. It is obvious then also $e(nh+a) = x(nh+a) - \hat{x}(nh+a)$. Moreover,

$$E\left({}_a\Upsilon^{\alpha}e\right)(nh) = E\left({}_a\Upsilon^{\alpha}x\right)(nh) - E\left({}_a\Upsilon^{\alpha}\hat{x}\right)(nh)$$

and by (3a)-(3b) and (4) we obtain

$$E\left({}_a\Upsilon^{\alpha}e\right)(nh) = (A - HC)x(nh+a) - F\hat{x}(nh+a) + (B - G + D)u(nh).$$

Then

$$E\left({}_a\Upsilon^{\alpha}e\right)(nh) = Fe(nh+a)$$

if and only if

$$F = A - HC \quad \text{and} \quad G = B + D. \tag{6}$$

On the other hand, it is known (see for example in [23]) that any sequence of elementary row operations on a given matrix is equivalent to premultiplication (left multiplication) of this matrix by an appropriate matrix. Then, using elementary row operations matrix E can be expressed in the following (row) equialent upper triangular form (similarly as in [13] where elementary row and columns operations are used):

$$P_1 E = \begin{pmatrix} 0 & E_1 \\ 0 & 0 \end{pmatrix}, \quad \text{with} \quad E_1 = \begin{pmatrix} e_{11} & e_{12} & \dots & e_{1r} \\ 0 & e_{22} & \dots & e_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_{rr} \end{pmatrix}$$

or lower triangular form

$$P_2 E = \begin{pmatrix} 0 & 0 \\ E_2 & 0 \end{pmatrix}, \quad \text{with} \quad E_2 = \begin{pmatrix} e_{11} & 0 & \dots & 0 \\ e_{21} & e_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{r1} & e_{r2} & \dots & e_{rr} \end{pmatrix}.$$

Therein P_1 and P_2 denote matrices of elementary row operations. Note that matrices $P_1 E$ and $P_2 E$ are nilpotent with nilpotent order $\nu = 2$.

Using some elementary row operations, system (4) can be rewritten in the equivalent form as

$$N \left({}_a\Upsilon^\alpha e \right)(nh) = \overline{F} e(nh + a) \tag{7}$$

with

$$N = PE \quad \text{and} \quad \overline{F} = PF \tag{8}$$

where P denotes a matrix of elementary row operations.

Remark 4.3. Using elementary row operations system (3) can be transformed to the equivalent form (7). Moreover, if matrix H is picked so that matrix $\overline{F}$ coincides with diagonal matrix Λ given in Lemma 4.2, then $e(nh) = 0$ and system (4) describes the perfect observer for fractional order system (3). Note that H can be picked in a such way that it can contains, but not necessary, elements depending on h and/or α.

Theorem 4.4. The observer (4) is the perfect fractional observer of the system (3) if and only if there exists the matrix $P \in \mathbb{R}^{p \times p}$ satisfying (8) and such that

$$\operatorname{rank} \begin{bmatrix} PA - \Lambda \\ C \end{bmatrix} = \operatorname{rank} C \tag{9}$$

where Λ is a diagonal matrix described in Corollary 4.3.

P r o o f. The reasoning is the same as given in [13] for the linear fractional continuous-time system. So, we present only sketch of the reasoning.

Let P denotes a nonsingular matrix of elementary row operations. In order to design the perfect observer (4) for the system (3) we should to choose matrices F, G, H such that condition (6) is fulfilled and matrix $\overline{F} = PF$ is represented in the diagonal form Λ described in Lemma 4.2. These requirements are fulfilled if and only if

$$PA - PHC = \Lambda \tag{10}$$

Equation (10) has solution H for given C and Λ if and only if (9) holds. □

Example 4.5. Let us consider the following system

$$\begin{aligned}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left({}_a\Upsilon^{\alpha} x\right)(nh) &= \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} x(nh+a) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(nh)\,, \\ y(nh) &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(nh+a) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(nh)\,.\end{aligned}$$

