ON APPROXIMATION OF STABILITY RADIUS FOR AN INFINITE-DIMENSIONAL FEEDBACK CONTROL SYSTEM

Hideki Sano

In this paper, we discuss the problem of approximating stability radius appearing in the design procedure of finite-dimensional stabilizing controllers for an infinite-dimensional dynamical system. The calculation of stability radius needs the value of H_∞-norm of a transfer function whose realization is described by infinite-dimensional operators in a Hilbert space. From the computational point of view, we need to prepare a family of approximate finite-dimensional operators and then to calculate the H_∞-norm of their transfer functions. However, it is not assured that they converge to the value of H_∞-norm of the original transfer function. The purpose of this study is to justify the convergence. In a numerical example, we treat parabolic distributed parameter systems with distributed control and distributed/boundary observation.

Keywords: distributed parameter system, finite-dimensional controller, stability radius, transfer function, semigroup

Classification: 93D15, 93C25

1. INTRODUCTION

In the field of control of distributed parameter systems, the linear system described by the following evolution equation with output equation has been used for a long time.

$$\dot{z}(t) = -Az(t) + Bu(t), \quad t > 0, \quad z(0) = z_0, \tag{1}$$

$$y(t) = Cz(t), \quad t > 0, \tag{2}$$

where $-A$ is the infinitesimal generator of a C_0-semigroup on a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. $B : \mathbf{R}^m \to H$ is a bounded input operator, and $C : D(C) \subset H \to \mathbf{R}^p$ is a bounded/unbounded output operator. $z(t) \in H$ is the state variable, $u(t) \in \mathbf{R}^m$ the input variable, and $y(t) \in \mathbf{R}^p$ the output variable. For system (1), (2), the stabilization problem/the optimal control problem by static controllers have been investigated by many researchers (see e. g. [4, 7] and the references therein). Also, the stabilization problem by finite-dimensional dynamic controllers has been widely studied. In the following, we briefly survey several works related to the latter stabilization problem. In this paper, we especially treat a problem that remains in the design method based on stability radius.

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In general, when one constructs a finite-dimensional model for an infinite-dimensional system and applies a finite-dimensional controller designed for it to the original infinite-dimensional system, spillover phenomenon may be occured by the influence of unmodeled modes. Sakawa firstly introduced two kinds of finite-dimensional observers for linear diffusion systems to reduce the influence of unmodeled modes for the closed-loop system with the finite-dimensional controller [12]. After that, Balas called one of them as the residual mode filter (RMF), and clarified that the RMF plays an essential role for the construction of finite-dimensional stabilizing controllers [1]. Moreover, it was shown in [13] that the results could be extended to the system with bounded input operator and A^{γ}-bounded output operator. On the other hand, Nambu gave the design method of infinite-dimensional stabilizing controllers applicable to linear parabolic systems under boundary control and observation, and further accomplished finite-dimensionalization of the obtained controllers [8]. Schumacher gave the direct design method of finite-dimensional controllers for a wide class of linear time-invariant systems [16], in which the eigenfunctions of the operator $-A + BF$ were used. Moreover, Schumacher's design method was extended to linear parabolic systems with unbounded control and observation by Curtain [3]. Also, Lasiecka gave finite element approximation of Luenberger's observer based controllers for linear parabolic systems with unbounded input and output operators [7, Chapter 4]. Thus, the existence of finite-dimensional controllers was assured theoretically for the systems mentioned above, however, these desgin methods had a common weak point that one could not give the order of controllers a priori, that is, the order was supposed to be taken sufficiently large, except the design method based on stability radius by El Jai and Pritchard [5, Chapter 1].

Although the method based on stability radius [5, Chapter 1] is very simple, in order to calculate stability radius we need the value of H_∞-norm of a transfer function whose realization is described by infinite-dimensional operators in a Hilbert space. From the computational point of view, we need to prepare a family of approximate finite-dimensional operators and then to calculate the H_∞-norm of their transfer functions. Then, we have a question of whether or not they converge to the value of H_∞-norm of the original transfer function, which gives a motivation of this paper. The purpose of this study is to justify the convergence, that is, to show a theory of approximation for Theorem 1.1 below, and further to give a numerical example to illustrate the assertion. Here, we note that in [6] the method via numerical analysis has been proposed for approximation of stability radii for high order finite-dimensional systems.

