

PROVING THE CHARACTERIZATION OF ARCHIMEDEAN COPULAS VIA DINI DERIVATIVES

JUAN FERNÁNDEZ-SÁNCHEZ AND MANUEL ÚBEDA-FLORES

In this note we prove the characterization of the class of Archimedean copulas by using Dini derivatives.

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1. INTRODUCTION

Copulas are n -dimensional distribution functions that concentrate the probability mass on $[0, 1]^n$ and whose univariate margins are uniformly distributed on $[0, 1]$. A (bivariate) *copula* is a function $C: [0, 1]^2 \rightarrow [0, 1]$ which satisfies:

- (C1) the boundary conditions $C(t, 0) = C(0, t) = 0$ and $C(t, 1) = C(1, t) = t$ for all $t \in [0, 1]$;
- (C2) the *2-increasing property*, i. e., $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$ for all u_1, u_2, v_1, v_2 in $[0, 1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

In particular, copulas are Lipschitz continuous functions in each variable with constant 1.

The importance of copulas comes from *Sklar's Theorem* [17], which shows that the joint distribution H of a pair of random variables and the corresponding marginal distributions F and G are linked by a copula C in the following manner: $H(x, y) = C(F(x), G(y))$ for all x, y in $[-\infty, \infty]$. If F and G are continuous, then the copula is unique; otherwise, the copula is uniquely determined on $\text{Range } F \times \text{Range } G$ [2]. For a complete review on copulas and some of their applications, we refer to [6, 9, 15].

Let $\varphi: [0, 1] \rightarrow [0, \infty]$ be a continuous strictly decreasing function such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the *pseudo-inverse* of φ , i. e., $\varphi^{[-1]}(x) = \varphi^{-1}(\min(\varphi(0), x))$ for $x \in [0, \infty]$, and consider the function given by

$$C_\varphi(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)), \quad (u, v) \in [0, 1]^2. \tag{1}$$

The following result provides a characterization of the function given by (1) to be a copula [16].

Theorem 1.1. The function C_φ given by (1) is a copula if, and only if, φ is convex.

Copulas given by (1) are called *Archimedean* – the name is due to a property of associative operations [10, 15] – and φ is the *generator* of C_φ – for another different characterization of Archimedean copulas, see [10]. Archimedean copulas became popular since they model the dependence structure between risk factors, and are used in many applications, such as finance, insurance, or reliability (see, for example, [4, 13]) due to their simple forms and nice properties.

In [18], the author provides three characterizations of n -dimensional Archimedean copulas: algebraic, differential and diagonal. Our purpose in this note is to provide a new proof of Theorem 1.1 (Section 3) by using Dini derivatives, a known result of Lebesgue from Real Analysis (Section 2) – which allow to reconstruct a function from the Dini derivative D^+f when this is finite – and a characterization of copulas given by Jaworski and Durante [5].

2. PRELIMINARY RESULTS FROM REAL ANALYSIS

Derived numbers play an important role in several results on the differentiability of monotone functions. We recall their definition [14].

Definition 2.1. The number λ (finite or infinite) is said to be a *derived number* of the function f at the point x_0 if there exists a sequence h_1, h_2, h_3, \dots ($h_n \neq 0$ for all n) such that $h_n \rightarrow 0$ and

$$\lim_{n \rightarrow \infty} \frac{f(x_0 + h_n) - f(x_0)}{h_n} = \lambda.$$

Symbolically, we say $\lambda = Df(x_0)$. If the (finite or infinite) derivative $f'(x_0)$ exists at the point x_0 , then it will be a derived number $Df(x_0)$, and in this case, the function f will have no other derived numbers at the point x_0 .

We note that in Definition 2.1 it is possible to use the term *derived number to the right* by imposing $h_n > 0$.

There are some particularly important derived numbers, the Dini derivatives, whose definition we recall now [11].

Definition 2.2. Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function, with $a < b$, and let x be a point in $[a, b[$. The limit

$$D^+f(x) = \limsup_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}$$

is called the (*rightside upper*) *Dini derivative* of f at x . When it is substituted \limsup by \liminf , we obtain the (*rightside lower*) *Dini derivative* D_+f .

The following result provides conditions for which a function can be recovered as a definite integral of one of its Dini derivatives [8].

