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GROUP SYNCHRONIZATION OF DIFFUSIVELY COUPLED HARMONIC OSCILLATORS

Liyun Zhao, Jun Liu, Lan Xiang and Jin Zhou

This paper considers group synchronization issue of diffusively directed coupled harmonic oscillators for two cases with nonidentical and identical agent dynamics. For the case of coupled nonidentical harmonic oscillators with positive coupling, it is demonstrated that distributed group synchronization can always be achieved under two kinds of network structures, i. e., the strongly connected graph and the acyclic partition topology with a directed spanning tree. It is interesting to find that the group synchronization states under acyclic partition are some periodic orbits with the same frequency and are simply related with the initial values of certain group regardless of ones of the other groups. For the case of coupled identical harmonic oscillators with positive and negative coupling, some generic algebraic criteria on group synchronization with both local continuous and instantaneous interaction are established respectively. In particular, an explicit expression of group synchronization states in terms of initial values of the agents can be obtained by the property of acyclic partition topology, and so it is very convenient to yield the desired group synchronization in practical application. Finally, numerical examples illustrate and visualize the effectiveness and feasibility of theoretical results.

Keywords: group synchronization, coupled harmonic oscillators, directed topology, acyclic partition

Classification: 74H65, 70K40

1. INTRODUCTION

In recent years, consensus or synchronization of networked multi-agent systems has received increasing attention due to its broad applications in a variety of fields including distributed computation, sensor networks, and coordination control, etc. [2, 7, 8, 11]. In particular, remarkable effort has been devoted to the synchronization and control of coupled harmonic oscillators [1, 3, 15, 18, 19, 27, 29], in part because it plays a significant role in various engineering applications of coupled multi-agent systems involving cooperative behaviors including mapping, sampling, patrol or surveillance [1, 15, 18]. As a consequence, a large quantity of synchronization protocols (or algorithms) have recently been proposed for coupled harmonic oscillators from various perspectives. For example, Ren investigated the synchronization problem of n coupled harmonic oscillators in a continuous-time setting, and showed that only if the directed graph has a spanning tree,

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all the states of coupled harmonic oscillators can synchronize to a periodic oscillatory motion [15]. Later on, Ballard et al. developed a discrete-time distributed algorithm for coupled harmonic oscillators [1]. Moreover, Su et al. addressed the same issue in a dynamic proximity network without any network connectivity assumption [18]. In addition, Cheng et al. [3] focused on the infinite-time and finite-time synchronization of coupled harmonic oscillators with the external disturbance. More recently, Zhou et al. also studied the impulsive or sampled-data synchronization of coupled harmonic oscillators based on some discontinuous or hybrid control schemes [27, 29].

All the above-mentioned works focused mainly on complete synchronization of coupled harmonic oscillators, i. e., all of agent dynamics finally converge to the same trajectory. However, when a complex cooperative task is completely implemented, a network of agents must be able to sense and respond to unexpected situations or any changes. This might result in stating that a network of agents evolve into several groups, i. e., all agents in the same group reach complete synchronization, but the motions of different groups may not coincide. This kind of synchronization is usually known as group or cluster synchronization, which may be viewed as an extended synchronization problem containing complete synchronization as a special case. Accordingly, group synchronization is more suitable to deal with cooperative control in complex multi-agent systems in practice. For instance, in the formation flight of Unmanned Air Vehicles (UAV), the designed cooperative scheme is effectively implemented to divide a large set of UAV into multiple groups such that all the agents in different groups display the corresponding synchronization patterns [14, 25]. As a result, group or cluster synchronization problem for different kinds of multi-agent systems has recently been a rather significant topic in both theoretical research and practical applications. It is reported that, in general, two designed strategies have been effectively employed to realize group or cluster synchronization of networked multi-agent systems modelled by first-order or second-order integrator dynamics [12, 14, 20, 21, 22, 25]. The first strategy is to aim at nonidentical agent dynamics in different groups with positive couplings [14, 19, 21, 25]. The other is to focus on identical agent dynamics with positive and negative couplings among the groups [12, 20, 22]. However, so far, there has been very little work to fully address group synchronization of coupled harmonic oscillators with directed interaction topology. These observations motivated the research work reported in the present article.

Given the above comments, in this paper, we investigate group synchronization problem of diffusively directed coupled harmonic oscillators for two cases. For the case of coupled nonidentical harmonic oscillators with positive coupling, we demonstrate that distributed group synchronization can always be guaranteed by two kinds of network structures, i. e., a strongly connected graph and an acyclic partition topology with a directed spanning tree. It is interesting to find that the group synchronization states under acyclic partition are some periodic orbits with the same frequency and are simply related with the initial values of certain group regardless of ones of the other groups. For the case of coupled identical harmonic oscillators with positive and negative coupling, we present some generic algebraic criteria on group synchronization of coupled harmonic oscillators with local continuous and instantaneous interaction, respectively. In particular, an explicit expression of group synchronization states in terms of initial values of the agents can be obtained. By the property of acyclic partition topology, it is very convenient to yield the desired group synchronization in practical application. Finally, numerical simulations are given to demonstrate the effectiveness of theoretical results.

The outline of this paper is as follows. Some preliminaries and problem formulation are presented in Section 2. Different group synchronization conditions under two control strategies are discussed in Sections 3 and 4 in detail, respectively. Application examples and their simulations are presented in Section 5. Finally, the concluding remarks are drawn in Section 6.

2. PRELIMINARIES AND PROBLEM STATEMENT

2.1. Notation

Throughout this paper, the following notations are used: R, $\mathbb{N} = \{1, 2, \ldots\}$, R^n and $R^{n\times n}$ stand for the set of real numbers, the set of positive integers, the set of n dimension column vectors and the set of $n \times n$ real matrices, respectively. $\mathbf{i}$ is imaginary unit. The subscript "T" stands for matrix transposition. $\mathbf{0}$ and $\mathbf{1}$ are the suitable vectors with all entries being 0 and 1, respectively. The identity matrix and zero matrix with appropriate dimensions are denoted as I and O, respectively. $\mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \in R^{n\times n}$ is the diagonal matrix with diagonal entries γ_i $(i = 1, 2, \ldots, n)$. $\mathrm{diag}(M_1, M_2, \ldots, M_n)$ denotes a block diagonal matrix whose diagonal blocks are given by $M_1, M_2, \ldots, M_n$. " $\otimes$ " represents the Kronecker product of two matrices. Unless otherwise specified, the dimension of matrixes are assumed to be compatible for algebraic operations in the sequel.