To explain the existing result [5] briefly, we shall consider the case where the operator C is bounded, i. e., $D(C) = H$, and the operator A is defined by

$$Af = \sum_{i=1}^{\infty} \lambda_i \langle f, \varphi_i \rangle \varphi_i, \quad f \in D(A), \tag{3}$$

$$D(A) = \left\{ f \in H \,;\, \sum_{i=1}^{\infty} \lambda_i^2 \langle f, \varphi_i \rangle^2 < +\infty \right\},$$

where $\{\lambda_i, i \geq 1\}$ is a sequence of real numbers such that $\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$, $\lim_{i\to\infty} \lambda_i = \infty$, and $\{\varphi_i, i \geq 1\}$ forms a complete orthogonal system in H. It is clear that the operator A is self-adjoint. Then, it follows from Hille-Yosida's theorem [9] that $-A$ generates a C_0-semigroup e^{-tA} on H whose expression is given by

$$e^{-tA}f = \sum_{i=1}^{\infty} e^{-\lambda_i t}\langle f, \varphi_i\rangle \varphi_i, \quad t \geq 0,\ f \in H.$$

In order to decompose system (1), (2), we use the orthogonal projection P_k defined by $P_k f = \sum_{i=1}^{k} \langle f, \varphi_i\rangle \varphi_i$. Using the operators P_l and $I - P_l$, where l is a positive integer, we decompose the state variable $z(t)$ as $z(t) = z_1(t) + z_2(t)$, where $z_1(t) := P_l z(t)$, $z_2(t) := (I - P_l)z(t)$. Also, the space H is expressed as

$$H = \overbrace{P_l H}^{\dim = l} \oplus \overbrace{(I - P_l)H}^{\dim = \infty}.$$

Then, system (1), (2) is equivalently expressed as follows (see e. g. [1]):

$$\begin{cases} \dot{z}_1(t) = -A_1 z_1(t) + B_1 u(t), & z_1(0) = z_{01}, \\ \dot{z}_2(t) = -A_2 z_2(t) + B_2 u(t), & z_2(0) = z_{02}, \\ y(t) = C_1 z_1(t) + C_2 z_2(t), & \end{cases}$$

where

$$\begin{array}{ll} A_1 := P_l A P_l, & A_2 := (I - P_l)A(I - P_l), \\ B_1 := P_l B, & B_2 := (I - P_l)B, \\ C_1 := C P_l, & C_2 := C(I - P_l), \\ z_{01} := P_l z_0, & z_{02} := (I - P_l)z_0. \end{array}$$

In the above, note that the operator A_2 is unbounded, whereas all the other operators are bounded. Since the finite-dimensional Hilbert space $P_l H$ is identified with the Euclidean space $\mathbf{R}^l$ with respect to the basis $\{\varphi_1, \varphi_2, \ldots, \varphi_l\}$, each element in $P_l H$ is identified with an l-dimensional vector, and the operators A_1, B_1, and C_1 are identified with matrices with appropriate size.

Assumption 1.

(i) The integer l is chosen such that the eigenvalues of the matrix $-A_1$, $\sigma(-A_1)$ contains all unstable eigenvalues of the operator $-A$.

(ii) The pair $(-A_1, B_1)$ is stabilizable and the pair $(C_1, -A_1)$ is detectable (see e. g. [17] for the definitions and the related theorems).

Under (ii) of Assumption 1, it is possible to choose a matrix F_1 such that $-A_1 - B_1 F_1$ is Hurwitz stable [17], since the pair $(-A_1, B_1)$ is stabilizable. Similarly, it is possible to choose a matrix G_1 such that $-A_1 - G_1 C_1$ is Hurwitz stable, since the pair $(C_1, -A_1)$ is detectable. Here, let us consider the following observer-based controller [17] for system (1), (2):

$$\begin{cases} \dot{w}_1(t) = -A_1 w_1(t) + B_1 u(t) + G_1(y(t) - C_1 w_1(t)), & w_1(0) = w_{10}, \\ u(t) = -F_1 w_1(t). & \end{cases} \tag{4}$$

Then, by introducing the error vector $e_1(t)^T := z_1(t)^T - w_1(t)^T$, the closed-loop system consisting of system (1), (2) and the controller (4) is written as

$$\dot{\xi}(t) = (\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C})\xi(t), \quad \xi(0) = \xi_0, \tag{5}$$

where the state $\xi(t) := [e_1(t)^T, z_1(t)^T, z_2(t)]^T$ is in the real Hilbert space $Z := \mathbf{R}^l \times \mathbf{R}^l \times (I - P_l)H$, and the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{K}$ are defined by

$$\mathcal{A} = \begin{bmatrix} -A_1 - G_1C_1 & 0 & 0 \\ B_1F_1 & -A_1 - B_1F_1 & 0 \\ B_2F_1 & -B_2F_1 & -A_2 \end{bmatrix}, \tag{6}$$

$$\mathcal{B} = \begin{bmatrix} -G_1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & C_2 \end{bmatrix}, \quad \mathcal{K} = 1.$$

Then, the following theorem is known.