Then $P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is the such matrix that $PE = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Picking $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & h^{\alpha} & 0 \\ 0 & 0 & 2 \end{bmatrix}$ we obtain $PA - \Lambda = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1-h^{\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since rank $\begin{bmatrix} PA - \Lambda \\ C \end{bmatrix} = 2 =$ rank$C$, then solving the equation $PA - \Lambda = PHC$ we obtain $H = \begin{bmatrix} 1 & h^{\alpha} \\ a_1 & a_2 \\ 1 & 1 \end{bmatrix}$ where a_1, a_2 are real numbers. Since

$$F = A - HC = \begin{bmatrix} 2 & h^{\alpha} & 0 \\ 1 & 1-a_2 & -a_1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad G = B + D = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

the perfect observer is of the considered system is of the form

$$\begin{aligned}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left({}_a\Upsilon^{\alpha} x\right)(nh) &= \begin{bmatrix} 2 & h^{\alpha} & 0 \\ 1 & 1-a_2 & -a_1 \\ 1 & 0 & 1 \end{bmatrix} \hat{x}(nh+a) \\ &+ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u(nh) + \begin{bmatrix} 1 & h^{\alpha} \\ a_1 & a_2 \\ 1 & 1 \end{bmatrix} y(nh).\end{aligned}$$

Theorem 4.6. Assume that rank$C = p$. Then the full order perfect observer (4) exists if and only of system (3) with $E = I$ is observable.

Proof. The result follows from the same facts as ones given in [20]. Namely, if C has a full column rank, then its inverse it is Moore-Penrose matrix C^{+} and directly from definition of perfect observer and (3) it follows that $\hat{x} = C^{-1}y - C^{-1}Du$ (i. e. system is observable). □

Since the procedure of finding the perfect observer requires matrix operation, for fractional discrete-time case it is same as [20].

Remark 4.7. If rank$C < p$, then the order perfect observer (4) still can exists under condition that the pair (A, C) is observable. Similarly, as previous, the procedure of finding this observer requires matrix operation and is same as [20].

The presented result can be easily also valid for the linear multi–parameter fractional order control system with the Caputo–, Riemann–Liouville and the Grünwald–Letnikov–h-difference operator.

5. DETECTABILITY OF FRACTION ORDER SYSTEM

One of the applications of the classical observer is the problem od detection of the input-output control system, see for example [4, 21].

In order to state conditions for detectability of fractional order systems let us recall, see [24], that the constant vector $x^{\text{eq}} = (x_1^{\text{eq}}, \dots, x_n^{\text{eq}})$ is an *equilibrium point* of the fractional difference system

$$E\left({}_a\Upsilon^\alpha x\right)(nh) = \overline{A}x(nh + a) \tag{11}$$

if and only if

$$E\left({}_a\Upsilon^\alpha x^{\text{eq}}\right)(nh) = \overline{A}x^{\text{eq}}$$

for all $n \in \mathbb{N}_0$. The equilibrium $x^{\text{eq}} = 0$ of (11) is said to be

(a) *stable* if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|\overline{x}(nh + a)\| < \epsilon$, for all $k \in \mathbb{N}_0$.

(b) *attractive* if there exists $\delta > 0$ such that $\|x_0\| < \delta$ implies $\lim_{n\to\infty} \overline{x}(nk) = 0$.

(c) *asymptotically stable* if it is stable and attractive.

The fractional difference system (11) is called *stable/asymptotically stable* if their equilibrium points $x^{\text{eq}} = 0$ are stable/asymptotically stable.

Proposition 5.1. (Mozyrska and Wyrwas [17]) Let $\alpha \in (0, 1]$ and $\beta < \alpha + 1$. Let R be the set of all roots of the equation $(z - 1)^\alpha = \lambda z^{\alpha-1}$. If all elements from R are strictly inside the unite circle, then $\lim_{n\to\infty} E_{(\alpha,\beta)}(\lambda, n) = 0$.

Proposition 5.2. Let us consider the system (11). Let R be the set of all roots of the equation

$$det\left(E - \frac{h^\alpha}{z}\left(\frac{z}{z-1}\right)^\alpha \overline{A}\right) = 0. \tag{12}$$

If all elements from R are strictly inside the unit circle, then the system (11) is asymptotically stable.

P r o o f. The some reasoning as in the proof of Proposition 4.2, leads us to

$$\begin{aligned} X(z) &= \left(\frac{z}{z-1}\right)^\beta \left[E - z^{-1}h^\alpha\left(\frac{z}{z-1}\right)^\alpha \overline{A}\right]^{-1} Ex_0 \\ &= \left(\frac{z}{z-1}\right)^\beta \frac{\operatorname{adj}\left[E - z^{-1}h^\alpha\left(\frac{z}{z-1}\right)^\alpha \overline{A}\right]E}{\det\left[E - z^{-1}h^\alpha\left(\frac{z}{z-1}\right)^\alpha \overline{A}\right]} x_0 \end{aligned} \tag{13}$$

where adj$M$ denotes a disjoint matrix of M and $\alpha = \beta$ for Riemann–Liouville or Grünwald-Letnikov-type operators and $\beta = 1$ for Caputo-type operator.