2.2. Graph theory

Assume n harmonic oscillators with a graph denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, 2, \ldots, n\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, $A = (a_{ij})_{n\times n}$ represent the set of nodes, the set of edges and the weighted adjacency matrix, respectively. A directed edge (i, j) means the node j receives direct information from the node i. $a_{ij} \neq 0$ if and only if there is a directed edge (j, i) in $\mathcal{G}$; otherwise $a_{ij} = 0$. Directed graph $\mathcal{G}$ is said to have a directed spanning tree if there exists a node k so that the node k has a directed path to any other node of the graph. The graph $\mathcal{G}$ is called as strongly connected if there exists a directed path in any two different nodes. The elements of the Laplacian matrix $L = (l_{ij}) \in R^{n\times n}$ associated with graph $\mathcal{G}$ are defined as: $l_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$, where $i \neq j$ [8, 13]. Obviously, this condition guarantees that the inter-agent couplings are diffusive, and hence such networks are also called diffusively coupled networks [5, 16, 22]. We say that $\{P_1, P_2, \ldots, P_q\}$ is a partition of the set $\mathcal{V} = \{1, 2, \ldots, n\}$, if, for any $1 \leq i, j \leq q$, $P_i \neq \emptyset, P_i \bigcap P_j = \emptyset (i \neq j)$, and $\bigcup_{i=1}^{q} P_i = \mathcal{V}$. Let $\hat{i}$ denote the subscript of the subset to which the node i belongs, i. e., $i \in P_{\hat{i}}$, clearly, $\hat{i} = 1, \ldots, q$. Assume $P_i = \{\sum_{j=0}^{i-1} n_j + 1, \sum_{j=0}^{i-1} n_j + 2, \ldots, \sum_{j=0}^{i} n_j\}$ with $n_0 = 0, \sum_{k=1}^{q} n_k = n$, then the Laplacian matrix L can be rewritten as follows:

$$\begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1q} \\ L_{21} & L_{22} & \cdots & L_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ L_{q1} & L_{q2} & \cdots & L_{qq} \end{pmatrix} \tag{1}$$

with $L_{ij} \in R^{n_i \times n_j} (1 \leq i, j \leq q)$. The sequent discussion is always based on the assumption that the row sum of L_{ij} is a constant r_{ij}. Besides, $\mathcal{G}_1, \ldots, \mathcal{G}_q$ denote the underlying graphs of node set $P_1, \ldots, P_q$ in $\mathcal{G}$, respectively.

2.3. Problem formulation

Now we consider a team of n agents moving in a one-dimensional Euclidean space, the equation of motion for each agent can be represented by a harmonic oscillator of the form

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = -\alpha_{\hat{i}}\, x_i(t) + u_i(t), \quad i = 1, 2, \ldots, n, \end{cases} \tag{2}$$

where $x_i(t) \in R$ is the position of the agent i at time t and $v_i(t) \in R$ is its corresponding velocity, $u_i(t)$ is the corresponding control input, and $\sqrt{\alpha_{\hat{i}}}$ is the frequency of oscillators in the $\hat{i}$th group. Generally, the definition of group synchronization is presented as follows:

Definition 2.1. n diffusively coupled harmonic oscillators (2) are said to achieve group synchronization with the partition $\{P_1, P_2, \ldots, P_q\}$ asymptotically, if the states of agents satisfy

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0, \quad \lim_{t\to\infty} |v_i(t) - v_j(t)| = 0, \quad \text{when} \quad \hat{i} = \hat{j}$$

for any initial values of all the states of (2).

Remark 2.2. Note that the condition that every row sum of the Laplacian matrix L is zero guarantees that the inter-agent couplings are diffusive. In addition, the concept of group synchronization in Definition 2.1 is equivalent to the one of group consensus of multi-agent system in [26], where the motions of some agents in different groups may coincide as time goes to infinity. Clearly, Definition 2.1 is a little weaker than the concept of general cluster synchronization defined in [14].

In this paper, we are mainly interested in group synchronization problem for diffusively coupled harmonic oscillators (2) with directed network topology in the sense of Definition 2.1. To do so, we propose the control input for the agent i described as

$$u_i(t) = -\sum_{j=1}^{n} l_{ij} v_j(t). \tag{3}$$

Using the control protocol (3), the dynamics of diffusively coupled harmonic oscillators (2) can be written as:

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = -\alpha_{\hat{i}}\, x_i(t) - \sum\limits_{j=1}^{n} l_{ij} v_j(t), \quad i = 1,\, 2,\, \dots,\, n. \end{cases} \tag{4}$$

3. GROUP SYNCHRONIZATION OF NONIDENTICAL AGENTS WITH POSITIVE COUPLINGS

In this section, we shall give the group synchronization analysis of coupled nonidentical harmonic oscillators with positive couplings, namely, where it is also assumed that the frequency $\sqrt{\alpha_{\hat{i}}}$ of agents in different groups are nonidentical, and the elements a_{ij} of adjacent matrix are nonnegative.

3.1. Strongly connected topology case

Theorem 3.1. Suppose that the directed topology graph $\mathcal{G}$ is strongly connected, then the system (2) with the control input (3) can solve group synchronization problem asymptotically.

Proof. Since $\mathcal{G}$ is strongly connected, $\mathcal{G}$ has a positive left eigenvector $\xi = (\xi_1, \xi_2, \dots, \xi_n)^{\mathrm{T}}$ associated with zero eigenvalue [24]. Define the average of position and velocity in the $\hat{i}$th group as

$$\bar{x}^{(\hat{i})}(t) = \frac{1}{\sum\limits_{k \in P_{\hat{i}}} \xi_k} \sum_{k \in P_{\hat{i}}} \xi_k x_k(t) \quad \text{and} \quad \bar{v}^{(\hat{i})}(t) = \frac{1}{\sum\limits_{k \in P_{\hat{i}}} \xi_k} \sum_{k \in P_{\hat{i}}} \xi_k v_k(t),$$

respectively. Let $\tilde{x}_i(t) = x_i(t) - \bar{x}^{(\hat{i})}(t)$ and $\tilde{v}_i(t) = v_i(t) - \bar{v}^{(\hat{i})}(t)$ be the position and velocity error of agent i, respectively. It follows that

$$\begin{aligned} \sum_{l \in P_{\hat{i}}} \xi_l \tilde{v}_l(t) &= \sum_{l \in P_{\hat{i}}} \xi_l \big[v_l(t) - \frac{1}{\sum\limits_{k \in P_{\hat{l}}} \xi_k} \sum_{k \in P_{\hat{l}}} \xi_k x_k(t) \big] \\ &= \sum_{l \in P_{\hat{i}}} \xi_l v_l(t) - \big(\sum_{l \in P_{\hat{i}}} \xi_l \frac{1}{\sum\limits_{k \in P_{\hat{l}}} \xi_k} \big) \sum_{k \in P_{\hat{l}}} \xi_k x_k(t) = 0. \end{aligned} \tag{5}$$

Note that $P_{\hat{i}} = P_{\hat{l}}$, it then follows that

$$\sum_{l \in P_{\hat{i}}} \xi_l \tilde{v}_l(t) \big[\frac{1}{\sum\limits_{k \in P_{\hat{l}}} \xi_k} \sum_{k \in P_{\hat{l}}} \xi_k \sum_{j=1}^{n} l_{kj} v_j(t) \big] = 0. \tag{6}$$

Choose Lyapunov functional candidate as

$$V(t) = \sum_{\hat{i}=1}^{q} V_{\hat{i}}(t) = \frac{1}{2} \sum_{\hat{i}=1}^{q} \sum_{l \in P_{\hat{i}}} \xi_l (\alpha_{\hat{i}} \tilde{x}_l^2(t) + \tilde{v}_l^2(t))$$

where $V_{\hat{i}}(t) = \frac{1}{2}\sum_{l\in P_{\hat{i}}} \xi_l(\alpha_{\hat{i}}\tilde{x}_l^2(t) + \tilde{v}_l^2(t))$.

