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LINEAR OPTIMIZATION WITH BIPOLAR MAX-PARAMETRIC HAMACHER FUZZY RELATION EQUATION CONSTRAINTS

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In this paper, the linear programming problem subject to the Bipolar Fuzzy Relation Equation (BFRE) constraints with the max-parametric hamacher composition operators is studied. The structure of its feasible domain is investigated and its feasible solution set determined. Some necessary and sufficient conditions are presented for its solution existence. Then the problem is converted to an equivalent programming problem. Some rules are proposed to reduce the dimensions of problem. Under these rules, some of the optimal variables are found without solving the problem. An algorithm is then designed to find an upper bound for its optimal objective value. With regard to this algorithm, a modified branch and bound method is extended to solve the problem. We combine the rules, the algorithm, and the modified branch and bound method in terms of an algorithm to solve the original problem.

Keywords: bipolar fuzzy relation equations, bipolar variables, linear optimization, modified branch and bound method, max-parametric hamacher compositions

Classification: 90-xx, 90Cxx, 90C70

1. INTRODUCTION

The main idea of Fuzzy Relation Equations (FREs) was firstly proposed by Sanchez [29] in 1976 and then extended by Pedrycz [26, 27] and Miyakoshi and Shimbo [25]. Its structure of solution set was determined by Sanchez [30] in 1977 and extended to max-T FREs by Di Nola et al. [5, 7]. The complete solution set of a consistent finite system of max-T FREs can be determined by a maximum solution and a finite number of minimal solutions. The computation of the maximum solution is not difficult and can be applied to check the consistency of the system. However, the detection of all the minimal solutions is equivalent to the set covering problem and hence it is an NP-hard problem [2, 3, 23, 24]. Various approaches have been designed to compute the minimal solutions. A comprehensive review about analytical approaches of FRE resolution can be found in [4, 8]. Good overviews about theory and applications of FREs can also be found in [6, 12, 13, 16, 18, 26, 27, 28]. The problem of minimizing a linear function with max-min FRE constraints was first investigated by Fang and Li [9] in 1999. To solve it, they decomposed the problem into two subproblems by separating the nonnegative and negative cost

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coefficients with the same constraints. The optimal solution of the original problem can be found by combining the optimal solutions of the sub-problems. The second subproblem obtains its optimum at the maximum solution and the first subproblem assumes its optimum at one of the minimal solutions of its feasible domain which can be determined by solving a 0-1 integer programming problem. The branch and bound method with jump-tracking technique was applied to solve the integer programming problem. This problem was improved by Wu et al. [35] and Wu and Guu [34] by providing the proper upper bounds for the branch and bound procedure. Fang and Li's problem was studied with the max-product composition by Loetamonphong and Fang [22]. They applied a similar idea to Fang and Li's idea to solve the problem. Guu and Wu [14] provided a necessary condition for an optimal solution in terms of the maximum solution derived from FREs. This necessary condition was adopted to provide an efficient procedure for resolution of the problem. The necessary condition was extended to the situation with max-strict t-norm composition [33]. This linear optimization problem has been studied by many researchers with other operators. Some other generalizations on this linear optimization problem can be found in [15, 31, 32, 36].

In the above problems, the fuzzy relation compositions create FRE constraints which are increasing with respect to each variable.

In certain fields of applications, for instance, in the covering and investing problem, where the human judgment plays a central role, there is a need for variables with a bipolar character that they should satisfy FRE constraints with two kinds of different composition operators, simultaneously. This kind of system is called Mixed Bipolar Fuzzy Relation Equations (MBFREs). Recently, Li et al. [17] investigated a kind of nonlinear programming problem with a non-differential objective function subject to a system of MFREs with the max-min and the max-product composition operator. They presented some properties of the optimization problem and designed a polynomial-time algorithm to solve this problem based on the properties. Feng et al. [10] then studied a similar problem to Li et al.'s problem with the max-min and the max-average operators and its properties. Then a polynomial-time algorithm was proposed for its resolution. Abbasi Molai [1] studied the linear optimization problem with the Mixed Fuzzy Relation Inequality (MFRI) constraints with two max-pseudo t-norms. He investigated to the structure of its feasible domain and designed an algorithm to solve the problem. On the other hands, Freson et al. [11] first described and investigated the system of bipolar max-min fuzzy relation equations and the associated linear minimization problem with a potential application of product public awareness in revenue management. They showed that the solution set of their system can be characterized by a finite set of maximal and minimal solution pairs. Li and Jin [20] also showed that the determination of its consistency is NP-complete. Consequently, the resolution of the optimization problem is NP-hard. Li and Liu [21] discussed the problem with max-T composition, where the involved triangular norm is the Lukasiewicz t-norm. This problem can be equivalently reformulated in polynomial time into a 0-1 integer linear optimization problem and then solved by integer programming techniques.

With regard to the importance of MFREs [1, 10, 17] and BFREs [11, 20, 21], we consider the linear optimization problem with BFRE constraints with max-parametric hamacher composition operators and bipolar variables, simultaneously. In this problem, the composition operator of each its constraint can be associated to a different members of the parametric hamacher family. We firstly investigate the structure of its feasible domain and determine its feasible solution set. Some necessary and sufficient conditions are presented for its solution existence. Then the original problem is converted to an equivalent programming problem. Also, some simplification procedures or rules are proposed to reduce the dimensions of the optimization problem. Under the rules, some of the optimal variables are found without solving the problem. Finally, the reduced problem is solved by integer programming techniques.

The rest of this paper is organized as follows. Section 2 introduces the linear optimization with the bipolar max-parametric hamacher fuzzy relation equation constraints and investigates the characterizations of its feasible domain. In Section 3, we present an equivalent problem to the optimization problem and propose five rules to simplify and reduce it. Also, we suggest an algorithm to compute an initial upper bound on the optimal objective value of the equivalent problem. In Section 4, an algorithm for solving the problem is designed. In Section 5, the algorithm is outlined and illustrated by two examples. Finally, conclusions are presented in Section 6.

2. LINEAR OPTIMIZATION WITH BIPOLAR MAX-PARAMETRIC HAMACHER FUZZY RELATION EQUATION CONSTRAINTS

This section is divided to two subsections. In the first subsection, we formulate the linear optimization problem with bipolar max-parametric hamacher fuzzy relation equation constraints. In the second subsection, we investigate its feasible domain.

2.1. Problem formulation

In this subsection, we formulate the linear optimization problem with bipolar max-parametric hamacher fuzzy relation equation constraints as follows:

$$\begin{aligned} &\text{Minimize} \quad Z(x) = \sum_{j=1}^{n} c_j x_j, \\ &\text{Subject to} \quad a_i^+ \circ_{\gamma_i} x \vee a_i^- \circ_{\gamma_i} \neg x = d_i, \qquad i = 1, \dots, m, \\ &\qquad\qquad\quad x \in [0,1]^n, \end{aligned} \tag{1}$$

where $a_i^+ = \left(a_{ij}^+\right)_{1\times n}$ and $a_i^- = \left(a_{ij}^-\right)_{1\times n}$ are fuzzy relation vectors on $[0,1]$. Also, assume that $d_i \in [0,1]$, for $i \in I = \{1, \dots, m\}$, and $c = (c_1, \dots, c_n)$ is a vector of cost coefficients where $c_j > 0$, for each $j \in J = \{1, \dots, n\}$.