Following the reasoning presented in [17], we can see there exists invertible matrix P such that $M = E - z^{-1}h^{\alpha}\left(\frac{z}{z-1}\right)^{\alpha}\overline{A}$ can be expressed in the form $M = PEP^{-1} - \frac{h^{\alpha}}{z}\left(\frac{z}{z-1}\right)^{\alpha} PJP^{-1}$ where $\overline{A} = PJP^{-1}$ and $J = \mathrm{diag}(J_1, \dots, J_s)$ with Jordan's blocks J_i of order r_i, $i = 1, \dots, s$. The number of blocks corresponding to eigenvalue λ_i of $\overline{A}$ is p_i. So, $\det M = \det(PEP^{-1}) - \frac{h^{\alpha}}{z}\left(\frac{z}{z-1}\right)\prod_{i=1}^{s}\det J_i$. If PEP^{-1} is a singular matrix, then using the inverse of $\mathcal{Z}$-transform to (13) and Proposition 5.1 we obtain the thesis. If PEP^{-1} is the identity matrix, then using the some arguments as in [17], we also obtain thesis. □

The equation (12) is called the *characteristic equation* associated with the system (11). Note that this characteristic equation is not longer a polynomial equation (as in the classical case), it can be also irrational equation. Hence the function on the right hand side of the equation (12) is a multivalued function. So, we should to use complex analysis tools for finding its solution instead of classical algebraic ones.

Definition 5.3. We say that system (3) is *detectable* if there exists a matrix $K \in \mathbb{R}^{p\times r}$ such that system $E\left({}_a\Upsilon^{\alpha}e\right)(nh) = (A - CK)e(nh + a)$ is asymptotically stable.

Theorem 5.4. If system (3) is detectable then there exists matrix $K \in \mathbb{R}^{p\times r}$ such that system

$$E\left({}_a\Upsilon^{\alpha}\right)(nh) = (A - C)\hat{x}(nh + a) + (B - KD)u(nh) + Ky(nh) \tag{14}$$

is the observer of system (3).

P r o o f. Let $e(nh) := x(nh) - \hat{x}(nh)$. Then it is easy to check that using (3a)-(3b) we get

$$E\left({}_a\Upsilon^{\alpha}e\right)(nh) = (A - CK)x(nh + a). \tag{15}$$

Since system (3) is detectable, we can choose matrix K in such way that system (15) is asymptotically stable, so $\lim_{n\to\infty} e(nh) = 0$. Hence $\lim_{n\to\infty}\hat{x}(nh) = x(nh)$. This means that (14) is the observer of system (3). □

Example 5.5. Let us consider the following system:

$$\begin{aligned}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\left({}_a\Upsilon^{\alpha}\right)(nh) &= \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} x(nh + a) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(nh) \\ y(nh) &= \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} x(nh + a) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(nh)\end{aligned} \tag{16}$$

with $h > 0$ and $\alpha \in (0, 1]$. Let us also consider the linear state-feedback controller gain $K = \begin{bmatrix} 0.5 & 1.5 \\ 0.5 & -0.5 \end{bmatrix}$. Then the closed loop system is of the form

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\left({}_a\Upsilon^{\alpha}\right)(nh) = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} x(nh + a). \tag{17}$$

This system is asymptotically stable if all elements from the set of roots of the equation

$$\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \frac{h^\alpha}{z}\left(\frac{z}{z-1}\right)^\alpha \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}\right) = 0$$

i.e. of the equation

$$2\frac{h^\alpha}{z}\left(\frac{z}{z-1}\right)^\alpha \left(1 - 2\frac{h^\alpha}{z}\left(\frac{z}{z-1}\right)^\alpha\right) = 0, \tag{18}$$

be strictly inside the unit circle (see Proposition 5.2). Note that $z = 0$ and $z = 2h^\alpha$ are solutions of this equation. For chosen h and α solutions of (18) are presented in tables below.