Thus, in view of (6), the derivative of $V_{\hat{i}}$ with respect to time is

$$\begin{aligned}\dot{V}_{\hat{i}}(t) &= -\sum_{l\in P_{\hat{i}}} \xi_l\tilde{v}_l(t)\Big(\sum_{j=1}^{n} l_{lj}v_j(t) - \frac{1}{\sum_{k\in P_{\hat{l}}}\xi_k}\sum_{k\in P_{\hat{l}}}\xi_k\sum_{j=1}^{n} l_{kj}v_j(t)\Big)\\ &= -\sum_{l\in P_{\hat{i}}} \xi_l\tilde{v}_l(t)\Big(\sum_{j=1}^{n} l_{lj}v_j(t)\Big).\end{aligned}$$

Let $\tilde{x}(t) = (\tilde{v}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t))^{\mathrm{T}}$, $\tilde{v}(t) = (\tilde{v}_1(t), \tilde{v}_2(t), \ldots, \tilde{v}_n(t))^{\mathrm{T}}$. Then, the derivative of V with respect to time gives

$$\begin{aligned}\dot{V}(t) &= -\sum_{\hat{i}=1}^{q}\Big[\sum_{l\in P_{\hat{i}}}\xi_l\tilde{v}_l(t)\Big[\sum_{j=1}^{n} l_{lj}(v_j(t) - \bar{v}^{(\hat{j})}(t)) + \sum_{j=1}^{n} l_{lj}\bar{v}^{(\hat{j})}(t)\Big]\Big]\\ &= -\sum_{\hat{i}=1}^{q}\Big[\sum_{l\in P_{\hat{i}}}\xi_l\tilde{v}_l(t)\Big[\sum_{j=1}^{n} l_{lj}(v_j(t) - \bar{v}^{(\hat{j})}(t))\Big]\Big]\\ &= -\sum_{l=1}^{n}\sum_{j=1}^{n}\xi_l l_{lj}\tilde{v}_l(t)\tilde{v}_j(t)\\ &= -\tfrac{1}{2}\tilde{v}^{\mathrm{T}}(t)(\Xi L + L^{\mathrm{T}}\Xi)\tilde{v}(t)\end{aligned}$$

where $\Xi = \mathrm{diag}(\xi_1, \ldots, \xi_n)$.

Based on the well-known LaSalle's invariance principle [17], we can conclude that all of the solutions of system (4), starting from any initial value, approach the largest invariant set $\tilde{M} = \{(\tilde{x}(t)^{\mathrm{T}}, \tilde{v}(t)^{\mathrm{T}})^{\mathrm{T}} | \dot{V}(t) = 0\}$. It is obvious that $\dot{V}(t) = 0$ if and only if $\tilde{v}_1(t) = \tilde{v}_2(t) = \cdots = \tilde{v}_n(t)$ [9, 24]. As a consequence, it is straight to derive that $v_i(t) = v_j(t)$ for $\hat{i} = \hat{j}$ from the definition of $\tilde{v}_i(t), \tilde{v}_j(t)$. So this completes the proof of Theorem 3.1. □

3.2. Acyclic partition topology case

It can be seen from Theorem 3.1 that the structure of strong connected topology can make diffusively coupled harmonic oscillators with the partition $\{P_1, P_2, \ldots, P_q\}$ to achieve distributed group synchronization. The conclusion is also applicable for directed network topology $\mathcal{G}$ with acyclic partition $\{P_1, P_2, \ldots, P_q\}$. In this case, by relabeling the indices of $\mathcal{G}$, the Laplacian matrix associated with $\mathcal{G}$ can take the form as

$$\begin{pmatrix} L_{11} & O & \cdots & O\\ L_{21} & L_{22} & \cdots & O\\ \vdots & \vdots & \ddots & \vdots\\ L_{q1} & L_{q2} & \cdots & L_{qq}\end{pmatrix}. \tag{7}$$

Without loss of generality, we still assume that the ith group P_i corresponds to the diagonal block L_{ii}.

Theorem 3.2. Assume that the directed topology graph $\mathcal{G}$ with acyclic partition has a spanning tree, then the system (2) with the control input (3) can solve group synchronization problem asymptotically. Furthermore, all coupled harmonic oscillators in different groups always synchronize to some specific periodic motion with the same frequency, which are only related with the initial values of the first group.

P r o o f. The proof procedure of Theorem 3.2 is mainly based on the structure of the Laplacian matrix with acyclic partition $\{P_1, P_2, \ldots, P_q\}$, which will be summarized as follows:

As for the first group P_1, it is obvious that the graph $\mathcal{G}_1$ associated with P_1 has a directed spanning tree. Following Theorem 3.1 in [15], we can conclude that all the harmonic oscillators in the first group P_1 will asymptotically converge to the synchronization state $(\bar{x}^{(1)}(t), \bar{v}^{(1)}(t))^{\mathrm{T}}$ given explicitly by

$$\begin{pmatrix} \bar{x}^{(1)}(t) \\ \bar{v}^{(1)}(t) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{\alpha_{\hat{1}}}t)\bar{\mathbf{p}}_1^{\mathrm{T}} x_0^{(1)} + \frac{1}{\sqrt{\alpha_{\hat{1}}}} \sin(\sqrt{\alpha_{\hat{1}}}t)\bar{\mathbf{p}}_1^{\mathrm{T}} v_0^{(1)} \\ -\sqrt{\alpha_{\hat{1}}} \sin(\sqrt{\alpha_{\hat{1}}}t)\bar{\mathbf{p}}_1^{\mathrm{T}} x_0^{(1)} + \cos(\sqrt{\alpha_{\hat{1}}}t)\bar{\mathbf{p}}_1^{\mathrm{T}} v_0^{(1)} \end{pmatrix}, \tag{8}$$

where $\bar{\mathbf{p}}_1$ is left eigenvector of L_{11} associated with the eigenvalue zero, and $(x_0^{(1)\mathrm{T}}, v_0^{(1)\mathrm{T}})^{\mathrm{T}} = (x_1(0), \ldots, x_{n_1}(0), v_1(0), \ldots, v_{n_1}(0))^{\mathrm{T}}$ is the initial value of all the states in the first group P_1.

Next, for the second group P_2, it is easy to see from (7) that the dynamics of all the states of coupled harmonic oscillators in the second group P_2 can be written as

$$\begin{cases} \dot{x}_{n_1+1:n_1+n_2}(t) = v_{n_1+1:n_1+n_2}(t), \\ \dot{v}_{n_1+1:n_1+n_2}(t) = -\alpha_{\hat{2}}\, x_{n_1+1:n_1+n_2}(t) - L_{22} v_{n_1+1:n_1+n_2}(t) - L_{21} v_{1:n_1}(t), \end{cases} \tag{9}$$

where $x_{n_1+1:n_1+n_2}(t), v_{n_1+1:n_1+n_2}(t)$ are the column stack vectors of $x_i(t)$ and $v_i(t) (i = n_1+1, \ldots, n_1+n_2)$, respectively, and $v_{1:n_1}(t)$ is the column stack vector of $v_i(t) (i = 1, \ldots, n_1)$, respectively. Accordingly, it is easy to verify that the synchronized state $(\bar{x}^{(2)}(t), \bar{v}^{(2)}(t))^{\mathrm{T}}$ in the second group P_2 must satisfy

$$\begin{cases} \dot{\bar{x}}^{(2)}(t) = \bar{v}^{(2)}(t), \\ \dot{\bar{v}}^{(2)}(t) = -\alpha_{\hat{2}}\, \bar{x}^{(2)}(t) - r_{22}\bar{v}^{(2)}(t) - r_{21}\bar{v}^{(1)}(t). \end{cases} \tag{10}$$