Lemma 2.3. If f is a continuous function that has a finite Dini derivative $D^+f(x)$ at every point x of \mathbb{R} , then

$$f(b) - f(a) = \int_a^b D^+f(x) \, dx$$

for each interval $[a, b]$.

Observe that, as an immediate consequence of Lemma 2.3, we have that if $D^+f \equiv 0$ then f is constant.

We understand a *strictly increasing (decreasing) singular function* as a continuous and strictly increasing (decreasing) function with derivative zero almost everywhere. Since the Dini derivatives of a decreasing function cannot be positive, Lemma 2.3 implies that a strictly decreasing singular function on an interval has a dense set of points in which D^+f is equal to $-\infty$. Both Lemma 2.3 and these last observations remain true if we replace D^+f by D_+f . Furthermore, we have the following lemma [3].

Lemma 2.4. If f is a strictly singular function, then the inverse f^{-1} is also strictly singular.

3. A NEW PROOF OF THE CHARACTERIZATION OF ARCHIMEDEAN COPULAS

We begin this section with some additional notation. For every function $K: [x_1, x_2] \times [y_1, y_2] \rightarrow \mathbb{R}$ and every $y \in [y_1, y_2]$, let K_y denote the function from $[x_1, x_2]$ onto \mathbb{R} given by $K_y(x) = K(x, y)$.

The following result – whose proof can be found in [5] – provides a characterization of copulas in terms of Dini derivatives.

Lemma 3.1. A function $C: [0, 1]^2 \rightarrow [0, 1]$ is a copula if, and only if, C satisfies (C1) and the following conditions:

1. C is continuous;
2. there exists a countable set $S \subset [0, 1]$ such that, for every $u \in [0, 1] \setminus S$, the following conditions hold:
 - (a) $D^+C_v(u)$ is finite for every $v \in [0, 1]$;
 - (b) $D^+C_{v_1}(u) \leq D^+C_{v_2}(u)$ whenever $0 \leq v_1 \leq v_2 \leq 1$.

We are now in position to provide a new proof of Theorem 1.1 by using Dini derivatives and derived numbers – compare, for example, with the ones given in [1, 7, 12].

Proof of Theorem 1.1. Suppose C_φ is a copula given by (1). Since φ is monotone, we have that φ is derivable almost everywhere. Let $u \in]0, 1[$ be a point such that $\varphi'(u)$ exists and $\varphi'(u) \neq 0$, and suppose $(C_\varphi)_v(u) \neq 0$. By writing

$$\frac{(C_\varphi)_v(u+h) - (C_\varphi)_v(u)}{h} = \frac{(C_\varphi)_v(u+h) - (C_\varphi)_v(u)}{\varphi(u+h) - \varphi(u)} \cdot \frac{\varphi(u+h) - \varphi(u)}{h},$$

and by taking supremum limits when $h \rightarrow 0^+$ in both sides, the existence of the derivative of φ in u assures

$$D^+(C_\varphi)_v(u) = \lambda \cdot \varphi'(u),$$

where λ is the inverse of a derived number of φ at $(C_\varphi)_v(u)$. To be precise, with $\tilde{h} := (C_\varphi)_v(u+h) - (C_\varphi)_v(u)$, we get $\tilde{h} \rightarrow 0^+$ for $h \rightarrow 0^+$ - as C_φ is non-decreasing - and

$$\begin{aligned} \lambda &= \limsup_{h \rightarrow 0^+} \frac{(C_\varphi)_v(u+h) - (C_\varphi)_v(u)}{\varphi(u+h) - \varphi(u)} = \limsup_{h \rightarrow 0^+} \frac{(C_\varphi)_v(u+h) - (C_\varphi)_v(u)}{\varphi(u+h) + \varphi(v) - [\varphi(u) + \varphi(v)]} \\ &= \limsup_{h \rightarrow 0^+} \frac{(C_\varphi)_v(u+h) - (C_\varphi)_v(u)}{\varphi((C_\varphi)_v(u+h)) - \varphi((C_\varphi)_v(u))} \\ &= \limsup_{h \rightarrow 0^+} \frac{(C_\varphi)_v(u+h) - (C_\varphi)_v(u)}{\varphi((C_\varphi)_v(u) + (C_\varphi)_v(u+h) - (C_\varphi)_v(u)) - \varphi((C_\varphi)_v(u))} \\ &= \limsup_{\tilde{h} \rightarrow 0^+} \frac{\tilde{h}}{\varphi((C_\varphi)_v(u) + \tilde{h}) - \varphi((C_\varphi)_v(u))} \\ &= \frac{1}{\liminf_{\tilde{h} \rightarrow 0^+} \frac{\varphi((C_\varphi)_v(u) + \tilde{h}) - \varphi((C_\varphi)_v(u))}{\tilde{h}}} = \frac{1}{D_+\varphi((C_\varphi)_v(u))}. \end{aligned}$$

