ON ADMISSIBILITY OF LINEAR ESTIMATORS IN MODELS WITH FINITELY GENERATED PARAMETER SPACE

Ewa Synówka-Bejenka and Stefan Zontek

The paper refers to the research on the characterization of admissible estimators initiated by Cohen [2]. In our paper it is proved that for linear models with finitely generated parameter space the limit of a sequence of the unique locally best linear estimators is admissible. This result is used to give a characterization of admissible linear estimators of fixed and random effects in a random linear model for spatially located sensors measuring intensity of a source of signals in discrete instants of time.

Keywords: linear model, linear estimation, linear prediction, admissibility, admissibility among an affine set, locally best estimator

Classification: 62F10, 62J10

1. INTRODUCTION

A characterization of admissible linear estimators in a general linear model is a rather complicated problem. An explicit characterization has been obtained only for special cases. This problem was solved by Cohen [2] for estimation of the mean vector in Gauss–Markov model with identity covariance matrix. His characterization is based on algebraic properties of matrices. Further generalizations have been done, among others, by Rao [16], Stępniak [20], Zontek [25], Klonecki and Zontek [11], Baksalary and Markiewicz [1], Groß and Markiewicz [6] and Stępniak [22]. Using other technique Olsen, Seely and Birkes [15] have described admissible unbiased quadratic estimators in two variance components model. Further generalizations have been done, among others, by Gnot and Kleffe [3].

LaMotte [12], inspired by the paper of Olsen et al. [15], elaborated a method of characterization of admissible linear estimators based on verification in a finite number of steps whether or not a linear estimator is locally best at a point belonging to properly extended parameter space. This method is general but its direct application is not easy.

Another way of investigations of admissibility of linear estimators used a connection between the closure of the set of the unique locally best estimators (ULBE) and the set of admissible linear estimators. The first set contains the second one. Using Bayes

DOI: 10.14736/kyb-2016-5-0724

approach this was proved by Stępniak [21] but his proof was not constructive. LaMotte [13] presented a construction of a sequence of unique locally best linear estimators that converged to the given admissible linear estimator. Similar result under some additional assumptions imposed on the model was obtained by Zontek [26]. These methods can not be applied directly when we are interested in admissible estimation of random and fixed effects.

The problem of the estimation of both effects was initiated by Henderson [8], who described the best linear unbiased prediction (BLUP) as being ,,joint maximum likelihood estimates". The simultaneous linear estimation of fixed and random effects was considered by, among others, Goldberger [5], Henderson [9], Harville [7] and Rao [17]. The long history of BLUP and its applications were been extensively described by Robinson [18]. Futher results on the derivations of BLUP were obtained, among others, by Jiang [10], Liu et al. [14] and Tian [24].

Synówka-Bejenka and Zontek [23] have shown that a problem of admissibility for a linear function of fixed and random effects could be restarted as a problem of admissibility for a linear function of the expected value only, in another properly defined linear model (called the dual model). Basing on LaMotte's results [12, 13] they have given in an explicit form a characterization of linear admissible estimators of a linear function of expected value in the models dual to balanced nested and crossed classification random models (see also Shiqing et al. [19]). For these models the parameter space is finitely generated. In this paper we shall show that for any model with finitely generated parameter space the class of all admissible linear estimators consists of all ULBE's and their limits. Our basic theoretical result will be used to characterize admissible simultaneous linear estimators of effects in a special random model frequently assumed for measurements of an intensity of a source of signals by a number of sensors.

Throughout this paper, $\mathcal{M}_{m\times q}$ denotes the space of $m\times q$ real matrices. The symbols A' and $\mathcal{R}(A)$ stand for the transpose and column space of $A\in\mathcal{M}_{m\times q}$, respectively. Denote the trace of a square matrix A by $\mathrm{tr}(A)$. For $A_1\in\mathcal{M}_{m_1\times q_1}$ and $A_2\in\mathcal{M}_{m_2\times q_2}$ the symbols $A_1\otimes A_2$ and $\mathrm{diag}(A_1,A_2)$ denote the Kronecker product and the matrix whose the diagonal consists of A_1 and A_2, respectively. The minimal closed convex cone containing a set $A\subset\mathcal{M}_{m\times m}\times\mathcal{M}_{m\times m}$ will be denoted by $[A]$. Let $\|a\|$ be the length of the vector $a\in\mathcal{R}^m$.