By introducing the synchronization errors $\tilde{v}_{1:n_1}(t) = v_{1:n_1}(t) - \mathbf{1}_{n_1}\bar{v}^{(1)}(t)$, $\tilde{x}_{n_1+1:n_1+n_2}(t) = x_{n_1+1:n_1+n_2}(t) - \mathbf{1}_{n_2}\bar{x}^{(2)}(t)$ and $\tilde{v}_{n_1+1:n_1+n_2}(t) = v_{n_1+1:n_1+n_2}(t) - \mathbf{1}_{n_2}\bar{v}^{(2)}(t)$, we obtain the synchronization error system

$$\begin{cases} \dot{\tilde{x}}_{n_1+1:n_1+n_2}(t) = \tilde{v}_{n_1+1:n_1+n_2}(t), \\ \dot{\tilde{v}}_{n_1+1:n_1+n_2}(t) = -\alpha_{\hat{2}}\, \tilde{x}_{n_1+1:n_1+n_2}(t) - L_{22}\tilde{v}_{n_1+1:n_1+n_2}(t) - L_{21}\tilde{v}_{1:n_1}(t), \end{cases} \tag{11}$$

which can be written in matrix form as

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}}_{n_1+1:n_1+n_2}(t) \\ \dot{\tilde{v}}_{n_1+1:n_1+n_2}(t) \end{pmatrix} &= \begin{pmatrix} O & I_{n_2} \\ -\alpha_{\hat{2}} I_{n_2} & -L_{22} \end{pmatrix} \begin{pmatrix} \tilde{x}_{n_1+1:n_1+n_2}(t) \\ \tilde{v}_{n_1+1:n_1+n_2}(t) \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathbf{0} \\ -L_{21}\tilde{v}_{1:n_1}(t) \end{pmatrix}. \end{aligned} \tag{12}$$

Note that the eigenvalues of the matrix $\begin{pmatrix} O & I_{n_2} \\ -\alpha_{\hat{2}} I_{n_2} & -L_{22} \end{pmatrix}$ are given by $\mu_i = \frac{-\lambda_i \pm \sqrt{\lambda_i^2 - 4\alpha_{\hat{2}}}}{2}$ with λ_i being the ith eigenvalue of L_{22}. Thus, it is easy to see that λ_i has the positive real part since $\mathcal{G}$ has a directed spanning tree, which implies that the eigenvalues μ_i have the negative real part. Based on the stability theory of linear dynamical systems, it is easy to know that the zero solution of the synchronization error system (12) is asymptotically stable, which leads to $(\tilde{x}_{n_1+1:n_1+n_2}(t)^{\mathrm{T}}, \tilde{v}_{n_1+1:n_1+n_2}(t)^{\mathrm{T}})^{\mathrm{T}} \to \mathbf{0}$ as $t \to \infty$. It then follows that all the states of the agents in the second group will asymptotically converge to $(\bar{x}^{(2)}(t), \bar{v}^{(2)}(t))^{\mathrm{T}}$.

Analogously, for the qth group P_q, and by using the same arguments as above, it is not hard to prove that all the states of agents in the qth group will asymptotically converge to $(\bar{x}^{(q)}(t), \bar{v}^{(q)}(t))^{\mathrm{T}}$, where $(\bar{x}^{(q)}(t), \bar{v}^{(q)}(t))^{\mathrm{T}}$ is just the solution of the following system

$$\begin{cases} \dot{\bar{x}}^{(q)}(t) = \bar{v}^{(q)}(t), \\ \dot{\bar{v}}^{(q)}(t) = -\alpha_{\hat{q}}\, \bar{x}^{(q)}(t) - r_{qq}\bar{v}^{(q)}(t) - r_{q1}\bar{v}^{(1)}(t) - r_{q2}\bar{v}^{(2)}(t) - \cdots - r_{q,q-1}\bar{v}^{(q-1)}(t). \end{cases} \tag{13}$$

Finally, we shall show that the synchronized oscillatory motion in different group is actually a specific periodic orbit with the same frequency $\sqrt{\alpha_{\hat{1}}}$. In fact, it follows from the synchronized state equation (10) with respect to the second group P_2 that

$$\ddot{\bar{x}}^{(2)}(t) + r_{22}\dot{\bar{x}}^{(2)}(t) + \alpha_{\hat{2}}\, \bar{x}^{(2)}(t) = -r_{21}\bar{v}^{(1)}(t). \tag{14}$$

According to the feature of harmonically excited vibration, it is easy to observe that the state response of (14) is actually a simple harmonic oscillation with the frequency $\sqrt{\alpha_{\hat{1}}}$, It is obvious that this frequency is just one of the forced exciting force $-r_{21}\bar{v}^{(1)}(t)$.

In what follows, we assume that the above conclusions still hold for the 3th, ..., $(q-1)$th groups, respectively. Since the synthesis of harmonic vibration with the same frequency is still a harmonic vibration with this frequency, the specific term $-r_{q1}\bar{v}^{(1)}(t) - r_{q2}\bar{v}^{(2)}(t) - \cdots - r_{q,q-1}\bar{v}^{(q-1)}(t)$ in (13) is also an exciting force with the frequency $\sqrt{\alpha_{\hat{1}}}$. Therefore, by repeating the previous inductions, it is straight to conclude from (10) that the synchronized oscillatory motion in the qth group is also a periodic orbit with the frequency $\sqrt{\alpha_{\hat{1}}}$. Actually, it is easy to see that its corresponding amplitude and initial phase is only related to the initial values of the first group, but not the others. Consequently, this immediately implies that the dynamics of synchronized periodic orbit is only related to the initial values of the first group. In summary, this completes the proof of Theorem 3.2. □

Remark 3.3. The results of Theorems 3.1 and 3.2 show that, coupled nonidentical harmonic oscillators with positive coupling can always achieve distributed group synchronization oscillatory motion over two kinks of network structures, i. e., the strongly connected graph and the acyclic partition topology with a directed spanning tree. It is interesting to find that coupled nonidentical harmonic oscillator under acyclic partition can synchronize to a specific periodic motion. Furthermore, the final synchronized periodic orbit is only related to the initial values of the first group, but not the others. This point will be further elaborated in numerical simulations section.