Theorem 1.1. [5] Suppose that Assumption 1 is satisfied. Then, the operator $\mathcal{A}$ defined by (6) generates an exponentially stable C_0-semigroup $e^{t\mathcal{A}}$ on Z. In addition, if the condition

$$\|\mathcal{C}(\cdot I - \mathcal{A})^{-1}\mathcal{B}\|_\infty := \sup_{\omega \in \mathbf{R}} \|\mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B}\| < 1, \tag{7}$$

that is,

$$\sup_{\omega \in \mathbf{R}} \|C_2(j\omega I + A_2)^{-1}B_2F_1(j\omega I + A_1 + B_1F_1)^{-1}(j\omega I + A_1)(j\omega I + A_1 + G_1C_1)^{-1}G_1\| < 1$$

is satisfied, the closed-loop operator $\mathcal{A} + \mathcal{B}\mathcal{K}\mathcal{C}$ of (5) also generates an exponentially stable C_0-semigroup $e^{t(\mathcal{A}+\mathcal{B}\mathcal{K}\mathcal{C})}$ on Z. In other words, the control law (4) becomes a finite-dimensional stabilizing controller for system (1), (2).

The proof of Theorem 1.1 is due to the result with respect to the stability radius [2, 11]. As shown in [11, 2], the stability radius $r_c(\mathcal{A}; \mathcal{B}, \mathcal{C})$ of the closed-loop system (5) is calculated as

$$r_c(\mathcal{A}; \mathcal{B}, \mathcal{C}) = \frac{1}{\sup_{\omega \in \mathbf{R}} \|G(j\omega)\|} = \frac{1}{\|G(\cdot)\|_\infty},$$

where $G(j\omega) := \mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B}$. Therefore, when $r_c(\mathcal{A}; \mathcal{B}, \mathcal{C}) > \|\mathcal{K}\| = 1$, that is, the condition (7) holds, the conclusion of the theorem immediately follows.

In Theorem 1.1, we note that the algorithm needs iteration of infinite times to check the condition (7), since it contains the infinite-dimensional operators A_2, B_2, and C_2. In Section 2, we discuss whether or not it is possible to approximate the operators A_2, B_2, and C_2 of the theorem by finite-dimensional operators. The novelty of this paper is the point that it proves the convergence of approximate stability radius to the original one in the feedback control system. Moreover, instead of (2), we discuss the case with unbounded output operator such as

$$y(t) = \tilde{C}(A + c)^\gamma z(t), \quad 0 < \gamma < 1, \tag{8}$$

where A is the unbounded operator defined by (3), $\tilde{C} : H \to \mathbf{R}^p$ is a bounded linear operator, and c is a constant chosen such that $\lambda_1 + c > 0$.

Remark 1.2. In [10], the finite-dimensional version of this theorem was given. That is, the design method of low order stabilizing controllers was proposed for high order finite-dimensional systems, by using stability radius.

Remark 1.3. The condition (7) is a sufficient condition for spillover phenomenon not to be occured for the feedback control system consisting of system (1), (2) and the control law (4).

2. MAIN RESULT

By using the orthogonal projection P_k defined in Section 1, we decompose the state variable $z(t)$ as $z(t) = z_1(t)+z_{2a}(t)+z_{2b}(t)$, where $z_1(t) := P_l z(t)$, $z_{2a}(t) := (P_n-P_l)z(t)$, $z_{2b}(t) := (I-P_n)z(t)$, $n > l$. Note that $z_{2a}(t) + z_{2b}(t) = z_2(t)$. Also, the space H is expressed as

$$H = \overbrace{P_l H}^{\dim = l} \oplus \underbrace{\overbrace{(P_n - P_l)H}^{\dim = n-l} \oplus \overbrace{(I - P_n)H}^{\dim = \infty}}_{=(I-P_l)H}.$$