2. MODELS WITH FINITELY GENERATED PARAMETER SPACE

Let $Y\in\mathcal{R}^m$ be a random vector with an unknown distribution belonging to $\mathcal{P}$. It is assumed that the expected value $\mathrm{E}Y$ and the covariance $\mathrm{cov}(Y)$ exist for all distributions in $\mathcal{P}$. We are interested in an admissible estimation of $K'\mathrm{E}Y$, where $K\in\mathcal{M}_{m\times q}$, in the class of linear estimators $L'Y$, where L belongs to an affine subset $\mathcal{L}$ of $\mathcal{M}_{m\times q}$, under the quadratic risk function

$$\mathrm{E}[(L'Y-K'\mathrm{E}Y)'(L'Y-K'\mathrm{E}Y)]=\mathrm{tr}[L'\mathrm{cov}(Y)L+(L-K)'\mathrm{E}Y(\mathrm{E}Y)'(L-K)].$$

Following LaMotte [12], consider the set

$$\mathcal{T}=\{(\mathrm{cov}(Y),\mathrm{E}Y(\mathrm{E}Y)'):P\in\mathcal{P}\}\tag{1}$$

as a new space of parameters and a point $(W_1, W_2) \in [\mathcal{T}]$ as an argument of an extended quadratic risk function of $L'Y$, i. e.,

$$\mathrm{R}(L'Y; (W_1, W_2)) = \mathrm{tr}[L'W_1L + (L-K)'W_2(L-K)].$$

An estimator $L'Y$ with $L \in \mathcal{L}$ is called locally best among $\mathcal{L}$ at a point $(W_1, W_2) \in [\mathcal{T}]$ if $\mathrm{R}(L'Y; (W_1, W_2)) \leq \mathrm{R}(N'Y; (W_1, W_2))$ for all $N \in \mathcal{L}$. If $\mathcal{L}$ is an affine subset of $\mathcal{M}_{m\times q}$ then exist $L_o \in \mathcal{L}$ and $\Pi \in \mathcal{M}_{m\times m}$ such that

$$\mathcal{L} = \{L_o + \Pi M : M \in \mathcal{M}_{m\times q}\}.$$

It can be shown (see Theorem 3.1 in LaMotte [12]) that an estimator $L'Y = (L_o + \Pi M)'Y$ is locally best among $\mathcal{L}$ at a point $(W_1, W_2) \in [\mathcal{T}]$ iff

$$\Pi'(W_1 + W_2)L = \Pi'W_2K$$

or equivalently iff

$$\Pi'(W_1 + W_2)\Pi M = \Pi'W_2(K - L_o) - \Pi'W_1L_o.$$

The above equation has exactly one solution iff

$$\mathcal{R}(\Pi'(W_1 + W_2)\Pi) = \mathcal{R}(\Pi').$$

In this section we assume that $[\mathcal{T}]$ is a finitely generated closed convex cone, i. e.,

$$[\mathcal{T}] = \left\{ \sum_{i=0}^{k+1} t_i(W_{1i}, W_{2i}) : t_o \geq 0, \dots, t_{k+1} \geq 0 \right\}, \tag{2}$$

where W_{1i} and W_{2i}, $i = 0, \dots, k+1$, are nonnegative matrices. To avoid some trivialities we also assume that

$$\mathcal{R}\left(\sum_{i=0}^{k+1} \Pi'(W_{1i} + W_{2i})\Pi \right) = \mathcal{R}(\Pi') \tag{3}$$

and that

$$\Pi'(W_{1i} + W_{2i})\Pi \neq 0 \ \text{ for } \ i = 0, \dots, k+1.$$

Note that under condition (3) the set of unique locally best estimators is nonempty. The proof of the main result of this paper is based on the following lemma.