Moreover, $x = (x_1, \dots, x_n)^T$ is a vector of decision variables to be determined and $\neg x$ denotes the negation of x, i.e., $\neg x = (1 - x_1, \dots, 1 - x_n)^T$. The operations "$\circ_{\gamma_i}$", $i = 1, \dots, m$, represent the max-parametric hamacher compositions with the parameters $\gamma_i \geq 0$.

2.2. The characterizations of its feasible domain

Now, the structure of the feasible domain of problem (1) will briefly be discussed. The constraint part of problem (1) is to find a set of solution vectors $x \in [0,1]^n$ for the following system of bipolar fuzzy relation equations

$$\begin{cases} a_1^+ \circ_{\gamma_1} x \vee a_1^- \circ_{\gamma_1} \neg x & = & d_1 \\ \vdots & & \vdots \\ a_m^+ \circ_{\gamma_m} x \vee a_m^- \circ_{\gamma_m} \neg x & = & d_m, \end{cases} \tag{2}$$

which can be rewritten as follows:

$$\max_{j\in J} \max \left\{ \frac{a_{ij}^+ x_j}{\gamma_i + (1-\gamma_i)\left(a_{ij}^+ + x_j - a_{ij}^+ x_j\right)}, \frac{a_{ij}^- (1-x_j)}{\gamma_i + (1-\gamma_i)\left(a_{ij}^- + (1-x_j) - a_{ij}^- (1-x_j)\right)} \right\} = d_i, \tag{3}$$

for each $i \in I$, or equivalently

$$\max_{j\in J} \max \left\{ \frac{a_{ij}^+ x_j}{\gamma_i + (1-\gamma_i)\left(a_{ij}^+ + x_j - a_{ij}^+ x_j\right)}, \frac{a_{ij}^- (1-x_j)}{\gamma_i + (1-\gamma_i)\left(a_{ij}^- x_j + 1 - x_j\right)} \right\} = d_i, \tag{4}$$

for each $i \in I$, where γ_i is a parameter and $\gamma_i \geq 0$, for each $i \in I$.

Let $A^+ = \left(a_{ij}^+\right)_{m\times n}$, $A^- = \left(a_{ij}^-\right)_{m\times n}$, and $d = (d_1, \ldots, d_m)^T$ be fuzzy relation matrices on $[0,1]$ and $\gamma = (\gamma_1, \ldots, \gamma_m)^T$ where $\gamma_i \geq 0$, for each $i \in I$. A system of bipolar max-parametric hamacher fuzzy relation equations is called consistent if its solution set, i.e., $S = X(A^+, A^-, \gamma, d)$, is nonempty and inconsistent otherwise.

Now, we will discuss on the necessary and sufficient conditions for existence of solution of bipolar fuzzy relation equations (4).

Lemma 2.1. A vector $x \in [0,1]^n$ is a solution for a system of bipolar max-parametric hamacher fuzzy relation equations (4) if and only if the following conditions are satisfied.

1. $\max \left\{ \frac{a_{ij}^+ x_j}{\gamma_i+(1-\gamma_i)\left(a_{ij}^+ + x_j - a_{ij}^+ x_j\right)}, \frac{a_{ij}^-(1-x_j)}{\gamma_i+(1-\gamma_i)\left(a_{ij}^- x_j + 1 - x_j\right)} \right\} \leq d_i$, with parameter $\gamma_i \geq 0$, for each $i \in I$ and $j \in J$.
2. For each $i \in I$ with parameter $\gamma_i \geq 0$, there exists $j_i \in J$ such that $\max \left\{ \frac{a_{ij_i}^+ x_{j_i}}{\gamma_i+(1-\gamma_i)\left(a_{ij_i}^+ + x_{j_i} - a_{ij_i}^+ x_{j_i}\right)}, \frac{a_{ij_i}^-\left(1-x_{j_i}\right)}{\gamma_i+(1-\gamma_i)\left(a_{ij_i}^- x_{j_i} + 1 - x_{j_i}\right)} \right\} = d_i$.

Proof. It is obvious. □

Now, we are ready to discuss on the bipolar fuzzy relation equations (4).

Remark 2.2. For any $a^+, a^-, d \in [0,1]$, and $\gamma \geq 0$, if $a^+ = 0$ and $\gamma = 0$, then $x = 0$ cannot be a feasible solution for the inequality $\frac{a^+ x}{\gamma+(1-\gamma)(a^+ + x - a^+ x)} \leq d$. Also, if $a^- = 0$ and $\gamma = 0$, then $x = 1$ cannot be a feasible solution for the inequality $\frac{a^-(1-x)}{\gamma+(1-\gamma)(a^- x + 1 - x)} \leq d$. So, we can remove these cases from our consideration due to

Lemma 2.1. Note that if $a^+ \neq 0$ and $a^- \neq 0$, then d cannot be zero due to the inequality $\max\left\{\frac{a^+x}{\gamma+(1-\gamma)(a^++x-a^+x)}, \frac{a^-(1-x)}{\gamma+(1-\gamma)(a^-x+1-x)}\right\} \leq d$. Therefore, for any $i \in I$ with $\gamma_i = 0$, we can assume that $a_{ij}^+ \neq 0$ and $a_{ij}^- \neq 0$, for each $j \in J$. Also, if for a fixed $i \in I$, there exists $j \in J$ such that $a_{ij}^+ \neq 0$ and $a_{ij}^- \neq 0$, then we can assume that $d_i \neq 0$. Hence, we exclude the case $\gamma_i = a_{ij}^+ = a_{ij}^- = 0$, for each $i \in I$ and $j \in J$ from our consideration in this paper.

Remark 2.3. For any $a^+, a^-, d \in [0,1]$, and $\gamma \geq 0$, if $a^+ \leq d$, then we set 1 instead of $\frac{d\left(\gamma+(1-\gamma)a^+\right)}{a^+-d(1-\gamma)(1-a^+)}$. Also, if $a^- \leq d$, then we set zero instead of $\frac{a^--d}{a^--d(1-\gamma)(1-a^-)}$.