Then, the infinite-dimensional operators A_2, B_2, and C_2 are equivalently expressed as follows:

$$A_2 = \begin{bmatrix} A_{2a} & 0 \\ 0 & A_{2b} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{2a} \\ B_{2b} \end{bmatrix}, \quad C_2 = \begin{bmatrix} C_{2a} & C_{2b} \end{bmatrix},$$

where $A_{2a} := (P_n - P_l)A(P_n - P_l)$, $B_{2a} := (P_n - P_l)B$, $C_{2a} := C(P_n - P_l)$, $A_{2b} := (I - P_n)A(I - P_n)$, $B_{2b} := (I - P_n)B$, $C_{2b} := C(I - P_n)$. Here, note that the operators A_{2a}, B_{2a}, and C_{2a} are identified with matrices with appropriate size. Then, the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ of (6) are expressed as

$$\mathcal{A} = \begin{bmatrix} -A_1 - G_1 C_1 & 0 & 0 & 0 \\ B_1 F_1 & -A_1 - B_1 F_1 & 0 & 0 \\ B_{2a} F_1 & -B_{2a} F_1 & -A_{2a} & 0 \\ B_{2b} F_1 & -B_{2b} F_1 & 0 & -A_{2b} \end{bmatrix}, \tag{9}$$

$$\mathcal{B} = \begin{bmatrix} -G_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 0 & C_{2a} & C_{2b} \end{bmatrix}.$$

Further, we set the truncated operators as

$$\mathcal{A}_n = \begin{bmatrix} -A_1 - G_1 C_1 & 0 & 0 \\ B_1 F_1 & -A_1 - B_1 F_1 & 0 \\ B_{2a} F_1 & -B_{2a} F_1 & -A_{2a} \end{bmatrix}, \tag{10}$$

$$\mathcal{B}_n = \begin{bmatrix} -G_1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{C}_n = \begin{bmatrix} 0 & 0 & C_{2a} \end{bmatrix}.$$

Now, let us define two transfer functions as follows:

$$G(j\omega) = \mathcal{C}(j\omega I - \mathcal{A})^{-1}\mathcal{B}, \tag{11}$$

$$G_n(j\omega) = \mathcal{C}_n(j\omega I - \mathcal{A}_n)^{-1}\mathcal{B}_n. \tag{12}$$

The following theorem and remarks are our main result in this paper.

Theorem 2.1. Suppose that Assumption 1 is satisfied. Then, the operator $\mathcal{A}_n$ defined by (10) generates a C_0-semigroup $e^{t\mathcal{A}_n}$ with norm bound $\|e^{t\mathcal{A}_n}\| \le Me^{-\nu t},\ t \ge 0$ on the Euclidean space $Z_n := \mathbf{R}^l \times \mathbf{R}^l \times \mathbf{R}^{n-l}$, where $M \ge 1$ and $\nu > 0$ are some constants independent of the integer n. Moreover, there holds

$$\|G_n(\cdot)\|_\infty \to \|G(\cdot)\|_\infty \quad \text{as} \quad n \to \infty,$$

that is, $r_c(\mathcal{A}_n; \mathcal{B}_n, \mathcal{C}_n) \to r_c(\mathcal{A}; \mathcal{B}, \mathcal{C})$ as $n \to \infty$. Accordingly, if $\|G_n(\cdot)\|_\infty < 1$ is satisfied for sufficiently large n, the control law (4) works as a finite-dimensional stabilizing controller for system (1), (2).

Proof. By Assumption 1, the C_0-semigroup generated by the matrix

$$\mathcal{A}_1 := \begin{bmatrix} -A_1 - G_1C_1 & 0 \\ B_1F_1 & -A_1 - B_1F_1 \end{bmatrix}$$

has a norm bound $\|e^{t\mathcal{A}_1}\| \le M_1 e^{-\nu_1 t},\ t \ge 0$, where $M_1 \ge 1$ and $0 < \nu_1 < \lambda_{l+1}$ are some constants independent of the integer n. Also, the C_0-semigroup generated by the matrix $-A_{2a}$ has a norm bound $\|e^{-tA_{2a}}\| \le e^{-\lambda_{l+1}t},\ t \ge 0$. Here, noting that $\|B_{2a}F_1\| \le \|B\|\|F_1\|$, we see that the first assertion holds with $M = M_1(1 + \frac{2M_1\|B\|\|F_1\|}{\lambda_{l+1} - \nu_1})$ and $\nu = \nu_1$.