4. GROUP SYNCHRONIZATION OF IDENTICAL AGENTS WITH POSITIVE AND NEGATIVE COUPLINGS

In this section, we consider the group synchronization problem of coupled identical harmonic oscillators with positive and negative couplings. In this case, all the frequencies $\sqrt{\alpha}$ of coupled harmonic oscillators are identical, and we further assume that each row sum of $L_{ij}(1 \le i,j \le q)$ is zero. It is easy to observe from the structure of the Laplacian matrix L that it has at least q zero eigenvalues. Furthermore, let $\mathbf{q}_1 = (\mathbf{1}_{n_1}^{\mathrm{T}}, \mathbf{0}_{n_2}^{\mathrm{T}}, \ldots, \mathbf{0}_{n_q}^{\mathrm{T}})^{\mathrm{T}}$, $\mathbf{q}_2 = (\mathbf{0}_{n_1}^{\mathrm{T}}, \mathbf{1}_{n_2}^{\mathrm{T}}, \ldots, \mathbf{0}_{n_q}^{\mathrm{T}})^{\mathrm{T}}$, $\ldots$, $\mathbf{q}_q = (\mathbf{0}_{n_1}^{\mathrm{T}}, \mathbf{0}_{n_2}^{\mathrm{T}}, \ldots, \mathbf{1}_{n_q}^{\mathrm{T}})^{\mathrm{T}}$ be the q linearly independent right eigenvectors of L associated with zero eigenvalue respectively, and the corresponding q linearly independent left eigenvectors of L associated with zero eigenvalue can be taken as

$$\begin{aligned} \mathbf{p}_1 &= (\theta_1^{\mathrm{T}}, \varphi_1^{(2)\mathrm{T}}, \ldots, \varphi_1^{(q)\mathrm{T}})^{\mathrm{T}}, \\ \mathbf{p}_2 &= (\varphi_2^{(1)\mathrm{T}}, \theta_2^{\mathrm{T}}, \ldots, \varphi_2^{(q)\mathrm{T}})^{\mathrm{T}}, \\ &\vdots \\ \mathbf{p}_q &= (\varphi_q^{(1)\mathrm{T}}, \varphi_q^{(2)\mathrm{T}}, \ldots, \varphi_q^{(q-1)\mathrm{T}}, \theta_q^{\mathrm{T}})^{\mathrm{T}}, \end{aligned} \tag{15}$$

satisfying $\theta_k^{\mathrm{T}}\mathbf{1}_{n_k} = 1 (k = 1, \ldots, q)$ and $\varphi_j^{(i)\mathrm{T}}\mathbf{1}_{n_i} = 0 (i \ne j)$ for each of $j = 1, \ldots, q$.

4.1. Continuous coupled case

Theorem 4.1. If L has exactly q zero eigenvalues and all the other eigenvalues have positive real parts, then the system (2) with the control input (3) can solve group synchronization problem asymptotically, and the synchronization state of the lth group can be explicitly expressed as

$$\begin{pmatrix} \bar{x}^{(l)}(t) \\ \bar{v}^{(l)}(t) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{\alpha}t)\mathbf{p}_l^{\mathrm{T}} x_0 + \frac{1}{\sqrt{\alpha}} \sin(\sqrt{\alpha}t)\mathbf{p}_l^{\mathrm{T}} v_0 \\ -\sqrt{\alpha} \sin(\sqrt{\alpha}t)\mathbf{p}_l^{\mathrm{T}} x_0 + \cos(\sqrt{\alpha}t)\mathbf{p}_l^{\mathrm{T}} v_0 \end{pmatrix}, \tag{16}$$

with the initial value $(x_0^{\mathrm{T}}, v_0^{\mathrm{T}})^{\mathrm{T}} = (x_1(0), \ldots, x_n(0), v_1(0), \ldots, v_n(0))^{\mathrm{T}}$.

Proof. Let $x(t)$, $v(t)$ be the column stack vectors of $x_i(t)$, $v_i(t)$ for $i = 1, \ldots, n$, respectively, then the system (2) with the control input (3) can be rewritten in a compact matrix form as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} O & I_n \\ -\alpha I_n & -L \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}. \tag{17}$$

Similar to the proof of Theorem 3.1 in [15], it can be obtained that the matrix $\begin{pmatrix} O & I_n \\ -\alpha I_n & -L \end{pmatrix}$ have the $2q$ eigenvalues $\pm\sqrt{\alpha}\mathbf{i}$, whose corresponding right and left eigenvectors are $\check{\mathbf{q}}_{i\pm} = (\mathbf{q}_i^{\mathrm{T}}, \pm\sqrt{\alpha}\mathbf{i}\mathbf{q}_i^{\mathrm{T}})^{\mathrm{T}}$, $\check{\mathbf{p}}_{i\pm} = (\frac{1}{2}\mathbf{p}_i^{\mathrm{T}}, \pm\frac{1}{2\sqrt{\alpha}\mathbf{i}}\mathbf{p}_i^{\mathrm{T}})^{\mathrm{T}} (i = 1, \ldots, q)$, respectively, and the other eigenvalues have the negative real parts.

By some elementary operations, we obtain the Jordan decomposition form below

$$\begin{pmatrix} O & I_n \\ -\alpha I_n & -L \end{pmatrix} = \check{\mathbf{Q}} \underline{J} \check{\mathbf{Q}}^{-1}, \tag{18}$$

where $\check{\mathbf{Q}} = (\check{\mathbf{q}}_{1+}, \check{\mathbf{q}}_{1-}, \ldots, \check{\mathbf{q}}_{q+}, \check{\mathbf{q}}_{q-}, \check{\mathbf{q}}_{2q+1}, \ldots, \check{\mathbf{q}}_{2n})$, $\check{\mathbf{Q}}^{-1} = (\check{\mathbf{p}}_{1+}, \check{\mathbf{p}}_{1-}, \ldots, \check{\mathbf{p}}_{q+}, \check{\mathbf{p}}_{q-}, \check{\mathbf{p}}_{2q+1}, \ldots, \check{\mathbf{p}}_{2n})$ with $\check{\mathbf{q}}_{2q+1}, \ldots, \check{\mathbf{p}}_{2n}$ and $\check{\mathbf{p}}_{2q+1}, \ldots, \check{\mathbf{p}}_{2n}$ being the other right and left eigenvectors or generalized eigenvectors, respectively, and the matrix

$$\underline{J} = \begin{pmatrix} \sqrt{\alpha}\mathbf{i} & O & \cdots & O & O & O \\ O & -\sqrt{\alpha}\mathbf{i} & & O & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & \cdots & \sqrt{\alpha}\mathbf{i} & O & O \\ O & O & \cdots & O & -\sqrt{\alpha}\mathbf{i} & O \\ O & O & \cdots & O & O & \underline{J}^* \end{pmatrix}$$

is the Jordan upper diagonal block matrix, and the diagonal entries of $\underline{J}^*$ are the other eigenvalues with the negative real parts of the matrix $\begin{pmatrix} O & I_n \\ -\alpha I_n & -L \end{pmatrix}$.