Lemma 2.1. If $L^{*\prime}Y$ is the limit of a sequence of ULBE among $\mathcal{L}$, then for any

$$\mathcal{L}_\Lambda = \{L^* + \Pi\Lambda M : M \in \mathcal{M}_{m\times q}\} \subseteq \mathcal{L},$$

where $\Lambda \in \mathcal{M}_{m\times m}$, there exists a point $W_\Lambda = (W_1, W_2) \in [\mathcal{T}]$ such that

$$\Lambda'\Pi'(W_1 + W_2)\Pi\Lambda \neq 0 \tag{4}$$

and that $L^{*\prime}Y$ is locally best among $\mathcal{L}_\Lambda$ at W_Λ.

The proof of this lemma is included in Appendix.

Theorem 2.2. If $L^{*\prime}Y$ is the limit of a sequence of ULBE among $\mathcal{L}$ in a model with finitely generated parameter space satisfying condition (3), then $L^{*\prime}Y$ is admissible for $K'EY$ among $\mathcal{L}$.

Proof. The step-wise procedure elaborated by LaMotte [12] can be successfuly applied, since by Lemma 2.1 a point satisfying (4) is nontrivial in each step. □

Remark 2.3. Another proof of this theorem can be given by showing that a linear model with parameter space of the form (2) is regular in the sense defined in Zontek [26].

Note that using Theorem 2.2 we get all admissible linear estimators of $K'\mathrm{E}Y$ among $\mathcal{L}$. This is due to the fact that every ULBE is admissible among $\mathcal{L}$ (see LaMotte [12]) and that all ULBE's and their limits constitute a complete class (see Stępniak [21] and LaMotte [13]).

3. APPLICATIONS TO RANDOM LINEAR MODELS

Let Y be a random n-vector having the following structure

$$Y = Z_o\beta + Z_1u_1 + \ldots + Z_ku_k + e,$$

where $Z_o \in \mathcal{R}^n$ is a known vector (usually the vector of ones); $\beta \in \mathcal{R}$ is the unknown parameter; $Z_1 \in \mathcal{M}_{n\times m_1}, \ldots, Z_k \in \mathcal{M}_{n\times m_k}$ are known nonzero matrices; $u_1 \in \mathcal{R}^{m_1}, \ldots, u_k \in \mathcal{R}^{m_k}$ are unobservable random vectors and e is a random n - vector of errors. We assume that $u_1, \ldots, u_k$ and e are uncorrelated vectors with zero expectations and covariance matrices of the form $\mathrm{cov}(u_1) = \sigma_1^2 I_{m_1}, \ldots, \mathrm{cov}(u_k) = \sigma_k^2 I_{m_k}, \mathrm{cov}(e) = \sigma_{k+1}^2 I_n$, respectively.
Clearly,

$$\mathrm{E}(Y) = Z_o\beta \ \text{ and } \ \mathrm{cov}(Y) = \sum_{i=1}^{k} \sigma_i^2 Z_i Z_i' + \sigma_{k+1}^2 I_n.$$

This will be schematically written as

$$Y \sim (Z_o\beta, \sum_{i=1}^{k+1} \sigma_i^2 Z_i Z_i'), \tag{5}$$

where $Z_{k+1} = I_n$.

We are interested in an admissible estimation of

$$\theta = [(K'Z_o\beta)', (Q_1'Z_1u_1)', \ldots, (Q_k'Z_ku_k)']' \tag{6}$$

in the class of linear estimators

$$L'Y = (L_0, L_1, \ldots, L_k)'Y, \tag{7}$$

where $K, L_0 \in \mathcal{M}_{n\times t_0}$; $Q_1, L_1 \in \mathcal{M}_{n\times t_1}$; … ; $Q_k, L_k \in \mathcal{M}_{n\times t_k}$. To compare the estimators, we use the ordinary quadratic risk function

$$\mathrm{E}\left[(L'Y-\theta)'(L'Y-\theta)\right].$$

Since random effects are also estimated, the risk function has a different structure than the risk function of linear estimator of fixed effects only. To give a characterization of linear admissible estimators of θ, Synówka-Bejenka and Zontek [23] reduced the problem to linear estimation of the fixed effects only in another properly defined dual model. We briefly recall this result.