Next, we estimate the H_∞-norm of $G(j\omega) - G_n(j\omega)$. From (9)–(12), we have

$$G(j\omega) = C_{2a}(j\omega I + A_{2a})^{-1}B_{2a}H(j\omega) + C_{2b}(j\omega I + A_{2b})^{-1}B_{2b}H(j\omega),$$

$$G_n(j\omega) = C_{2a}(j\omega I + A_{2a})^{-1}B_{2a}H(j\omega),$$

by straightforward calculation, where

$$H(j\omega) := -F_1(j\omega I + A_1 + B_1F_1)^{-1}(j\omega I + A_1)(j\omega I + A_1 + G_1C_1)^{-1}G_1.$$

From these, it follows that

$$G(j\omega) - G_n(j\omega) = C_{2b}(j\omega I + A_{2b})^{-1}B_{2b}H(j\omega).$$

By Assumption 1, it is easy to see that $\|H(\cdot)\|_\infty < +\infty$. Also, noting that $\|e^{-tA_{2b}}\| \le e^{-\lambda_{n+1}t},\ t \ge 0$, and that by Hille-Yosida's theorem [9, Theorem 1.5.3 and Remark 1.5.4], $\|(\lambda I + A_{2b})^{-k}\| \le \frac{1}{(\mathrm{Re}\,\lambda + \lambda_{n+1})^k},\ \mathrm{Re}\,\lambda > -\lambda_{n+1},\ k = 1, 2, \ldots$, we have

$$\|(\cdot I + A_{2b})^{-1}\|_\infty = \sup_{\omega \in \mathbf{R}} \|(j\omega I + A_{2b})^{-1}\| \le \frac{1}{\lambda_{n+1}} \to 0 \quad \text{as} \quad n \to \infty. \tag{13}$$

Moreover noting that $\|B_{2b}\|$, $\|C_{2b}\| \to 0$ as $n \to \infty$, we have

$$|\|G(\cdot)\|_\infty - \|G_n(\cdot)\|_\infty| \le \|G(\cdot) - G_n(\cdot)\|_\infty \le \|C_{2b}\|\|(\cdot I + A_{2b})^{-1}\|_\infty \|B_{2b}\|\|H(\cdot)\|_\infty \to 0$$

as $n \to \infty$, which implies that the second assertion holds.

From the second assertion, it follows that $\|G(\cdot)\|_\infty < 1$ if $\|G_n(\cdot)\|_\infty < 1$ for sufficiently large n, which implies from Theorem 1.1 that the third assertion holds. □

Remark 2.2. When the output equation (8) is used instead of (2), we obtain the similar result as in Theorem 2.1. In this case, the control law (4) is replaced as

$$\begin{cases} \dot{w}_1(t) = -A_1 w_1(t) + B_1 u(t) + G_1(y(t) - \tilde{C}_1(A_1 + c)^\gamma w_1(t)), \quad w_1(0) = w_{10}, \\ u(t) = -F_1 w_1(t), \end{cases} \tag{14}$$

as a result, the operators C_1, C_{2a}, and C_{2b} are replaced as $\tilde{C}_1(A_1 + c)^\gamma$, $\tilde{C}_{2a}(A_{2a} + c)^\gamma$, and $\tilde{C}_{2b}(A_{2b} + c)^\gamma$ in the operators (9), (10). Therefore, we need to use the following estimate instead of (13):

$$\|(A_{2b} + c)^\gamma (\cdot I + A_{2b})^{-1}\|_\infty \le \frac{(\lambda_{n+1} + c)^\gamma + \lambda_{n+1}^\gamma \Gamma(1-\gamma)}{\lambda_{n+1}} \to 0 \quad \text{as} \quad n \to \infty, \tag{15}$$

where $\Gamma(\cdot)$ is the gamma function. For the derivation of (15), see [14].

Remark 2.3. System (1), (2) is the parabolic distributed parameter system which contains a diffusion process and a transport-diffusion process, and so on. The assertions of Theorem 2.1 also hold in the case where the operator A is replaced by a Riesz-spectral operator, that is, for the system described by a flexible beam equation.

Remark 2.4. In Theorem 2.1, one cannot give a priori estimate with respect to n for assuring $\|G_n(\cdot)\|_\infty < 1$.