1) If $h = 0,1$, then

α	z	α	z	α	z	α	z
0,05	1,9545	0,3	1,0024	0,55	0,5637	0,8	0,317
0,1	1,5887	0,35	0,8934	0,6	0,5024	0,85	0,2825
0,15	1,4156	0,4	0,7962	0,65	0,4477	0,9	0,2518
0,2	1,2619	0,45	0,7096	0,7	0,399	0,95	0,2244
0,25	1,1247	0,5	0,6324	0,75	0,3557	1	0,2

2) If $h = 0,25$, then

α	z	α	z	α	z	α	z
0,05	1,8661	0,3	1,3195	0,55	0,933	0,8	0,6597
0,1	1,7411	0,35	1,2311	0,6	0,8705	0,85	0,6156
0,15	1,6245	0,4	1,1487	0,65	0,8122	0,9	0,5743
0,2	1,5157	0,45	1,0718	0,7	0,7579	0,95	0,5359
0,25	1,4142	0,5	-	0,75	0,7071	1	0,5

3) If $h = 0,5$, then

α	z	α	z	α	z	α	z
0,05	1,9319	0,3	1,6245	0,55	1,366	0,8	1,147
0,1	1,866	0,35	1.5692	0,6	1,3195	0,85	1,1096
0,15	1,8025	0,4	1,5157	0,65	1,2746	0,9	1,0718
0,2	1,7411	0,45	1,4641	0,7	1,2311	0,95	1,0353
0,25	1,6818	0,5	1,4142	0,75	1,1892	1	-

It is easy to see that detectebility of the system (16), for given gain matrix, depends on both h and α , even they are not appear in the matrix K. In the given example, if $h = 0,1$, then the system (16) is detectable for $\alpha \in (0,3;1]$, if $h = 0,25$ then this system is detectible for $\alpha \in (0,5;1]$, but if $h = 0,5$ (or $h > 0,5$) system is not detectible for any $\alpha \in (0;1]$. For h and α for which the system (16) is detectible the observer of the considered system is

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} ({}_a\Upsilon^\alpha)(nh) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \hat{x}(nh+a) + \begin{bmatrix} 0,5 \\ 1,5 \end{bmatrix} u(nh) + \begin{bmatrix} 0.5 & 1.5 \\ 0.5 & -0.5 \end{bmatrix} y(nh).$$

Proposition 5.6. If a function $\gamma : \mathbb{N}_0 \to \mathbb{R}^p$ is the solution of system $E\left({}_a\Upsilon^\alpha x\right)(nh) = Ax(nh+a) + Bu(nh)$ and $\lim_{n\to\infty} C\gamma(nh) = 0$ implies $\lim_{n\to\infty} \gamma(nh) = 0$ then system system (3) is detectable.

Proof. The result follows directly from definitions of detectability and asymptotic stability. □

The presented result can be easily also valid for the linear multi–parameter fractional order control system with the Caputo or the Riemann–Liouville or the Grünwald–Letnikov–h-difference operators.

6. CONCLUSIONS

In the paper the concept of the linear fractional order perfect (exact) observer for singular h-difference fractional system was introduced. The conditions of its existence were given. Since some definitions and facts that we discussed are the same for the Riemann–Liouville-, the Caputo–type and the Grünwald–Letnikov–type fractional h-difference operator are similar, the common symbol ${}_a\Upsilon^\alpha$ defined by its values has been used. It has been shown that for a detectable singular system one can design a fractional singular observer, but no longer a perfect one. This observer, for the given gain matrix, depends on sampling step h and order of the system. Note also that the function on the right hand side of the equation (12) is a multivalued function, so complex analysis methods are need, but not more tools for testing the existence of roots of matrix K as in the classical case.

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REFERENCES

[1] T. Abdeljawad and D. Baleanu: Fractional differences and integration by parts. J. Comput. Analysis Appl. *13* (2011), 3, 574–582.

[2] F. M. Atici and P. W. Eloe: A transform method in discrete fractional calculus. Int. J. Difference Equations *2* (2007), 165–176.

[3] N. R. O. Bastos, R. A. C. Ferreira, and D. F. M. Torres: Necessary optimality conditions for fractional difference problems of the calculus of variations. Discrete Contin. Dyn. Syst. *29* (2011), 2, 417–437. DOI:10.3934/dcds.2011.29.417

[4] L. Dai: Observers for discrete singular systems. IEEE Trans. Automat. Control *33* (1988), 2, 187–191. DOI:10.1109/9.387

[5] M. Darouach and L. Boutat-Baddas: Observers for a class of nonlinear singular systems. IEEE Trans. Automat. Control *53* (2008), 11, 2627–2633. DOI:10.1109/tac.2008.2007868

[6] M. A. Duarte-Mermoud, M. J. Mira, I. S. Pelissier, and J. C. Travieso-Torres: Evaluation of a fractional order PI controller applied to induction moror speed control. In: Proc. 8th IEEE Int. Conf. on Control and Automation, Xiamen 2010, pp. 573–577. DOI:10.1109/icca.2010.5524496