As a model dual to (5) they have considered the model

$$\boldsymbol{Y} = (Y', (Z_1u_1)', \ldots, (Z_ku_k)')' \sim \left(\boldsymbol{X}\beta, \sum_{i=1}^{k+1}\sigma_i^2\boldsymbol{V}_i\right), \tag{8}$$

where

$$\boldsymbol{X} = (Z_o', \mathbf{0}, \ldots, \mathbf{0})',$$
$$\boldsymbol{V}_i = (v_1+v_{i+1})(v_1+v_{i+1})' \otimes Z_iZ_i', \quad i=1,\ldots,k,$$

and

$$\boldsymbol{V}_{k+1} = v_1v_1' \otimes I_n,$$

while v_i is the ith versor in $\mathcal{R}^{k+1}$. Note that for

$$\boldsymbol{L}'\boldsymbol{Y} = \begin{bmatrix} L_0 & L_1 & \cdots & L_k \\ \mathbf{0} & -Q_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & -Q_k \end{bmatrix}' \begin{bmatrix} Y \\ Z_1u_1 \\ \vdots \\ Z_ku_k \end{bmatrix} \tag{9}$$

considered as an estimator of

$$\boldsymbol{K}'\mathrm{E}\boldsymbol{Y} = \begin{bmatrix} K & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \begin{bmatrix} Z_o\beta \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

we have

$$L'Y-\theta = \boldsymbol{L}'\boldsymbol{Y} - \boldsymbol{K}'\mathrm{E}\boldsymbol{Y}.$$

Hence the quadratic risk of $L'Y$ considered as an estimator of θ in the original model is equal to the quadratic risk function of $\boldsymbol{L}'\boldsymbol{Y}$ considered as an estimator of $\boldsymbol{K}'\mathrm{E}\boldsymbol{Y}$ in the dual model. Note that each estimator $L'Y$ of the form (7) defines exactly one estimator of the form (9) and therefore the class of considered linear estimators of $\boldsymbol{K}'\mathrm{E}\boldsymbol{Y}$ in model (8) is restricted to the set

$$\boldsymbol{\mathcal{E}}_o = \{\boldsymbol{L}'\boldsymbol{Y} : \boldsymbol{L} \in \mathcal{L}_o\},$$

where $\mathcal{L}_o$ is an affine set given by

$$\mathcal{L}_o = \{\boldsymbol{L}_o + \boldsymbol{\Pi}_o\boldsymbol{M} : \boldsymbol{M} \in \mathcal{M}_{(k+1)n\times(t_0+\ldots+t_k)}\},$$

while $\boldsymbol{L}_o = \operatorname{diag}(\mathbf{0}, -Q_1, \ldots, -Q_k)$ and $\boldsymbol{\Pi}_o = v_1 v_1' \otimes I_n$. This means that a linear estimator $L'Y$ of θ is admissible in the model (5) if and only if the corresponding estimator $\boldsymbol{L}'\boldsymbol{Y}$ of $\boldsymbol{K}'\mathrm{E}\boldsymbol{Y}$ is admissible among $\mathcal{L}_o$ in model (8). Note that the parameter space given by (1) corresponding to the dual model (8) is a finitely generated closed convex cone defined by

$$
\begin{aligned}
(W_{1o}, W_{2o}) &= (\mathbf{0}, \boldsymbol{X}\boldsymbol{X}'), \\
(W_{1i}, W_{2i}) &= (\boldsymbol{V}_i, \mathbf{0}), \quad i = 1, \ldots, k+1.
\end{aligned}
$$

Using the rule of duality, Synówka-Bejenka and Zontek [23] obtained a characterization of linear admissible estimators of a linear function of fixed and random effects in the k-way balanced nested classification random model and the k-way balanced crossed classification random model. To prove that in the considered models each limit of ULBE's is admissible, they applied a step-wise procedure of LaMotte [12]. By Theorem 2.2, we do not need to describe in details all steps of LaMotte's procedure. It is enough to present formulas on ULBE's in the form for which their limits can be characterized. In the next section we illustrate this approach to a special model which was used by Gnot et al. [4] for measurement descriptions provided by several sensors.