By the use of general linear differential equation theory, it is followed that

$$\begin{aligned} & \lim_{t\to\infty} \exp\left\{ \begin{pmatrix} O & I_n \\ -\alpha I_n & -L \end{pmatrix} t \right\} \\ = \quad & \sum_{i=1}^{q} \left(\exp(\sqrt{\alpha}\mathbf{i}t)\check{\mathbf{q}}_{i+}\check{\mathbf{p}}_{i+}^{\mathrm{T}} + \exp(-\sqrt{\alpha}\mathbf{i}t)\check{\mathbf{q}}_{i-}\check{\mathbf{p}}_{i-}^{\mathrm{T}} \right) \\ = \quad & \sum_{i=1}^{q} \begin{pmatrix} \cos(\sqrt{\alpha}t)\mathbf{q}_i\mathbf{p}_i^{\mathrm{T}} & \frac{1}{\sqrt{\alpha}}\sin(\sqrt{\alpha}t)\mathbf{q}_i\mathbf{p}_i^{\mathrm{T}} \\ -\sqrt{\alpha}\sin(\sqrt{\alpha}t)\mathbf{q}_i\mathbf{p}_i^{\mathrm{T}} & \cos(\sqrt{\alpha}t)\mathbf{q}_i\mathbf{p}_i^{\mathrm{T}} \end{pmatrix}, \end{aligned} \tag{19}$$

which yields that

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} \to \sum_{i=1}^{q} \begin{pmatrix} \cos(\sqrt{\alpha}t)\mathbf{q}_i\mathbf{p}_i^{\mathrm{T}}x_0 + \frac{1}{\sqrt{\alpha}}\sin(\sqrt{\alpha}t)\mathbf{q}_i\mathbf{p}_i^{\mathrm{T}}v_0 \\ -\sqrt{\alpha}\sin(\sqrt{\alpha}t)\mathbf{q}_i\mathbf{p}_i^{\mathrm{T}}x_0 + \cos(\sqrt{\alpha}t)\mathbf{q}_i\mathbf{p}_i^{\mathrm{T}}v_0 \end{pmatrix}.$$

As a consequence, it is straightforward to show that all the states $(x_l(t),\, v_l(t)\,)^{\mathrm{T}}$ in the lth group will converge asymptotically to the synchronization state $(x^{(l)}(t),\, v^{(l)}(t)\,)^{\mathrm{T}}$. So the proof of Theorem 4.1 is completed. □

Specifically, if the network topology $\mathcal{G}$ has a directed acyclic partition $\{P_1, P_2, \ldots, P_q\}$, and the corresponding subgraph $\mathcal{G}_i$ with respect to P_i has a directed spanning tree $(i = 1, \ldots, q)$, it is easy to check that the conditions of Theorem 4.1 are satisfied. In this case, the q linearly independent left eigenvectors of L associated with zero eigenvalue can be taken as

$$\begin{aligned} \omega_1 &= (\theta_1^{\mathrm{T}}, \mathbf{0}_{n-n_1}^{\mathrm{T}})^{\mathrm{T}}, \\ \omega_2 &= (\varphi_2^{(1)\mathrm{T}}, \theta_2^{\mathrm{T}}, \mathbf{0}_{n-n_1-n_2}^{\mathrm{T}})^{\mathrm{T}}, \\ &\vdots \\ \omega_q &= (\varphi_q^{(1)\mathrm{T}}, \varphi_q^{(2)\mathrm{T}}, \ldots, \varphi_q^{(q-1)\mathrm{T}}, \theta_q^{\mathrm{T}})^{\mathrm{T}}, \end{aligned}$$

with $\theta_k^{\mathrm{T}}\mathbf{1}_{n_k} = 1(k = 1, \ldots, q)$ and $\varphi_j^{(i)\mathrm{T}}\mathbf{1}_{n_i} = 0(j = 2, \ldots, q; i < j)$.

By Theorem 4.1 and Lemma 2 in [10], we can obtains a simple and practical criterion on group synchronization presented by the following corollary.

Corollary 4.2. For the directed graph $\mathcal{G}$ with the acyclic partition $\{P_1, P_2, \dots, P_q\}$, if each of the subgraph $\mathcal{G}_i$ with respect to P_i has a directed spanning tree, then the system (2) with the control input (3) can solve group synchronization problem asymptotically, and the synchronization state of the lth group can be explicitly given by

$$\begin{pmatrix} \bar{x}^{(l)}(t) \\ \bar{v}^{(l)}(t) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{\alpha}t)\omega_l^{\mathrm{T}} x_0 + \frac{1}{\sqrt{\alpha}} \sin(\sqrt{\alpha}t)\omega_l^{\mathrm{T}} v_0 \\ -\sqrt{\alpha}\sin(\sqrt{\alpha}t)\omega_l^{\mathrm{T}} x_0 + \cos(\sqrt{\alpha}t)\omega_l^{\mathrm{T}} v_0 \end{pmatrix}. \tag{20}$$

4.2. Instantaneous coupled case

For the system (1), we propose the following impulsive control input

$$u_i(t) = -\mu \sum_{k=1}^{\infty} \sum_{j=1}^{n} l_{ij} v_j(t) \delta(t - t_k), \tag{21}$$

where $i = 1, 2, \dots, n$, $\mu > 0$ stands for coupling strength, and $\delta(t)$ is the Dirac impulsive function depicted the effects of instantaneous interaction among oscillators only at certain moments, and the impulsive time sequence $\{t_k\} (k \in \mathbb{N})$ is strictly increasing sequence, i. e. $t_1 < \dots < t_k < \dots$ (see [6] and relevant references therein). For simplicity, we assume $t_0 = 0$, and $t_k - t_{k-1} \equiv h > 0$ (h is a constant sampling period).

Based on the property of the Dirac function, the system (2) with the control input (21) can be formulated as the following impulsive dynamical system

$$\begin{cases} \dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = -\alpha x_i(t), \quad t \neq t_k, \\ \triangle x_i(t_k) = 0, \\ \triangle v_i(t_k) = -\mu \sum_{j=1}^{n} l_{ij} v_j(t_k), \quad t = t_k, \end{cases} \tag{22}$$

where $\triangle x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ and $\triangle v_i(t_k) = v_i(t_k^+) - v_i(t_k^-)$, respectively. Without loss of generality, we further assume that $x_i(t)$ and $v_i(t)$ are left continuous at time moments $t = t_k$. Thus, the impulsive dynamical system (22) can be rewritten as

$$\begin{cases} \begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} O & I_n \\ -\alpha I_n & O \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}, \quad t \neq t_k, \\ \begin{pmatrix} x(t_k^+) \\ v(t_k^+) \end{pmatrix} = \begin{pmatrix} I_n & O \\ O & I_n - \mu L \end{pmatrix} \begin{pmatrix} x(t_k) \\ v(t_k) \end{pmatrix}, \quad t = t_k. \end{cases} \tag{23}$$

Theorem 4.3. If L has exactly q zero eigenvalues, all the other eigenvalues $\lambda_r (r = q+1, \dots, n)$ have positive real parts, and the following conditions are satisfied:

(i) $0 < \mu < \dfrac{2\mathrm{Re}(\lambda_r)}{|\lambda_r|^2}$,

(ii) $0 < h < \dfrac{\pi}{\sqrt{\alpha}}$.

Then, the system (2) with the control input (21) can solve group synchronization problem asymptotically, and the synchronization state of the lth group can be explicitly given by (16).

P r o o f. By considering the conditions in this theorem, it is easy to obtain that there exists a nonsingular matrix P such that $L = PJP^{-1}$ is the Jordan decomposition of L, where the first q columns of P are $\mathbf{q}_1, \ldots, \mathbf{q}_q$, the first q rows of P^{-1} are $\mathbf{p}_1, \ldots, \mathbf{p}_q$, $J = \operatorname{diag}(\underbrace{0, \ldots, 0}_{q}, J^*)$ is a block diagonal matrix, and J^* is the Jordan upper diagonal block matrix corresponding to the other eigenvalues λ_r $(r = q+1, \ldots, n)$.