[7] A. Dzielinski, D. Sierociuk, and G. Sarwas: Some applications of fractional order calculus. Bull. Pol. Acad. Sci. Tech. Sci. *58* (2010), 4, 583-59. DOI:10.2478/v10175-010-0059-6

[8] R. A. C. Ferreira and D. F. M. Torres: Fractional h-difference equations arising from the calculus of variations. Appl. Anal. Discrete Math. *5* (2011), 1, 110–121. DOI:10.2298/aadm110131002f

[9] M. Fiacchini and G. Millerioux: Deat-beat functional observers for discrete-time LVP systems with unknown inputs. IEEE Trans. Automat. Control *58* (2013), 12, 3230–3235. DOI:10.1109/tac.2013.2261712

[10] E. Girejko, D. Mozyrska, and M. Wyrwas: Advances in the theory and applications of non-integer order systems. In: Comparison of h-difference fractional operators (W. Mitkowski, J. Kacprzyk, and J. Baranowski, eds.), Springer *257* (2013), pp. 191–197. DOI:10.1007/978-3-319-00933-9_17

[11] A. Isidori: Nonlinear Control Theory. Springer, 1991.

[12] T. Kaczorek: Full-order perfect observers for continuous-time linear systems. Pomiary, Automatyka, Kontrola *1* (2001), 3–6.

[13] T. Kaczorek: Advances in Modelling and control of non-integer-order systems. In: Perfect Observers of Fractional Descriptor Continuous-Time Linear System (K. J. Latawiec, M. Lukaniszyn and R. Stanislawski, eds.), Lecture Notes in Electrical Engineering, Springer International Publishing *320* (2015), pp. 3–12. DOI:10.1007/978-3-319-09900-2_1

[14] K. S. Miller and B, Ross : Fractional difference calculus. In: Proc. Int. Symp. on Univalent Functions, Fractional Calculus and their Applications, Nihon University, Kōriyama 1988, pp. 139–152.

[15] D. Mozyrska and E. Girejko: Advances in Harmonic Analysis and Operator Theory: The Stefan Samko Anniversary. In: Overview of the fractional h-difference operators, Springer *229* (2013), pp. 253–267. DOI:10.1007/978-3-0348-0516-2_14

[16] D. Mozyrska, E. Pawluszewicz, and M. Wyrwas: Local observability and controllability of nonlinear discrete-time fractional order systems based on their linearization. Int. J. Syst. Sci. *48* (2017), 4, 788–794.

[17] D. Mozyrska and M. Wyrwas: The $\mathcal{Z}$-transform method and delta-type fractional difference operators. Discrete Dynamics in Nature and Society 2015, pp. 47–58. DOI:10.1007/978-3-319-09900-2_5

[18] D. Mozyrska, M. Wyrwas, and E. Pawluszewicz: Stabilization of linear multi-parameter fractional difference control systems. In: Proc. 20th Int. Conf. on Methods and Models in Automation and Robotics MMAR'2015, Miedzyzdroje 2915, pp. 315-319. DOI:10.1109/mmar.2015.7283894

[19] I. N'Doye, M. Darouach, M. Zasadzinski, and N.-E. Radhy: Observers design for singular fractional-order system. In: Proc. 50th Int. Conf. on Decision and Control and European Control Conference CDC-ECC'2011, Orlando 2011, pp. 4017–4022. DOI:10.1109/cdc.2011.6161336

[20] M. Slawinski and T. Kaczorek: Perfect observers for continuous time linear systems. Pomiary, Automatyka, Kontrola *1* (2004), 39-44.

[21] E. D. Sontag: Mathematical Control Theory. Springer 1998. DOI:10.1007/978-1-4612-0577-7

[22] J. C. Trigeassou, T. Poinot, J. Lin, A. Oustaloup, and F. Levron: Modelling and identification of a non integer order system. In: Proc. European Control Conference ECC'1999, Karlsruhe 1999, pp. 2453–2458.

[23] W. A. Wolowich: Linear Multivariable Systems. Springer-Verlag, 1974. DOI:10.1007/978-1-4612-6392-0

[24] M. Wyrwas, E. Pawluszewicz, and E. Girejko: Stability of nonlinear h- difference systems with n fractional orders. Kybernetika *51* (2015), 1, 112–136. DOI:10.14736/kyb-2015-1-0112

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