4. EXAMPLE

Let us consider the following model

$$y_j = v k_j + e_j, \quad j = 1, \ldots, n_2,$$

where v denotes the intesity of the source of random signal, k_j denotes the influence of the source on the jth sensor (known positive constant) and e_j is the random error. We assume that $v \sim N(\beta, \sigma_1^2)$ and $e_j \sim N(0, \sigma_2^2)$ are independent random variables. Under these assumptions the vector $Y_i = (y_{i1}, \ldots, y_{in_2})'$, where y_{ij} is the ith measurement provided by the jth sensor, has the normal distribution with the expectation and the covariance matrix having the form

$$\mathrm{E}(\mathrm{Y_i}) = \beta k, \;\; \operatorname{cov}(\mathrm{Y_i}) = \sigma_1^2 k k' + \sigma_2^2 I_{n_2}, \quad i = 1, \ldots, n_1.$$

Therefore, to describe the n_1 independent measurements provided by each of the sensors we can use

$$Y = (Y_1', \ldots, Y_{n_1}')', \tag{10}$$

which is a special case of model (5). Note that Y has an multivariate normal distribution with the following parameters

$$\mathrm{E}(Y) = (\mathbf{1}_{n_1} \otimes k)\beta = Z_o \beta,$$

$$\operatorname{cov}(Y) = \sigma_1^2 (I_{n_1} \otimes kk') + \sigma_2^2 I_n = \sigma_1^2 Z_1 Z_1' + \sigma_2^2 Z_2 Z_2',$$

where $n = n_1 n_2$. For $k = \mathbf{1}_{n_2}$ model (10) reduces to so the called one-way balanced random model.

Following Synówka-Bejenka and Zontek [23], to obtain explicit formulas for ULBE in model (8) corresponding to model (10), we define the following matrices

$$\begin{aligned} E_0 &= \frac{1}{p_0} Z_0 Z_0', \\ E_1 &= \frac{1}{p_1} Z_1 Z_1' - E_0, \\ E_2 &= \frac{1}{p_2} Z_2 Z_2' - (E_0 + E_1), \end{aligned}$$

where $p_o = n_1 k'k$, $p_1 = k'k$ and $p_2 = 1$. Note that E_o, E_1, E_2 are idempotent and orthogonal matrices such that

$$Z_i Z_i' = p_i \sum_{j=0}^{i} E_j \quad \text{for} \quad i = 0, 1, 2. \tag{11}$$

To characterize admissible estimators $\boldsymbol{L}'\boldsymbol{Y}$ of $\boldsymbol{K}'\mathrm{E}\boldsymbol{Y}$ among $\mathcal{L}_o$ in model (8) corresponding to model (10) we give the following lemma, which is proved in Appendix.

Lemma 4.1. An estimator $\boldsymbol{L}'\boldsymbol{Y}$ is ULBE at a point $\left(s_1\boldsymbol{V}_1 + s_2\boldsymbol{V}_2, s_0\boldsymbol{X}\boldsymbol{X}'\right)$ in $\mathcal{T}$ among $\mathcal{L}_o$ in model (8) corresponding to model (10) if and only if $s_0 \geq 0$, $s_1 \geq 0$, $s_2 > 0$ and

$$\boldsymbol{L} = \begin{bmatrix} L_0 & L_1 \\ \mathbf{0} & -Q_1 \end{bmatrix},$$

where

$$\begin{aligned} L_0 &= \frac{s_0 p_0}{s_0 p_0 + s_1 p_1 + s_2 p_2} E_0 K, \\ L_1 &= \frac{s_1 p_1}{s_1 p_1 + s_2 p_2} \left(E_1 + \frac{s_1 p_1 + s_2 p_2}{s_0 p_0 + s_1 p_1 + s_2 p_2} E_0 \right) Q_1. \end{aligned}$$

Theorem 4.2. An estimator $\boldsymbol{L}'\boldsymbol{Y}$ of $\boldsymbol{K}'\mathrm{E}\boldsymbol{Y}$ is admissible among $\mathcal{L}_o$ in model (8) corresponding to model (10) if and only if $\boldsymbol{L}$ belongs to the set

$$\left\{ \begin{bmatrix} a_0 E_0 K & a_1[E_1 + (1-a_0)E_0]Q_1 \\ \mathbf{0} & -Q_1 \end{bmatrix} : a_0 \in [0,1] \text{ and } a_1 \in [0,1] \right\}. \tag{12}$$