According to extending concept of master stability function introduced in [12], and by introducing $(\bar{x}(t), \bar{v}(t))^{\mathrm{T}} = (I_2 \otimes P^{-1})(x(t), v(t))^{\mathrm{T}}$, the systems (23) can be decomposed into the group synchronization state equation

$$\begin{pmatrix} \dot{\bar{x}}^{(1:q)}(t) \\ \dot{\bar{v}}^{(1:q)}(t) \end{pmatrix} = \begin{pmatrix} O & I_q \\ -\alpha I_q & O \end{pmatrix} \begin{pmatrix} \bar{x}^{(1:q)}(t) \\ \bar{v}^{(1:q)}(t) \end{pmatrix} \tag{24}$$

and the transverse variational equation

$$\begin{cases} \begin{pmatrix} \dot{\bar{x}}^{(q+1:n)}(t) \\ \dot{\bar{v}}^{(q+1:n)}(t) \end{pmatrix} = \begin{pmatrix} O & I_{n-q} \\ -\alpha I_{n-q} & O \end{pmatrix} \begin{pmatrix} \bar{x}^{(q+1:n)}(t) \\ \bar{v}^{(q+1:n)}(t) \end{pmatrix}, & t \neq t_k, \\ \begin{pmatrix} \bar{x}^{(q+1:n)}(t_k^+) \\ \bar{v}^{(q+1:n)}(t_k^+) \end{pmatrix} = \begin{pmatrix} I_{n-q} & O \\ O & I_{n-q} - \mu J^* \end{pmatrix} \begin{pmatrix} \bar{x}^{(q+1:n)}(t_k) \\ \bar{v}^{(q+1:n)}(t_k) \end{pmatrix}, & t = t_k, \end{cases} \tag{25}$$

respectively.

It then follows that the analytical solution of (25) can be expressed as

$$\begin{pmatrix} \bar{x}^{(q+1:n)}(t) \\ \bar{v}^{(q+1:n)}(t) \end{pmatrix} = (\Phi(t-t_k) \otimes I_{n-q}) M^k \begin{pmatrix} \bar{x}^{(q+1:n)}(0) \\ \bar{v}^{(q+1:n)}(0) \end{pmatrix}, \tag{26}$$

where

$$M = \begin{pmatrix} I_{n-q} & O \\ O & I_{n-q} - \mu J^* \end{pmatrix} (\Phi(h)) \otimes I_{n-q})$$

with the fundamental solution matrix $\Phi(t)$ being given by $\begin{pmatrix} \cos\sqrt{\alpha}t & \frac{1}{\sqrt{\alpha}}\sin\sqrt{\alpha}t \\ -\sqrt{\alpha}\sin\sqrt{\alpha}t & \cos\sqrt{\alpha}t \end{pmatrix}$.

Therefore, by employing the similar analysis procedure of [28], we can prove that all the solutions of (25) will asymptotically converge to zero if the conditions (i) and (ii) of Theorem 4.3 are satisfied.

Finally, by $(x(t), v(t))^{\mathrm{T}} = (I_2 \otimes P)(\bar{x}(t), \bar{v}(t))^{\mathrm{T}}$, it is easy to see that

$$\begin{aligned} (x(t)^{\mathrm{T}}, v(t)^{\mathrm{T}})^{\mathrm{T}} \to \Big(& \underbrace{\bar{x}^{(1)}(t), \ldots, \bar{x}^{(1)}(t)}_{n_1}, \ldots, \underbrace{\bar{x}^{(q)}(t), \ldots, \bar{x}^{(q)}(t)}_{n_q}, \\ & \underbrace{\bar{v}^{(1)}(t), \ldots, \bar{v}^{(1)}(t)}_{n_1}, \ldots, \underbrace{\bar{v}^{(q)}(t), \ldots, \bar{v}^{(q)}(t)}_{n_q} \Big)^{\mathrm{T}}. \end{aligned} \tag{27}$$

Accordingly, it is naturally to conclude that the synchronization state of the lth group can be explicitly given by

$$\begin{pmatrix} \bar{x}^{(l)}(t) \\ \bar{v}^{(l)}(t) \end{pmatrix} = \begin{pmatrix} \cos\sqrt{\alpha}t & \frac{1}{\sqrt{\alpha}}\sin\sqrt{\alpha}t \\ -\sqrt{\alpha}\sin\sqrt{\alpha}t & \cos\sqrt{\alpha}t \end{pmatrix} \begin{pmatrix} \mathbf{p}_l^{\mathrm{T}} x_0 \\ \mathbf{p}_l^{\mathrm{T}} v_0 \end{pmatrix}. \tag{28}$$

Consequently, it is immediately to show that the conclusion of Theorem 4.3 is true. □

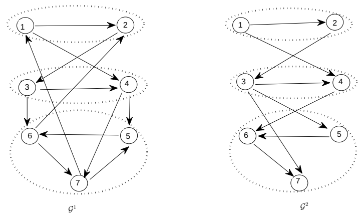

Fig. 1. Network topologies $\mathcal{G}^1$, $\mathcal{G}^2$ for seven oscillators, $a_{ij} = 1$ if there is an arrow from oscillator j to i, $a_{ij} = 0$ otherwise. The nodes in the same dot ellipse belong to the same group and the different dot ellipses indicate the different groups. The nodes in the 1th, 2th and 3th groups are $\{1, 2\}, \{3, 4\}, \{5, 6, 7\}$, respectively.

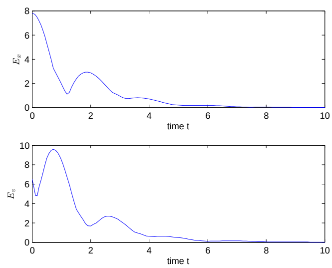

Fig. 2. Evolution of the position group synchronization error E_x and the velocity group synchronization error E_v, respectively.

Similar to Corollary 4.2, we have the following consequence.

Corollary 4.4. For the directed graph $\mathcal{G}$ with the acyclic partition $\{P_1, P_2, \ldots, P_q\}$, if each of the subgraph $\mathcal{G}_i$ with respect to P_i has a directed spanning tree, and the conditions (i), (ii) of Theorem 4.3 hold, then the system (2) with the control input (3) can solve group synchronization problem asymptotically, and the synchronization state of the lth group can be explicitly given by (20).

Remark 4.5. Although Theorems 4.1 and 4.2 provide some generic algebraic criteria on group synchronization of coupled harmonic oscillators with local continuous and instantaneous interaction respectively, they are not convenient to verity in practical application for the higher-dimensional Laplacian matrix. In addition, it should be noted from Corollaries 4.1 and 4.2 that an explicit expression of group synchronization states in terms of initial values of the agents can be obtained by the property of acyclic partition topology, and so this property will be used to yield the desired group synchronization of coupled identical oscillator systems in practice.

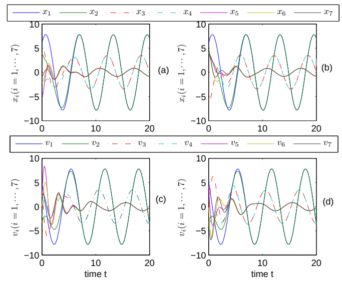

Fig. 3. The nonidentical harmonic oscillators over $\mathcal{G}^2$ evolve into groups with the same frequency. Besides, the group synchronization states for position in the subgraph (a) are the same as ones in the subgraph (b), the group synchronization states for velocity in the subgraph (c) are the same as ones in the subgraph (d).