Lemma 2.4. For any $a^+, a^-, d \in [0,1]$, and $\gamma \geq 0$, the inequality

$$\max\left\{\frac{a^+x}{\gamma+(1-\gamma)\left(a^++x-a^+x\right)}, \frac{a^-\left(1-x\right)}{\gamma+(1-\gamma)\left(a^-x+1-x\right)}\right\} \leq d,$$

holds if and only if

$$\frac{a^--d}{a^--d\left(1-\gamma\right)\left(1-a^-\right)} \leq x \leq \frac{d\left(\gamma+\left(1-\gamma\right)a^+\right)}{a^+-d\left(1-\gamma\right)\left(1-a^+\right)}.$$

Proof. It is obvious that $\max\left\{\frac{a^+x}{\gamma+(1-\gamma)(a^++x-a^+x)}, \frac{a^-(1-x)}{\gamma+(1-\gamma)(a^-x+1-x)}\right\} \leq d$, holds if and only if $\frac{a^+x}{\gamma+(1-\gamma)(a^++x-a^+x)} \leq d$ and $\frac{a^-(1-x)}{\gamma+(1-\gamma)(a^-x+1-x)} \leq d$, which are considered in the following cases.

Case 1: $\dfrac{a^+x}{\gamma+\left(1-\gamma\right)\left(a^++x-a^+x\right)} \leq d.$

Since $a^+ \in [0,1]$, $x \in [0,1]$, and $\gamma \geq 0$, we have $\gamma + (1-\gamma)\left(a^+ + x - a^+x\right) = \left(a^+ + x - a^+x\right) + \gamma\left(1 - \left(a^+ + x - a^+x\right)\right) \geq 0$. So, with regard to Remark 2.2, the inequality $\frac{a^+x}{\gamma+(1-\gamma)(a^++x-a^+x)} \leq d$ implies that

$$x\left(a^+ - d\left(1-\gamma\right)\left(1-a^+\right)\right) \leq d\left(\gamma + \left(1-\gamma\right)a^+\right). \tag{5}$$

Now, we consider relation (5) in the following subcases.

1. If $a^+ > d$, then $a^+ - d + \gamma d + a^+d - \gamma a^+d > \gamma d + a^+d - \gamma a^+d$. Since $\gamma d + a^+d - \gamma a^+d = d\left(a^+ + \gamma\left(1-a^+\right)\right) \geq 0$, we have $a^+ - d\left(1-\gamma\right)\left(1-a^+\right) = a^+ - d + \gamma d + a^+d - \gamma a^+d > 0$. Therefore, the relation (5) can be simplified to $x \leq \frac{d\left(\gamma+(1-\gamma)a^+\right)}{a^+-d(1-\gamma)(1-a^+)} < 1$. Hence, in subcase (1), the relation (5) is satisfied for each $x \in \left[0, \frac{d\left(\gamma+(1-\gamma)a^+\right)}{a^+-d(1-\gamma)(1-a^+)}\right]$.

2. If $a^+ = d$, then $a^+ - d + \gamma d + a^+d - \gamma a^+d = \gamma d + a^+d - \gamma a^+d$. Since $\gamma d + a^+d - \gamma a^+d \geq 0$, we have $a^+ - d\left(1-\gamma\right)\left(1-a^+\right) \geq 0$. If $a^+ - d\left(1-\gamma\right)\left(1-a^+\right) > 0$, it follows

easily that $x \leq \frac{d\left(\gamma+(1-\gamma)a^{+}\right)}{a^{+}-d(1-\gamma)(1-a^{+})} = 1$. If $a^{+} - d\,(1-\gamma)\,(1-a^{+}) = 0$, the relation (5) is satisfied for any $x \in [0,1]$. Thus, in subcase (2), the relation (5) is satisfied for any $x \in [0,1]$.

3. If $a^{+} < d$, then $a^{+} - d + \gamma d + a^{+}d - \gamma a^{+}d < \gamma d + a^{+}d - \gamma a^{+}d$. So, one of the following cases occurs $a^{+} - d\,(1-\gamma)\,(1-a^{+}) > 0$, or $a^{+} - d\,(1-\gamma)\,(1-a^{+}) < 0$, or $a^{+} - d\,(1-\gamma)\,(1-a^{+}) = 0$. Considering $a^{+} - d\,(1-\gamma)\,(1-a^{+}) > 0$, we have $\frac{d\left(\gamma+(1-\gamma)a^{+}\right)}{a^{+}-d(1-\gamma)(1-a^{+})} > 1$ and also considering $a^{+} - d\,(1-\gamma)\,(1-a^{+}) < 0$, we have $\frac{d\left(\gamma+(1-\gamma)a^{+}\right)}{a^{+}-d(1-\gamma)(1-a^{+})} \leq 0$. Therefore, in subcase (3), the relation (5) is satisfied for each $x \in [0,1]$.

Hence, the inequality $\frac{a^{+}x}{\gamma+(1-\gamma)(a^{+}+x-a^{+}x)} \leq d$ holds if and only if $0 \leq x \leq \frac{d\left(\gamma+(1-\gamma)a^{+}\right)}{a^{+}-d(1-\gamma)(1-a^{+})}$ with regard to Remark 2.3.

Case 2: $\dfrac{a^{-}\,(1-x)}{\gamma+(1-\gamma)\,(a^{-}x+1-x)} \leq d.$

Since $a^{-} \in [0,1]$, $x \in [0,1]$, and $\gamma \geq 0$, we have

$$\gamma + (1-\gamma)\left(a^{-}x + 1 - x\right) = \left(a^{-}x + 1 - x\right) + \gamma\left(1 - \left(a^{-}x + 1 - x\right)\right) \geq 0.$$

So, with regard to Remark 2.2, the inequality $\frac{a^{-}(1-x)}{\gamma+(1-\gamma)(a^{-}x+1-x)} \leq d$ implies that

$$x\left(a^{-} - d\,(1-\gamma)\left(1-a^{-}\right)\right) \geq a^{-} - d. \tag{6}$$

In this case, since $d\,(a^{-} + \gamma\,(1-a^{-})) \geq 0$, then

$$a^{-} - d\,(1-\gamma)\left(1-a^{-}\right) = a^{-} - d + d\left(a^{-} + \gamma\left(1-a^{-}\right)\right) \geq a^{-} - d.$$

The remaining proof for this case is similar to the proof of Case 1. Therefore, the inequality $\frac{a^{-}(1-x)}{\gamma+(1-\gamma)(a^{-}x+1-x)} \leq d$ holds if and only if $\frac{a^{-}-d}{a^{-}-d(1-\gamma)(1-a^{-})} \leq x \leq 1$ with regard to Remark 2.3.