P r o o f. The necessary condition. Let $\boldsymbol{L}$ belong to the set (12) with $a_i \in [0,1)$ for $i = 0, 1$. Using Lemma 4.1 it can be checked that $\boldsymbol{L}'\boldsymbol{Y}$ is ULBE at point $W = \left(s_1\boldsymbol{V}_1 + s_2\boldsymbol{V}_2, s_0\boldsymbol{X}\boldsymbol{X}'\right)$ in $\mathcal{T}$ given by

$$\begin{aligned} s_0 &= \frac{a_0}{p_0(1-a_0)(1-a_1)} s_2, \\ s_1 &= \frac{a_1}{p_1(1-a_1)} s_2, \\ s_2 &> 0. \end{aligned}$$

Moreover, note that for any fixed values $a_1 \in [0,1)$ and $s_2 > 0$, s_0 runs over $[0,+\infty)$ when $a_0 \in [0,1)$. Similary, s_1 runs over $[0,+\infty)$ when $a_1 \in [0,1)$. So the set (12) is the closure of

$$\{\boldsymbol{L} : \boldsymbol{L}'\boldsymbol{Y} \text{ is ULBE at a point in } \mathcal{T} \text{ among } \mathcal{L}_o\}.$$

So the first part of the proof is completed by using the result of LaMotte [13] that each linear estimator of $\boldsymbol{K}'\mathrm{E}\boldsymbol{Y}$ admissible among $\mathcal{L}_o$ is the limit of a sequance of ULBE's at points in $\mathcal{T}$ among $\mathcal{L}_o$.

Sufficiency follows straightforwardly from Theorem 2.2. □

Remark 4.3. For $a_0 = 1$ an admissible estimator $\boldsymbol{L}'\boldsymbol{Y}$ is unbiased for $\boldsymbol{K}'\mathrm{E}\boldsymbol{Y}$ in model (8) corresponding to model (10). Hence, for the original model (10), the estimator of the form $(E_0K, a_1E_1Q_1)'Y$ is unbiased for $[(K'Z_o\beta)', (Q_1'Z_1u_1)']'$. So, under the assumption that $a_1 = \frac{\sigma_1^2 k'k}{\sigma_1^2 k'k+\sigma_2^2} \in [0,1)$ is known, the estimator

$$\left[(K'Z_0\widehat{\beta})', \left(a_1\frac{1}{k'k}Q_1'Z_1Z_1'(Y - Z_0\widehat{\beta})\right)'\right]'$$

with $\widehat{\beta} = \frac{1}{n_1 k'k}Z_0'Y = \frac{1}{n_1 k'k}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} k_j y_{ij}$ is BLUP. When $a_1 \notin [0,1]$ this estimator is still unbiased but inadmissible.

5. APPENDIX

P r o o f of Lemma 2.1.

Let $W_1(t) = \sum_{i=0}^{k+1} t_i W_{1i}$ and let $W_2(t) = \sum_{i=0}^{k+1} t_i W_{2i}$ for $t = (t_0, \ldots, t_{k+1})' \in \mathcal{R}_{\geq}^{k+1}$. Using this notation the set $[\mathcal{T}]$ can be written as

$$[\mathcal{T}] = \{(W_1(t), W_2(t)) : t \in \mathcal{R}_{\geq}^{k+1}\}.$$

Let

$$\mathcal{F} = \{t \in \mathcal{R}_{\geq}^{k+1} : \mathcal{R}(\Pi'(W_1(t) + W_2(t))\Pi) = \mathcal{R}(\Pi')\}.$$

For $t \in \mathcal{F}$ let $L(t)$ be a matrix in $\mathcal{L}_o$ such that $[L(t)]'Y$ is locally best at $(W_1(t), W_2(t))$. Of course $L(t)$ is uniquely defined.

Let $t^{(n)} = (t_0^{(n)}, \ldots, t_{k+1}^{(n)})' \in \mathcal{F}, n = 1, 2, \ldots$ be a sequence such that

$$L^* = \lim_{n\to\infty} L(t^{(n)}).$$

Define $t_\Lambda^{(n)} = (t_{\Lambda 0}^{(n)}, \ldots, t_{\Lambda k+1}^{(n)})'$ by

$$t_{\Lambda i}^{(n)} = \begin{cases} 0, & \text{when } \Lambda'\Pi'(W_{1i} + W_{2i})\Pi\Lambda = 0, \\ t_i^{(n)}, & \text{when } \Lambda'\Pi'(W_{1i} + W_{2i})\Pi\Lambda \neq 0, \end{cases}$$

and denote by

$$l_\Lambda = \lim_{n\to\infty} \frac{1}{||t_\Lambda^{(n)}||} t_\Lambda^{(n)}.$$

Of course $W_\Lambda = (W_1(l_\Lambda), W_2(l_\Lambda)) \in [\mathcal{T}]$ satisfies condition (4).