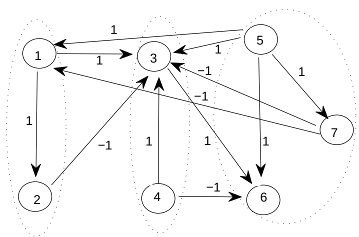

Fig. 4. The network topology in the subsection 5.2.

5. ILLUSTRATIVE EXAMPLES AND NUMERICAL SIMULATION

In this section, some examples and their simulations are worked out to demonstrate effectiveness of the theoretical results. Unless otherwise specified, in the following simulations, all initial values are randomly selected within the interval $[-5, 5]$.

5.1. Nonidentical agents case with positive coupling

This subsection considers the network topology involving three groups shown in Figure 1, and frequencies of the agents in the 1th, 2th and 3th groups are $1, \sqrt{3}, \sqrt{5}$.

Firstly, consider the network topology $\mathcal{G}^1$ shown in Figure 1, it can be seen that $\mathcal{G}^1$ is strongly connected. According to Theorem 3.1, group synchronization can be realized. Define the group synchronization errors with respect to the position and velocity respectively, i.e., $E_x = |x_1 - x_2| + |x_3 - x_4| + |x_5 - x_6| + |x_6 - x_7|$ and $E_v = |v_1 - v_2| + |v_3 - v_4| + |v_5 - v_6| + |v_6 - v_7|$. Figure 2 shows the evolution of synchronization error. Obviously, the graph $\mathcal{G}^2$ in Figure 1 is an acyclic partition topology. The subgraphs (a) and (c) in Figure 3 show the synchronization process of both position

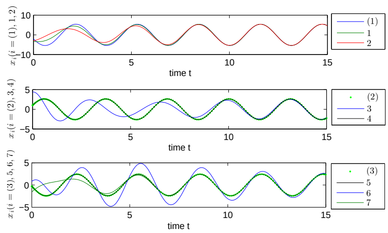

Fig. 5. Evolution of the position for three different groups. $(1), (2), (3)$ mean synchronization states of the 1th, 2th, 3th groups, respectively, $1, 2, \ldots, 7$ mean the harmonic oscillators $1, 2, \ldots, 7$, respectively.

and velocity with the initial value $(6, 4, 4, -2, -5.2, -2.1, -1, 5, -3.2, 4, -3, 2, 0.1, 2.1)^{\mathrm{T}}$, while the subgraphs (b) and (d) describe evolution of both position and velocity with the initial value $(6, 4, -6, 4, 3, 4.1, 4.8, 5, -3.2, -1, -4, 4.2, -0.2, -3.1)^{\mathrm{T}}$. From Figure 3, it can be seen that the harmonic oscillators finally evolve into three groups with the same frequency, which is obviously in consistence with the Theorem 3.2. In addition, because the initial positions of the agents 1 and 2 of the first group in the subgraph (a) are the same as ones in the the subgraph (b), the group synchronization states for position in the subgraph (a) are the same as ones in the the subgraph (b), there is the same result for the group synchronization states of velocity (see the subgraphs (c) and (d)), which further accounts for the final group synchronization states only depend on the initial values of the first group (the agents 1 and 2) but not others (the agents $3, 4, 5, 6,$ 7).

5.2. Identical agents case with positive and negative coupling

In this subsection, the frequency of all harmonic oscillators is chosen as 2. Consider the directed network topology shown in Figure 4, the weight a_{ij} are marked beside the edge (j, i), it is easy to verify the eigenvalues of L are $0, 0, 0, 1, 1, 1, 1$, the left eigenvectors associated with the eigenvalue 0 are $\mathbf{p}_1 = (1, 0, 0, 0, 1, 0, -1)^{\mathrm{T}}, \mathbf{p}_2 = (0, 0, 0, 1, 0, 0, 0)^{\mathrm{T}}, \mathbf{p}_3 = (0, 0, 0, 0, 1, 0, 0)^{\mathrm{T}}$, which satisfies (15). Simulation shows that the group synchronization can be realized (see Figure 5) and the group synchronization states can be expressed as the form (16). As for impulsive coupled case, we can calculate $\min\{\frac{2\mathrm{Re}(\lambda_r)}{|\lambda_r|^2}\} = 2$, where λ_r are the nonzero eigenvalues of L. Therefore, when we choose $0 < \mu < 2$, $0 < h < \frac{\pi}{2}$, the group synchronization can be realized according to Theorem 4.3. Figure 6 shows group synchronization process with $\mu = 0.5$, $h = 0.3$, which is accordance with the result of Theorem 4.4. For the case of the acyclic partition, the conclusion also can be verified, hence they are omitted due to the limitation of space.

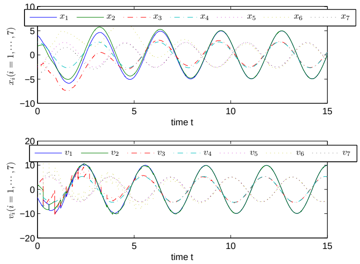

Fig. 6. Group synchronization process under the impulsive interaction with $\mu = 0.5$, $h = 0.3$. $1, 2, \ldots, 7$ mean the harmonic oscillators $1, 2, \ldots, 7$, respectively.

6. CONCLUDING REMARK

In this paper, we have investigated the group synchronization problem of diffusively coupled harmonic oscillators. *Compared with the existing works reported in the literature, the contribution of the present investigation includes*:

i) Group synchronization problem of diffusively coupled harmonic oscillators with directed network topology is firstly considered.

ii) Two control strategies are further developed to deal with the group synchronization of coupled harmonic oscillators for two cases with nonidentical and identical agent dynamics. Finally, simulation examples have been provided to verify effectiveness and correctness of the theoretical results. Future work will further consider the group synchronization problem of diffusively coupled harmonic oscillators with discontinuous or hybrid time setting under various physical and communication constraints, such as time delay, stochastic noise and different external disturbances.

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Liyun Zhao, Shanghai Institute of Applied Mathematics and Mechanics and Shanghai Key Laboratory of Mechanics in Energy Engineering, Shanghai University, Shanghai 200072 and School of Mathematics, Physics and Biological Engineering, Inner Mongolia University of Science and Technology, Baotou 014010. P. R. China.
e-mail: zhaoliyunxingyun@163.com

Jun Liu, Shanghai Institute of Applied Mathematics and Mechanics and Shanghai Key Laboratory of Mechanics in Energy Engineering, Shanghai University, Shanghai, 200072 and Department of Mathematics, Jining University, Qufu 273155, Shandong. P. R. China.
e-mail: sdwslj@163.com

Lan Xiang, Department of Physics, School of Science, Shanghai University, Shanghai 200444. P. R. China.
e-mail: xianglanhtu@126.com

Jin Zhou, Corresponding author. Shanghai Institute of Applied Mathematics and Mechanics and Shanghai Key Laboratory of Mechanics in Energy Engineering, Shanghai University, Shanghai 200072. P. R. China.
e-mail: jzhou@shu.edu.cn