Considering Cases 1 and 2, for any $a^{+}, a^{-}, d \in [0,1]$, and $\gamma \geq 0$, the inequality $\max\left\{\frac{a^{+}x}{\gamma+(1-\gamma)(a^{+}+x-a^{+}x)}, \frac{a^{-}(1-x)}{\gamma+(1-\gamma)(a^{-}x+1-x)}\right\} \leq d$, holds if and only if

$$\frac{a^{-} - d}{a^{-} - d\,(1-\gamma)\,(1-a^{-})} \leq x \leq \frac{d\,(\gamma + (1-\gamma)\,a^{+})}{a^{+} - d\,(1-\gamma)\,(1-a^{+})}.$$

□

Lemma 2.5. Assume that point x_0 is the meeting place of two curves $f_1(x) = \frac{a^{+}x}{\gamma+(1-\gamma)(a^{+}+x-a^{+}x)}$ and $f_2(x) = \frac{a^{-}(1-x)}{\gamma+(1-\gamma)(a^{-}x+1-x)}$ and $c = f_1(x_0) = f_2(x_0)$. For any $a^{+}, a^{-}, d \in [0,1]$, and $\gamma \geq 0$ the equation

$$\max\left\{\frac{a^{+}x}{\gamma+(1-\gamma)\,(a^{+}+x-a^{+}x)}, \frac{a^{-}\,(1-x)}{\gamma+(1-\gamma)\,(a^{-}x+1-x)}\right\} = d,$$

has a solution if and only if $c \le d \le \max\{a^+, a^-\}$, in which case its solution set $X(a^+, a^-, \gamma, d)$ is determined by

Case 1: If $a^- < d \le a^+$, then $X(a^+, a^-, \gamma, d) = \left\{ \frac{d(\gamma + (1-\gamma)a^+)}{a^+ - d(1-\gamma)(1-a^+)} \right\}$;

Case 2: If $a^+ < d \le a^-$, then $X(a^+, a^-, \gamma, d) = \left\{ \frac{a^- - d}{a^- - d(1-\gamma)(1-a^-)} \right\}$;

Case 3: If $c \le d \le \min\{a^+, a^-\}$, then

$$X(a^+, a^-, \gamma, d) = \left\{ \frac{d(\gamma + (1-\gamma)a^+)}{a^+ - d(1-\gamma)(1-a^+)}, \frac{a^- - d}{a^- - d(1-\gamma)(1-a^-)} \right\}.$$

i) Case $0 \le \gamma < 1$

ii) Case $\gamma = 1$

iii) Case $\gamma > 1$

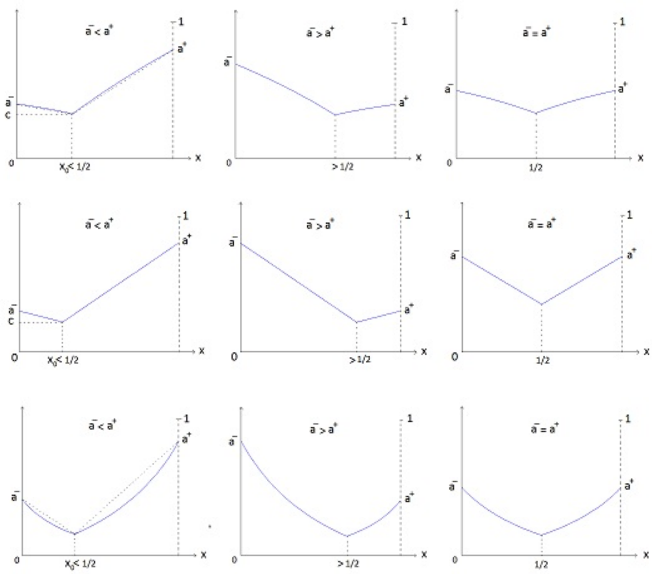

Fig. 1. The illustration of function

$$\max\left\{ \frac{a^+ x}{\gamma + (1-\gamma)(a^+ + x - a^+ x)}, \frac{a^-(1-x)}{\gamma + (1-\gamma)(a^- x + 1 - x)} \right\}.$$

P r o o f . For given $a^+, a^- \in [0, 1]$, and $\gamma \ge 0$, the range of the function $\max\{f_1(x), f_2(x)\}$ can be observed from Figure 1 and can be easily determined. □

Remark 2.6. If $a^+ = a^- = 0$ and $\gamma > 0$, then $c = 0$. So, if $a^+ = a^- = d = 0$ and $\gamma > 0$, then we define $X(a^+, a^-, \gamma, d) = [0, 1]$.

Now, considering Lemmas 2.1 and 2.4, the lower and upper bound on the solutions for the bipolar fuzzy relation equations (4) can be obtained in the following lemma. Note that the value of γ_i can be different for each $i \in I$.

Lemma 2.7. Considering Remark 2.3, the vector $\check{x} = (\check{x}_1, \dots, \check{x}_n)^T$ is the lower bound on the solution set of equations (4), where

$$\check{x}_j = \max_{i \in I} \left\{ \frac{a_{ij}^- - d_i}{a_{ij}^- - d_i (1-\gamma_i)\left(1-a_{ij}^-\right)} \right\}.$$

Also, the vector $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)^T$ is the upper bound on the solution set of equations (4), where

$$\hat{x}_j = \min_{i \in I} \left\{ \frac{d_i \left(\gamma_i + (1-\gamma_i) a_{ij}^+\right)}{a_{ij}^+ - d_i (1-\gamma_i)\left(1-a_{ij}^+\right)} \right\}.$$

Proof. It is obvious with regard to Lemmas 2.1 and 2.4. □

Remark 2.8. If $\max\left\{ \frac{a_{ij}^+ x_j}{\gamma_i + (1-\gamma_i)\left(a_{ij}^+ + x_j - a_{ij}^+ x_j\right)}, \frac{a_{ij}^-(1-x_j)}{\gamma_i + (1-\gamma_i)\left(a_{ij}^- x_j + 1 - x_j\right)} \right\} = d_i > 0$, holds for some x_j with $\check{x}_j \le x_j \le \hat{x}_j$, it holds by Lemma 2.5 and Remark 2.2 that $\max\left\{ \frac{a_{ij}^+ x_j}{\gamma_i + (1-\gamma_i)\left(a_{ij}^+ + x_j - a_{ij}^+ x_j\right)}, \frac{a_{ij}^-(1-x_j)}{\gamma_i + (1-\gamma_i)\left(a_{ij}^- x_j + 1 - x_j\right)} \right\} = d_i = 0$, for all $i \in M_0$ where $M_0 = \{i \in I \mid d_i = 0\}$. Consequently, the equations with zero right-hands can be discarded once $\check{x}$ and $\hat{x}$ have been obtained.

Remark 2.9. If $S \neq \emptyset$, then $\check{x} \le \hat{x}$. Moreover, if $x \in S$, then $\check{x} \le x \le \hat{x}$. But its converse is not true.

Remark 2.10. If there exists some $j \in J$ such that $\check{x}_j = \hat{x}_j$, then x_j takes the unique value in any possible solution. Therefore, the variable x_j and the equations for which

$$\max\left\{ \frac{a_{ij}^+ \hat{x}_j}{\gamma_i + (1-\gamma_i)\left(a_{ij}^+ + \hat{x}_j - a_{ij}^+ \hat{x}_j\right)}, \frac{a_{ij}^- (1-\check{x}_j)}{\gamma_i + (1-\gamma_i)\left(a_{ij}^- \check{x}_j + 1 - \check{x}_j\right)} \right\} = d_i,$$

can be deleted in further analysis. In such a case, the system of bipolar max-parametric hamacher fuzzy relation equations can be reduced to a system with a smaller dimension and different lower and upper bound on the solution set.