Since $[L(t^{(n)})]'Y$ is the unique locally best estimator at $(W_1(t^{(n)}), W_2(t^{(n)}))$, then

$$\Pi'[W_1(t^{(n)}) + W_2(t^{(n)})]L(t^{(n)}) = \Pi' W_2(t^{(n)})K.$$

Hence we have

$$\frac{1}{||t_\Lambda^{(n)}||}\Lambda'\Pi'[W_1(t_\Lambda^{(n)}) + W_2(t_\Lambda^{(n)})]L(t^{(n)}) = \frac{1}{||t_\Lambda^{(n)}||}\Lambda'\Pi' W_2(t_\Lambda^{(n)})K.$$

From this equality for $n \to \infty$ we get

$$\Lambda'\Pi'[W_1(l_\Lambda) + W_2(l_\Lambda)]L^* = \Lambda'\Pi' W_2(l_\Lambda)K.$$

This means that $(L^*)'Y$ is locally best at W_Λ. This finishes the proof. □

Proof of Lemma 4.1.

An estimator $\boldsymbol{L}'\boldsymbol{Y}$ is locally best at $\left(s_1\boldsymbol{V}_1 + s_2\boldsymbol{V}_2, s_0\boldsymbol{X}\boldsymbol{X}'\right)$ in $\mathcal{T}$ among $\mathcal{L}_o$ iff $s_j \geq 0$ for $j = 0, 1, 2$ and

$$\boldsymbol{\Pi}_o\left(s_1\boldsymbol{V}_1 + s_2\boldsymbol{V}_2 + s_0\boldsymbol{X}\boldsymbol{X}'\right)\boldsymbol{L} = s_0\boldsymbol{\Pi}_o\boldsymbol{X}\boldsymbol{X}'\boldsymbol{K}.$$

In more details this equation can be written as

$$\begin{aligned}\left(\sum_{i=0}^{2} s_i Z_i Z_i'\right) L_0 &= s_0 Z_0 Z_0' K,\\ \left(\sum_{i=0}^{2} s_i Z_i Z_i'\right) L_1 &= s_1 Z_1 Z_1' Q_1.\end{aligned}$$

Of course, the above equations have only one solution, with respect to L_0 and L_1 iff the matrix $\sum_{i=0}^{2} s_i Z_i Z_i'$ is nonsingular, that is iff $s_2 > 0$. The assertion follows from (11). □

(Received July 8, 2015)

REFERENCES

[1] J. K. Baksalary and A. Markiewicz: Admissible linear estimators in the general Gauss–Markov model. J. Statist. Plann. Inference *19* (1988), 349–359. DOI:10.1016/0378-3758(88)90042-0

[2] A. Cohen: All admissible estimates of the mean vector. Ann. Math. Statist. *37* (1966), 458–463. DOI:10.1214/aoms/1177699528

[3] S. Gnot and J. Kleffe: Quadratic estimation in mixed linear models with two variance components. J. Statist. Plann. Inference *8* (1983), 267–279. DOI:10.1016/0378-3758(83)90045-9

[4] S. Gnot, E. Rafajłowicz, and A. Urbańska-Motyka: Statistical inference in linear model for spatially located sensors and random inputs. Ann. Inst. Statist Math. *53* (2001), 370–379. DOI:10.1023/a:1012426923971

[5] A. S. Goldberger: Best linear unbiased prediction in the generalized linear regression model. J. Amer. Statist. Assoc. *57* (1962), 369–375. DOI:10.1080/01621459.1962.10480665

[6] J. Groß and A. Markiewicz: Characterization of admissible linear estimators in the linear model. Linear Algebra Appl. *388* (2004), 239–248. DOI:10.1016/s0024-3795(03)00459-2

[7] D. A. Harville: Extension of the Gauss-Markov theorem to include the estimation of random effects. Ann. Statist. *2* (1976), 384–395. DOI:10.1214/aos/1176343414

[8] C. R. Henderson: Estimation of genetic parameters (abstract). Ann. Math. Statist. *21* (1950), 309–310.