Remark 2.5. According to the procedure in [5], we wrote the closed-loop system consisting of system (1), (2) and the controller (4) as equation (5). But, we may consider the other expression such as

$$\dot{\xi}(t) = (\mathcal{A}' + \mathcal{B}'\mathcal{K}'\mathcal{C}')\xi(t), \quad \xi(0) = \xi_0, \tag{16}$$

where the operators $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$, and $\mathcal{K}'$ are defined by

$$\mathcal{A}' = \begin{bmatrix} -A_1 - G_1 C_1 & 0 & -G_1 C_{2a} & -G_1 C_{2b} \\ B_1 F_1 & -A_1 - B_1 F_1 & 0 & 0 \\ 0 & 0 & -A_{2a} & 0 \\ 0 & 0 & 0 & -A_{2b} \end{bmatrix}, \tag{17}$$

$$\mathcal{B}' = \begin{bmatrix} 0 \\ 0 \\ B_{2a} \\ B_{2b} \end{bmatrix}, \quad \mathcal{C}' = \begin{bmatrix} F_1 & -F_1 & 0 & 0 \end{bmatrix}, \quad \mathcal{K}' = 1.$$

Based on this expression, we can obtain the similar results as in Theorems 1.1 and 2.1. Especially, in the case of single input and single output system, noting that the transfer function $G'(j\omega) := \mathcal{C}'(j\omega I - \mathcal{A}')^{-1}\mathcal{B}'$ is equal to the transfer function $G(j\omega)$ defined by (11), we have the same stability radius for the closed-loop systems (5) and (16), i.e., $r_c(\mathcal{A}'; \mathcal{B}', \mathcal{C}') = r_c(\mathcal{A}; \mathcal{B}, \mathcal{C})$. As a result, we see that $r_c(\mathcal{A}'_n; \mathcal{B}'_n, \mathcal{C}'_n) = r_c(\mathcal{A}_n; \mathcal{B}_n, \mathcal{C}_n)$ holds for approximate operators.

3. NUMERICAL EXAMPLE

We consider the following parabolic distributed parameter system:

$$\begin{cases} z_t(t,x) = \varepsilon z_{xx}(t,x) + \alpha z_x(t,x) + \mu z(t,x) + b(x)u(t), \quad t > 0,\ x \in (0,1), \\ z_x(t,0) = 0, \quad z(t,1) = 0, \quad t > 0, \\ z(0,x) = z_0(x), \quad x \in [0,1], \end{cases} \tag{18}$$

where $z(t,x) \in \mathbf{R}$ is the temperature at time t and at the point $x \in [0,1]$, $u(t) \in \mathbf{R}$ is the control input, and, $\varepsilon > 0$ and α, $\mu \geq 0$ are physical parameters. $b(x) := \frac{1}{r}\mathbf{1}_{[x_0 - r/2, x_0 + r/2]}(x)$ denotes the actuator influence function, where $\mathbf{1}_{[\,\cdot\,,\,\cdot\,]}(x)$ denotes the characteristic function. We first consider the following observation for system (18):

$$y(t) = \int_0^1 c(x) z(t,x)\,\mathrm{d}x, \tag{19}$$

where $c(x) := \frac{1}{r}\mathbf{1}_{[x_1 - r/2, x_1 + r/2]}(x)$ is the sensor influence function.

Let $\beta := \frac{\alpha}{\varepsilon}$. We formulate system (18), (19) in a Hilbert space $L^2_\beta(0,1)$, where $L^2_\beta(0,1)$ is the weighted L^2-space with inner product

$$\langle \varphi, \psi \rangle_\beta := \int_0^1 \varphi(x)\psi(x)e^{\beta x}\,\mathrm{d}x, \quad \varphi, \psi \in L^2_\beta(0,1).$$

Setting $\mathcal{L}\varphi = -\varepsilon\varphi'' - \alpha\varphi' - \mu\varphi$, we define the unbounded operator $A : D(A) \subset L^2_\beta(0,1) \to L^2_\beta(0,1)$ as

$$\begin{aligned} &A\varphi = \mathcal{L}\varphi, \quad \varphi \in D(A), \\ &D(A) = \{\, \varphi \in H^2(0,1)\,;\, \varphi'(0) = 0,\ \varphi(1) = 0 \,\}. \end{aligned}$$