Without loss of generality, assume that $d_i > 0$, for each $i \in I$ and $\check{x}_j < \hat{x}_j$, for each $j \in J$ with regard to Remarks 2.8 and 2.10. Considering Lemma 2.1, we should focus on the values that they satisfy the equations in the set S. Since those equations hold only at the values specified in $\check{x}$ and $\hat{x}$, a generalized matrix $\tilde{Q} = (\tilde{q}_{ij})_{m \times n}$, called the characteristic matrix, can be constructed to record such information where

$$\tilde{q}_{ij} = \begin{cases} \{\check{x}_j\}, & \text{if } \frac{a_{ij}^-(1-\check{x}_j)}{\gamma_i + (1-\gamma_i)\left(a_{ij}^- \check{x}_j + 1 - \check{x}_j\right)} = d_i \neq \frac{a_{ij}^+ \hat{x}_j}{\gamma_i + (1-\gamma_i)\left(a_{ij}^+ + \hat{x}_j - a_{ij}^+ \hat{x}_j\right)}, \\ \{\hat{x}_j\}, & \text{if } \frac{a_{ij}^-(1-\check{x}_j)}{\gamma_i + (1-\gamma_i)\left(a_{ij}^- \check{x}_j + 1 - \check{x}_j\right)} \neq d_i = \frac{a_{ij}^+ \hat{x}_j}{\gamma_i + (1-\gamma_i)\left(a_{ij}^+ + \hat{x}_j - a_{ij}^+ \hat{x}_j\right)}, \\ \{\check{x}_j, \hat{x}_j\}, & \text{if } \frac{a_{ij}^-(1-\check{x}_j)}{\gamma_i + (1-\gamma_i)\left(a_{ij}^- \check{x}_j + 1 - \check{x}_j\right)} = d_i = \frac{a_{ij}^+ \hat{x}_j}{\gamma_i + (1-\gamma_i)\left(a_{ij}^+ + \hat{x}_j - a_{ij}^+ \hat{x}_j\right)}, \\ \emptyset, & \text{otherwise}, \end{cases} \tag{7}$$

for each $i \in I$ and $j \in J$.

Theorem 2.11. Consider the system of bipolar fuzzy relation equations (4). A vector $x \in [0,1]^n$ is a solution for these equations if and only if $\check{x} \le x \le \hat{x}$ and the induced binary matrix $Q^x = \left(q^x_{ij}\right)_{m\times n}$ has no any zero rows, where

$$Q^x = (q^x_{ij})_{m\times n} = \begin{cases} 1, & \text{if } \ x_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof is similar to the proof of Theorem 1 in [21]. □

Since each nonempty element in the characteristic matrix $\tilde{Q}$ contains at most two distinct values, Theorem 2.11 suggests that we can focus on the vectors that their component values are only from those in $\check{x}$ and $\hat{x}$ in order to determine the consistency of the system of bipolar fuzzy relation equations (4). Let $\{y_1, y_2, \dots, y_n\}$ be a set of boolean variables. For each $j \in J$, label the value $\hat{x}_j$ with the positive literal y_j and the value $\check{x}_j$ with the negative literal $\neg y_j$, respectively. Subsequently, for each $i \in I$, denote

$$N_i^+ = \{j \in J \mid \hat{x}_j \in \tilde{q}_{ij}\} \quad \text{and} \quad N_i^- = \{j \in J \mid \check{x}_j \in \tilde{q}_{ij}\}, \tag{8}$$

and the clause

$$C_i = \bigvee_{j\in N_i^+} y_j \vee \bigvee_{j\in N_i^-} \neg y_j. \tag{9}$$

The clause C_i is nothing but an alternative representation of the i^{th} row of $\tilde{Q}$ according to the construction of $\tilde{Q}$. Consequently, $\tilde{Q}$ can be represented by the boolean formula $C = \bigwedge_{i\in I} C_i$, called the characteristic boolean formula of the bipolar fuzzy relation equations (4), and Theorem 2.11 can be rewritten accordingly.

Theorem 2.12. A system of bipolar fuzzy relation equations (4) is consistent if and only if its characteristic boolean formula $C = \bigwedge_{i\in I} C_i$ is well-defined and satisfiable.

Proof. The proof is similar to the proof of Theorem 2.5 in [19]. □

Note that in $C = \bigwedge_{i\in I} C_i$, it is possible that $N_i^+ \cap N_i^- \neq \emptyset$ for some $i \in I$. In such a case, the corresponding clause C_i contains both y_j and $\neg y_j$ for $j \in N_i^+ \cap N_i^-$ and hence it becomes a tautology. It is clear that such a clause can be omitted to simplify the analysis if only the satisfiability of the characteristic boolean formula is concerned.

Note that for a system of bipolar fuzzy relation equations (4), all the critical information is expressed by the lower bound $\check{x}$, the upper bound $\hat{x}$, and the characteristic matrix $\tilde{Q}$. So, we can convert matrix $\tilde{Q}$ into two binary characteristic matrices Q^+ and Q^- in the following definition.

Definition 2.13. Define $Q^+ = \left(q^+_{ij}\right)_{m\times n}$ and $Q^- = \left(q^-_{ij}\right)_{m\times n}$ such that for each $i \in I$ and $j \in J$,

$$q^+_{ij} = \begin{cases} 1, & \text{if } \ \hat{x}_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$q_{ij}^{-} = \begin{cases} 1, & \text{if } \check{x}_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise.} \end{cases}$$

Notation 2.14. Considering Definition 2.13, we can also define N_i^+ and N_i^- for each $i \in I$ in relation (8) as follows:

$$N_i^+ = \{j \in J \mid q_{ij}^+ = 1\} \quad \text{and} \quad N_i^- = \{j \in J \mid q_{ij}^- = 1\}. \tag{10}$$

Definition 2.15. For the matrix Q^+, define

$$I_j^+(x) = \left\{i \in I \mid x_j = \hat{x}_j \quad \text{and} \quad q_{ij}^+ = 1\right\}$$

and

$$J_i^+(x) = \left\{j \in J \mid x_j = \hat{x}_j \quad \text{and} \quad q_{ij}^+ = 1\right\},$$

for each $i \in I$ and $j \in J$. Also, for the matrix Q^-, define

$$I_j^-(x) = \left\{i \in I \mid x_j = \check{x}_j \quad \text{and} \quad q_{ij}^- = 1\right\}$$

and

$$J_i^-(x) = \left\{j \in J \mid x_j = \check{x}_j \quad \text{and} \quad q_{ij}^- = 1\right\},$$

for each $i \in I$ and $j \in J$. Furthermore, set $I_j(x) = I_j^+(x) \cup I_j^-(x)$, for each $j \in J$.