[9] C. R. Henderson: Selection index and expected genetic advance. In: Statistical Genetics and Plant Breeding (W. D. Hanson and H. F. Robinson, eds.), NAS-NRC 982, Washington 1963, pp. 141–163.

[10] J. Jiang: A derivation of BLUP-Best linear unbiased predictor. Statist. Probab. Lett. *32* (1997), 321–324. DOI:10.1016/s0167-7152(96)00089-2

[11] W. Klonecki and S. Zontek: On the structure of admissible linear estimators. J. Multivariate Anal. *24* (1988), 11–30. DOI:10.1016/0047-259x(88)90098-x

[12] L. R. LaMotte: Admissibility in linear estimation. Ann. Statist. *10* (1982), 245–255. DOI:10.1214/aos/1176345707

[13] L. R. LaMotte: On limits of uniquely best linear estimators. Metrika *45* (1997), 197–211. DOI:10.1007/bf02717103

[14] X. Q. Liu, J. Y. Rong, and X. Y. Liu: Best linear unbiased prediction for linear combinations in general mixed linear models. J. Multivariate Anal. *99* (2008), 1503–1517. DOI:10.1016/j.jmva.2008.01.004

[15] A. Olsen, J. Seely, and D. Birkes: Invariant quadratic unbiased estimation for two variance components. Ann. Statist. *4* (1976), 878–890. DOI:10.1214/aos/1176343586

[16] C. R. Rao: Estimation of parameters in a linear model. Ann. Statist. *4* (1976), 1023–1037. DOI:10.1214/aos/1176343639

[17] C. R. Rao: Estimation in linear models with mixed effects: a unified theory. In: Proc. Second International Tampere Conference in Statistics (T. Pukkila and S. Puntanen, eds.), Dept. of Mathematical Sciences, Univ. of Tampere, Tampere 1987, pp. 73–98.

[18] G. K. Robinson: That BLUP is a good thing-the estimation of random effects. Statist. Sci. *6* (1991), 15–51. DOI:10.1214/ss/1177011933

[19] W. Shiqing, M. Ying, and F. Zhijun: Integral expression form of admissible linear estimators of effects in linear mixed models. In: Proc. 2010 International Conference on Computing, Control and Industrial Engineering, IEEE, Wuhan 2010, pp. 56–60. DOI:10.1109/ccie.2010.133

[20] C. Stępniak: On admissible estimators in a linear model. Biometrical J. *26* (1984), 815–816. DOI:10.1002/bimj.4710260725

[21] C. Stępniak: A complete class for linear estimation in a general linear model. Ann. Inst. Statist. Math. A *39* (1987), 563–573. DOI:10.1007/bf02491490

[22] C. Stępniak: Admissible invariant esimators in a linear model. Kybernetika *50* (2014), 310–321. DOI:10.14736/kyb-2014-3-0310

[23] E. Synówka-Bejenka and S. Zontek: A characterization of admissible linear estimators of fixed and random effects in linear models. Metrika *68* (2008), 157–172. DOI:10.1007/s00184-007-0149-0

[24] Y. Tian: A new derivation of BLUPs under random-effects model. Metrika *78* (2015), 905–918. DOI:10.1007/s00184-015-0533-0

[25] S. Zontek: On characterization of linear admissible estimators: an extension of a result due to C. R. Rao. J. Multivariate Anal. *23* (1987), 1–12. DOI:10.1016/0047-259x(87)90174-6

[26] S. Zontek: Admissibility of limits of the unique locally best linear estimators with application to variance components models. Probab. Math. Statist. *9* (1988), 29–44.

Ewa Synówka-Bejenka, Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, ul. prof. Z. Szafrana 4 a, 65-516 Zielona Góra. Poland.
e-mail: e.synowka@wmie.uz.zgora.pl

Stefan Zontek, Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, ul. prof. Z. Szafrana 4 a, 65-516 Zielona Góra. Poland.
e-mail: s.zontek@wmie.uz.zgora.pl