Then, A is a self-adjoint operator in $L^2_\beta(0,1)$ and it has the following eigenvalues and eigenfunctions:

$$\begin{aligned} \lambda_i &= \omega_i^2 \varepsilon + \frac{\alpha^2}{4\varepsilon} - \mu, \\ \varphi_i(x) &= \left(\frac{1}{2} + \frac{\varepsilon}{\alpha}\cos^2\omega_i \right)^{-\frac{1}{2}} e^{-\frac{\alpha}{2\varepsilon}x} \sin\omega_i(1-x), \end{aligned}$$

$i \geq 1$, where $\omega_1 < \omega_2 < \cdots < \omega_i < \cdots$ are the solutions of $\tan\omega = -\frac{2\varepsilon}{\alpha}\omega$ on $\omega > 0$, and $\{\varphi_i\}_{i=1}^\infty$ forms a complete orthogonal system in $L^2_\beta(0,1)$. Note that the operator $-A$

generates an analytic semigroup e^{-tA} on $L^2_\beta(0,1)$ whose growth bound is equal to $-\lambda_1$. If $-\lambda_1 > 0$, it is clear that system (18), (19) is unstable. Here, by defining the bounded operators $B : \mathbf{R} \to L^2_\beta(0,1)$ and $C : L^2_\beta(0,1) \to \mathbf{R}$ as

$$Bv = bv, \quad v \in \mathbf{R},$$
$$C\zeta = \langle e^{-\beta\cdot} c, \zeta \rangle_\beta, \quad \zeta \in L^2_\beta(0,1),$$

system (18), (19) is expressed as in (1), (2).

Next, we consider the following boundary observation for system (18):

$$y(t) = z_x(t,1). \tag{20}$$

In this case, we can formulate the observation equation (20) as

$$y(t) = \tilde{C}(A+c)^\gamma z(t), \tag{21}$$

where $\gamma := \frac{3}{4} + \epsilon' \in (\frac{3}{4}, 1)$, and $\tilde{C} : L^2_\beta(0,1) \to \mathbf{R}$ is the bounded operator defined by

$$\tilde{C}\xi = \langle -\tfrac{1}{\varepsilon}(A+c)^{\frac{1}{4}-\epsilon'} h, \xi \rangle_\beta, \quad \xi \in L^2_\beta(0,1).$$

In the above, $h \in H^2(0,1)$ is the unique solution of the boundary value problem

$$(\mathcal{L}+c)h = 0 \ \text{ in } (0,1), \quad h'(0) = 0, \ \ h(1) = e^{-\beta}.$$

Especially, when $c = \mu$, the solution is concretely given by $h(x) = e^{-\beta}$. For the derivation of (21), see e. g. [15].

Now, let $\varepsilon = 0.1$, $\alpha = 0$, $\mu = 1$, $x_0 = 0.8$, $x_1 = 0.4$, $r = 0.02$, and $\epsilon' = 0.15$. Then, the eigenvalues and eigenfunctions of the operator A become $\lambda_i = 0.1(i-\frac{1}{2})^2\pi^2 - 1$, $\varphi_i(x) = \sqrt{2}\cos(i-\frac{1}{2})\pi x$, $i \geq 1$. As a result, we see that $-A$ has one unstable eigenvalue. Next, by setting $l = 3$, we can derive two models $(-A_1, B_1, C_1)$ and $(-A_1, B_1, \tilde{C}_1(A_1+\mu)^\gamma)$ that satisfy Assumption 1. In fact, these models are controllable and observable [17], and they correspond to the low order finite-dimensional models of system (18), (19) and system (18), (20). For each model, let us choose F_1 as an optimal regulator gain [17] and choose G_1 as an optimal filter gain [17], with the weights $Q = 2I_3$ and $R = 1$. Then, for the model $(-A_1, B_1, C_1)$, we have

$$F_1 = \begin{bmatrix} 5.3149 & -0.1541 & 0.1733 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 2.3165 \\ -0.0533 \\ -0.1680 \end{bmatrix},$$
$$\sigma(-A_1 - B_1F_1) = \{ -0.8102, -1.9880, -5.5814 \},$$
$$\sigma(-A_1 - G_1C_1) = \{ -1.1507, -1.8157, -5.5804 \},$$

where $\sigma(-A_1 - B_1F_1)$ denotes the set of eigenvalues of the matrix $-A_1 - B_1F_1$. Similarly, for the model $(-A_1, B_1, \tilde{C}_1(A_1+\mu)^\gamma)$, we have

$$F_1 = \begin{bmatrix} 5.3149 & -0.1541 & 0.1733 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -4.3337 \\ 0.2356 \\ -0.5369 \end{bmatrix},$$
$$\sigma(-A_1 - B_1F_1) = \{ -0.8102, -1.9880, -5.5814 \},$$
$$\sigma(-A_1 - G_1\tilde{C}_1(A_1+\mu)^\gamma) = \{ -0.8075, -2.8760, -19.1124 \}.$$