Notation 2.16. Let $I_j^+ = I_j^+(\hat{x})$, $J_i^+ = J_i^+(\hat{x})$, $I_j^- = I_j^-(\check{x})$ and $J_i^- = J_i^-(\check{x})$ for each $i \in I$ and $j \in J$.

Corollary 2.17 can be easily derived from Theorem 2.11 and Notation 2.16.

Corollary 2.17. For each $x \in S$ and $i \in I$, we must have at least one of the following cases.

Case 1: There exists $j \in J_i^+$ such that $x_j = \hat{x}_j$,
Case 2: There exists $j \in J_i^-$ such that $x_j = \check{x}_j$.

We will introduce an equivalent problem to the optimization problem (1) and present some rules to reduce it in the next section.

3. SIMPLIFICATION OPERATIONS

In this section, we will introduce an equivalent problem to the optimization problem (1) and we will then present some rules for reducing the size of this problem. Applying these rules, we will determine some of the optimal variables of the equivalent problem (or problem (1)) without solving it. Then, we will develop an algorithm to compute an initial upper bound on the optimal objective value of the equivalent problem. With this upper bound, the optimal solution can eventually be obtained in a smaller search space.

In the rules and lemmas of this section, we assume that the bipolar system is consistent or equivalently, $S \neq \emptyset$. Moreover, we display the optimal solution with the notation $x^* = (x_1^*, \ldots, x_n^*)^T$.

The following lemma states that each component of an optimal solution x^* of problem (1) can be selected among the components of $\check{x}$ and $\hat{x}$.

Lemma 3.1. Consider the optimization problem (1). Then there exists an optimal solution $x^* = (x_1^*, \dots, x_n^*)^T$ such that for each $j \in J$ either $x_j^* = \hat{x}_j$ or $x_j^* = \check{x}_j$.

P r o o f . The proof is similar to the proof of Lemma 4 in [21]. □

Considering Lemma 3.1, an equivalent problem to the optimization problem (1) can be stated as follows.

$$\begin{aligned} \text{Minimize} \quad & Z'(x) = \sum_{j=1}^{n} c_j \left(x_j - \check{x}_j\right), \\ \text{Subject to} \quad & a_i^+ \circ_{\gamma_i} x \vee a_i^- \circ_{\gamma_i} \neg x = d_i, \quad i = 1, \dots, m, \\ & x \in [0,1]^n, \end{aligned} \tag{11}$$

where $\check{x}$ is defined in Lemma 2.7.

It is not difficult to see that solving problem (1) is equivalent to solving problem (11). Note that there exists only a difference $(\sum_{j \in J} c_j \check{x}_j)$ between the optimization problems (1) and (11). So, we focus on solving the optimization problem (11).

Remark 3.2. For any solution $x \in S$, each equation in the bipolar fuzzy relation equations (4) only needs to be satisfied by at least one of the values $\check{x}_j$ or $\hat{x}_j$. Once one equation is satisfied by one of these components, it is not considered again and can be deleted from the bipolar fuzzy relation equations (4). Furthermore, if the k^{th} component of an optimal solution x^* is selected in the process of solving problem (11), then $x_k^* = \check{x}_k$ or $x_k^* = \hat{x}_k$ with regard to Lemma 3.1. Therefore, if $x_k^* = \check{x}_k$ is selected, then the row(s) $i \in I_k^-$ and the column k can be deleted from the matrices Q^+ and Q^-. Also, if $x_k^* = \hat{x}_k$ is selected, then the row(s) $i \in I_k^+$ and the column k can be deleted from the matrices Q^+ and Q^-.

Now, we are ready to present some rules to reduce the size of problem (11) (or problem (1)) by fixing the decision variables.

Rule 1. If there exists some $K \subseteq J$ such that $\bigcup_{k \in K} I_k^+ \subseteq \bigcup_{k \in K} I_k^-$, then $x_k^* = \check{x}_k$ for each $k \in K$.

P r o o f . With regard to Lemma 3.1, we have $x_j^* = \hat{x}_j$ or $x_j^* = \check{x}_j$ for each $j \in J$. Let x^* be any optimal solution. Then we have $x_k^* = \check{x}_k$ for each $k \in K$. Otherwise, there exists an index set $\emptyset \neq K_0 \subseteq K$ such that $x_k^* = \hat{x}_k$ for each $k \in K_0$. Without loss of generality, suppose that $x_k^* = \check{x}_k$ for each $k \in K \setminus K_0$. We will establish a solution with a better objective value than x^*. A solution x^{**} can be built by putting $x_k^{**} = \check{x}_k$ for each $k \in K_0$ and $x_j^{**} = x_j^*$ for each $j \in J \setminus K_0$. It is obvious that $\check{x} \leq x^{**} \leq \hat{x}$. Since $\bigcup_{k \in K_0} I_k^+ \subseteq \bigcup_{k \in K} I_k^+ \subseteq \bigcup_{k \in K} I_k^-$ and $\bigcup_{k \in K \setminus K_0} I_k^- \subseteq \bigcup_{k \in K} I_k^-$, we have:

$$\begin{aligned}\bigcup_{j\in J} I_j(x^*) &= \left(\bigcup_{j\in J\setminus K} I_j(x^*)\right) \cup \left(\bigcup_{k\in K\setminus K_0} I_k^-\right) \cup \left(\bigcup_{k\in K_0} I_k^+\right)\\ &\subseteq \left(\bigcup_{j\in J\setminus K} I_j(x^*)\right) \cup \left(\bigcup_{k\in K} I_k^-\right) = \bigcup_{j\in J} I_j(x^{**}).\end{aligned}$$

Since the matrix Q^{x^*} has no any zero rows and $\bigcup_{j\in J} I_j(x^{**}) \supseteq \bigcup_{j\in J} I_j(x^*)$, then the matrix $Q^{x^{**}}$ has no any zero rows. So, x^{**} is a solution for the system of bipolar fuzzy relation equations (4) with regard to Theorem 2.11. Furthermore, we have:

$$\begin{aligned}\sum_{j\in J} c_j x_j^* &= \sum_{j\in J\setminus K} c_j x_j^* + \sum_{k\in K\setminus K_0} c_k \check{x}_k + \sum_{k\in K_0} c_k \hat{x}_k\\ &> \sum_{j\in J\setminus K} c_j x_j^* + \sum_{k\in K\setminus K_0} c_k \check{x}_k + \sum_{k\in K_0} c_k \check{x}_k = \sum_{j\in J} c_j x_j^{**}.\end{aligned}$$

The last inequality is derived from $c_k > 0$ and $\hat{x}_k > \check{x}_k$ for each $k \in K_0$. Therefore, we have $Z'(x^*) > Z'(x^{**})$. This result is a contradiction with the optimality of vector x^*. □

Lemma 3.3. If there exists $K \subset J$ and $t \in J \setminus K$ such that $I_t^+ \subseteq \bigcup_{k\in K} I_k^+$, then $x_t^* = \check{x}_t$ or there exists an index set $\emptyset \neq K_0 \subseteq K$ such that $x_k^* = \check{x}_k$ for each $k \in K_0$.