From Lemma 3.1, we have $D^+(C_\varphi)_{v_1}(u) \leq D^+(C_\varphi)_{v_2}(u)$ as long as $0 \leq v_1 \leq v_2 \leq 1$, which implies

$$\frac{\varphi'(u)}{D_+\varphi((C_\varphi)_{v_1}(u))} \leq \frac{\varphi'(u)}{D_+\varphi((C_\varphi)_{v_2}(u))},$$

and therefore $D_+\varphi((C_\varphi)_{v_1}(u)) \leq D_+\varphi((C_\varphi)_{v_2}(u))$ - since φ is strictly decreasing - i. e. $D_+\varphi$ is increasing in $]0, u[$.

We now prove that there exists a sequence $\{u_n\} \rightarrow 1$ as $n \rightarrow +\infty$ such that $\varphi'(u_n) \neq 0$ for every n . Suppose, on the contrary, this is not true, that is, we have $\varphi'(u) = 0$ almost everywhere in an interval $[a_0, 1] \subset [0, 1]$ and φ is not derivable in the rest of the points, i. e. φ is a strictly decreasing singular function. From Lemma 2.4, we have that φ^{-1} is a strictly decreasing singular function in $[0, \varphi(a_0)]$. In this case, there exists a set of real points $\{x_n : n \in \mathbb{N}\}$ such that $\{x_n\} \rightarrow 0$ as $n \rightarrow +\infty$ with $D_+(\varphi^{-1})(x_n) = -\infty$.

Now, let x be a real point such that $D_+(\varphi^{-1})(x) = -\infty$, and let u and v be two real points such that $\varphi(u) + \varphi(v) = x$, with u such that any derived number to the right of φ at u is different from 0 - we note that the existence of u and v is due to the continuity of φ and as a consequence of the fact that the derived numbers to the right cannot be greater than $D_+\varphi$.

Since $(C_\varphi)_v$ verifies the Lipschitz condition with constant 1, we have

$$\beta \cdot D_+\varphi^{-1}(\varphi(u) + \varphi(v)) \leq 1,$$

where β is a derived number to the right of φ at u . Since $D_+\varphi^{-1}(\varphi(u) + \varphi(v)) = -\infty$ and $\beta < 0$, that upper bound is not possible, so we obtain a contradiction; therefore, there exists a sequence $\{u_n\} \rightarrow 1$ as $n \rightarrow +\infty$ such that $\varphi'(u_n) \neq 0$ for every n .

All this reasoning leads to the fact that $D_+\varphi$ is non-decreasing in $]0, 1[$. Therefore, if s and s' are two numbers in $[0, 1]$ such that $s > s'$, from Lemma 2.3 we have

$$\begin{aligned} \frac{\varphi(s) + \varphi(s')}{2} - \varphi\left(\frac{s + s'}{2}\right) &= \frac{1}{2} \left[\varphi(s) - \varphi\left(\frac{s + s'}{2}\right) \right] - \frac{1}{2} \left[\varphi\left(\frac{s + s'}{2}\right) - \varphi(s') \right] \\ &= \frac{1}{2} \left(\int_{\frac{s+s'}{2}}^s D_+\varphi(t) dt - \int_{s'}^{\frac{s+s'}{2}} D_+\varphi(t) dt \right) \geq 0, \end{aligned}$$

and we conclude that φ is convex.

Conversely, we only need to follow the same steps backwards, which completes the proof. \square

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*Juan Fernández-Sánchez, Research Group of Mathematical Analysis, University of Almería, Carretera de Sacramento s/n, 04120 Almería. Spain.
e-mail: juanfernandez@ual.es*

*Manuel Úbeda-Flores, Department of Mathematics, University of Almería, Carretera de Sacramento s/n, 04120 Almería. Spain.
e-mail: mubeda@ual.es*