For the both cases, we use the same notation G_n to indicate the transfer functions of approximate feedback control systems. Figures 1 and 2 show that the value of $\|G_n(\cdot)\|_\infty$ converges to some value less than 1 as n goes to infinity, which means from Theorem 2.1 that the control law (4) (resp. the control law (14)) works as a finite-dimensional stabilizing controller for system (18), (19) (resp. system (18), (20)), by making choice of such l, F_1, and G_1. Here, note that the convergence speed of the case with boundary observation (20) is late compared with that of the case with distributed observation (19).

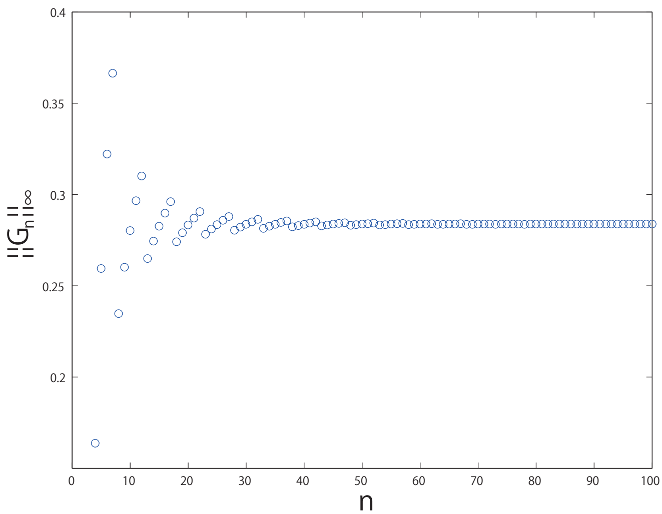

Fig. 1. The case of distributed observation (19). $\|G_{100}\|_\infty = 0.2838\ (< 1)$.

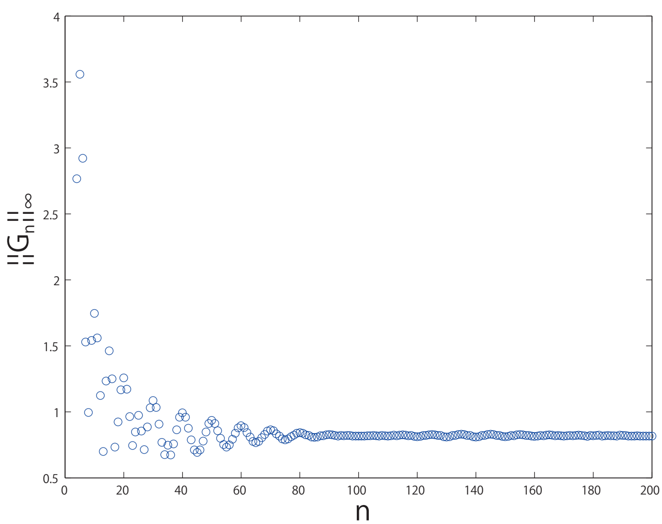

Fig. 2. The case of boundary observation (20). $\|G_{200}\|_\infty = 0.8160\ (< 1)$.

The difference is caused by the estimates (13) and (15). In the numerical simulation, we used MATLAB Control System Toolbox.

4. CONCLUSIONS

In this paper, in connection with the work of [5], we studied the problem of approximating stability radius appearing in the design of finite-dimensional stabilizing controllers for an infinite-dimensional dynamical system. From the computational point of view, we needed to prepare a family of approximate finite-dimensional operators and then to calculate the H_∞-norm of their transfer functions. Theorem 2.1 assures that they converge to the value of H_∞-norm of the original transfer function. In the future, we plan to study the similar problem for the case where the system operator is not expressed by a Riesz-spectral operator as well as for the case with input delay.

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Hideki Sano, Department of Applied Mathematics, Graduate School of System Informatics, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501. Japan.
e-mail: sano@crystal.kobe-u.ac.jp