P r o o f. Let x^* be an optimal solution. If there exists at least one $\emptyset \neq K_0 \subseteq K$ such that $x_k^* = \check{x}_k$, for each $k \in K_0$, then the result is obtained. Otherwise, assume that $x_k^* = \hat{x}_k$ for each $k \in K$. Since $x_k^* = \hat{x}_k$ for each $k \in K$, $I_t^+ \subseteq \bigcup_{k\in K} I_k^+$, $c_k > 0$, and the problem is minimization, then we have to assign the possible minimum value to the variable x_t^*, i. e. $x_t^* = \check{x}_t$. □

Now, we use Lemma 3.3 in the proof of Rule 2.

Rule 2. If there exists $K \subset J$ and $t \in J \setminus K$ such that $I_t^+ \subseteq \bigcup_{k\in K} I_k^+$, $I_t^- \supseteq \bigcup_{k\in K} I_k^-$, and $c_t\,(\hat{x}_t - \check{x}_t) > \sum_{k\in K} c_k\,(\hat{x}_k - \check{x}_k)$, then $x_t^* = \check{x}_t$.

P r o o f. Let x^* be any optimal solution. If $x_t^* = \check{x}_t$, then the result is correct. Now, suppose to the contrary that $x_t^* = \hat{x}_t$. Then with regard to Lemma 3.3, there exists $\emptyset \neq K_0 \subseteq K$ such that $x_k^* = \check{x}_k$ for each $k \in K_0$. Without loss of generality, suppose that $x_k^* = \hat{x}_k$ for each $k \in K \setminus K_0$. We will establish a solution with a better objective value than x^*. A solution x^{**} can be built by putting $x_t^{**} = \check{x}_t$ and $x_k^{**} = \hat{x}_k$ for each $k \in K_0$ and $x_j^{**} = x_j^*$ for each $j \in J \setminus K_0$, $j \neq t$. It is obvious that $\check{x} \le x^{**} \le \hat{x}$. Since

$\bigcup_{k\in K\setminus K_0} I_k^+ \subseteq \bigcup_{\substack{j\in J\setminus K_0\\ j\neq t}} I_j(x^*)$ and $\bigcup_{k\in K_0} I_k^- \subseteq \bigcup_{k\in K} I_k^- \subseteq I_t^-$, we have:

$$\bigcup_{j\in J} I_j(x^*) = \left(\bigcup_{\substack{j\in J\setminus K_0\\ j\neq t}} I_j(x^*)\right) \cup I_t^+ \cup \left(\bigcup_{k\in K_0} I_k^-\right) \subseteq \left(\bigcup_{\substack{j\in J\setminus K_0\\ j\neq t}} I_j(x^*)\right) \cup \left(\bigcup_{k\in K} I_k^+\right) \cup I_t^-$$

$$= \left(\bigcup_{\substack{j\in J\setminus K_0\\ j\neq t}} I_j(x^*)\right) \cup \left(\bigcup_{k\in K_0} I_k^+\right) \cup (\bigcup_{k\in K\setminus K_0} I_k^+) \cup I_t^-$$

$$\subseteq \left(\bigcup_{\substack{j\in J\setminus K_0\\ j\neq t}} I_j(x^*)\right) \cup \left(\bigcup_{k\in K_0} I_k^+\right) \cup I_t^- = \bigcup_{j\in J} I_j(x^{**}).$$

Since the matrix Q^{x^*} has no any zero rows and $\bigcup_{j\in J} I_j(x^{**}) \supseteq \bigcup_{j\in J} I_j(x^*)$, then the matrix $Q^{x^{**}}$ has no any zero rows. So, x^{**} can be a solution for the system of bipolar fuzzy relation equations (4) with regard to Theorem 2.11. Furthermore, we have:

$$\begin{aligned}\sum_{j\in J} c_j x_j^* &= \sum_{\substack{j\in J\setminus K_0\\ j\neq t}} c_j x_j^* + \sum_{k\in K_0} c_k \check{x}_k + c_t \hat{x}_t \\ &> \sum_{\substack{j\in J\setminus K_0\\ j\neq t}} c_j x_j^* + \sum_{k\in K_0} c_k \hat{x}_k + c_t \check{x}_t = \sum_{j\in J} c_j x_j^{**}.\end{aligned}$$

The last inequality is derived from $c_t\left(\hat{x}_t - \check{x}_t\right) > \sum_{k\in K} c_k\left(\hat{x}_k - \check{x}_k\right) \geq \sum_{k\in K_0} c_k\left(\hat{x}_k - \check{x}_k\right)$. Therefore, we have $Z'(x^*) > Z'(x^{**})$. This result is a contradiction with the optimality of vector x^*. □

Rule 3. If there exists $i_0 \in I$ and $j_0 \in J$ such that the following conditions are satisfied.

1. $q_{i_0 j_0}^+ = q_{i_0 j_0}^- = 0$,
2. $I_{j_0}^+ \setminus I_{j_0}^- \subseteq I_j^+$ for each $j \in J_{i_0}^+$,
3. $I_{j_0}^+ \setminus I_{j_0}^- \subseteq I_j^-$ for each $j \in J_{i_0}^-$,

then $x_{j_0}^* = \check{x}_{j_0}$.

P r o o f. Since every optimal solution is a feasible solution, then we must have at least one of the following cases:

Case 1: There exists $k \in J_{i_0}^+$ such that $x_k^* = \hat{x}_k$,

Case 2: There exists $k \in J_{i_0}^-$ such that $x_k^* = \check{x}_k$,

with regard to Corollary 2.17. If Case 1 occurs, then the row(s) $i \in I_k^+$ can be deleted from the matrices Q^+ and Q^- and if Case 2 occurs, then the row(s) $i \in I_k^-$ can be deleted from the matrices Q^+ and Q^- with regard to Remark 3.2. Now, consider conditions 2 and 3. With regard to the above cases, there exists $j \in J_{i_0}^+$ such that the row(s) $i \in I_j^+$ can be deleted or there exists $j \in J_{i_0}^-$ such that the row(s) $i \in I_j^-$ can be deleted from the matrices Q^+ and Q^-. Therefore, after updating the sets $I_{j_0}^+$ and $I_{j_0}^-$, we have $I_{j_0}^+ \setminus I_{j_0}^- = \emptyset$. Hence, we have $I_{j_0}^+ \subseteq I_{j_0}^-$ and $x_{j_0}^* = \check{x}_{j_0}$ with regard to Rule 1. □

Rule 4 can be easily derived from Corollary 2.17 and Rule 3.

Rule 4. If there exists $i \in I$ such that $|J_i^+| + |J_i^-| = 1$, i. e., $J_i^+ = \{j_1\}$ and $J_i^- = \emptyset$ (or $J_i^- = \{j_1\}$ and $J_i^+ = \emptyset$), then

a) $x_{j_1}^* = \hat{x}_{j_1}$ (or $x_{j_1}^* = \check{x}_{j_1}$),

b) Under the above assumptions, if there exists $k \in J \setminus \{j_1\}$ such that $I_k^+ \setminus I_k^- \subseteq I_{j_1}^+$ (or $I_k^+ \setminus I_k^- \subseteq I_{j_1}^-$), then $x_k^* = \check{x}_k$.

Now, we are ready to develop an algorithm to compute an initial upper bound on the optimal objective value of problem (11). In this algorithm, we first begin with one feasible solution $x = (x_1, \dots, x_n)^T$ which can easily be obtained by the characteristic boolean formula $C = \bigwedge_{i \in I} C_i$ in conjunctive normal form. Then, we determine an index set $J' = \{j \in J \mid x_j = \hat{x}_j\}$ and try to convert the components $\hat{x}_j$ in the feasible solution x to $\check{x}_j$ (if possible) where $j \in J'$ such that the vector x remains a feasible solution for the optimization problem (11). To do this, the index $j \in J'$ is selected such that the value $c_j\,(\hat{x}_j - \check{x}_j)$ have had the most value. We are now ready to present an algorithm to compute an initial upper bound on the objective optimal value of problem (11) based on the above points.

First of all, we express the following definition.

Definition 3.4. Suppose that $H := \{h_j = c_j\,(\hat{x}_j - \check{x}_j) \mid j \in J'\}$ and $\max_{j \in J'} h_j = L$. Then we define

$$\arg\max_{j \in J'} h_j = \{j \in J' \mid h_j = L\}.$$

Algorithm 1. An algorithm for computing an initial upper bound on the optimal objective value of problem (11).

Step 1. Obtain a feasible solution x from the reduced matrices Q^+ and Q^- applying characteristic boolean formula $C = \bigwedge_{i \in I} C_i$ in conjunctive normal form and relations (9) and (10).

Step 2. Obtain the index set $J' = \{j \in J \mid x_j = \hat{x}_j\}$.
Step 3. If $J' = \emptyset$, then go to Step 7. Otherwise, go to Step 4.
Step 4. Compute $c_j\left(\hat{x}_j - \check{x}_j\right)$ for each $j \in J'$.
Step 5. $H := \{h_j = c_j\left(\hat{x}_j - \check{x}_j\right) \mid j \in J'\}$. Go to the Procedure Decreasing order (H).
Step 6. For $k = 1 : |J'|$ do

6.1. If $I^-_{s_k} \cup \left(\bigcup_{j \in J \setminus \{s_k\}} I_j(x)\right) = I$, then set $x_{s_k} = \check{x}_{s_k}$.

Step 7. Compute the initial upper bound for the optimization problem (11) as follows:

$$U = \sum_{j \in J} c_j\left(x_j - \check{x}_j\right).$$

Step 8. End.

The used procedure in the Algorithm 1 is presented below.

Procedure. Decreasing order (H).
Step 1. $k = 1$.
Step 2. If $H = \emptyset$, then stop and return $\{s_k \mid k = 1, \ldots, |J'|\}$.
Step 3. If $|\arg\max_{j \in J'} h_j| = 1$, then choose s_k from $\arg\max_{j \in J'} h_j$.
Step 4. If $|\arg\max_{j \in J'} h_j| > 1$, then $s_k = \min(\arg\max_{j \in J'} h_j)$.
Step 5. Let $J' := J' \setminus \{s_k\}$ and update H.
Step 6. $k := k + 1$.
Step 7. Go to Step 2.

Theorem 3.5. The obtained vector x at end of Algorithm 1 is a feasible solution and U is an upper bound on the optimal objective value of problem (11).

Proof. In this algorithm, we first obtain a feasible solution x (Step 1). This vector x can change only in the during Step 6 of the algorithm. We prove that x remains feasible even if it changes. If x changes, then there exists some $k \in J$ such that $\hat{x}_k$ decreases to $\check{x}_k$. So, the resulting vector remains feasible because

$$I^-_k \cup \left(\bigcup_{j \in J \setminus \{k\}} I_j(x)\right) = I.$$

Furthermore, since the vector x is a feasible solution for the system of bipolar fuzzy relation equations (4), its objective value $U = \sum_{j \in J} c_j\left(x_j - \check{x}_j\right)$ is an upper bound on the optimal objective value of problem (11). □

From the theorem given, one has the following rule.

Rule 5. If there exists $k \in J$ such that $c_k\left(\hat{x}_k - \check{x}_k\right) > U$, then $x^*_k = \check{x}_k$.

Proof. It is obvious. □

4. AN ALGORITHM FOR RESOLUTION OF PROBLEM (1)

In this section, we define a value matrix M using the matrices Q^+ and Q^-. After it, we explain the modified branch and bound ($B\&B$) method with the jump-tracking technique to solve the optimization problem (11). Then we present an algorithm for resolution of problem (1). At first, we need to express the following theorem.

Theorem 4.1. Let $\check{x}$ and $\hat{x}$ be the lower and upper bound, respectively. Then

$$I_j(x) \subseteq I_j^+ \cup I_j^-, \quad \forall x \in S, \quad \forall j \in J.$$

Exactly, we have $I_j(x)=I_j^+$ (when $x_j = \hat{x}_j$) or $I_j(x)=I_j^-$ (when $x_j = \check{x}_j$) or $I_j(x) = \emptyset$ (when $\check{x}_j < x_j < \hat{x}_j$).

Proof. It is obvious from Definition 2.15 and the definition of $\tilde{Q}$ in relation (7) . □

It follows from Lemma 3.1 that $x_j^* = \hat{x}_j$ or $x_j^* = \check{x}_j$ for each $j \in J$. If $x_j^* = \hat{x}_j$, then $I_j(x^*)=I_j^+$ and if $x_j^* = \check{x}_j$, then $I_j(x^*)=I_j^-$ for each $j \in J$ with regard to Theorem 4.1. Since $I_j(x^*)=I_j^+$ or $I_j(x^*)=I_j^-$ for each $j \in J$, we can restrict our search within I_j^+ and I_j^-, to which we now turn. To do this, we have to determine the index set $I_j(x^*)$ for each $j \in J$ to compute the components x_j^*.

Define the value matrix M=$(m_{ij})_{m\times 2n}$ based on the optimization problem (11) where

$$m_{i,2j-1} = \begin{cases} c_j(\hat{x}_j - \check{x}_j), & \text{if } q_{ij}^+ = 1, \\ \infty, & \text{otherwise,} \end{cases} \quad \text{and} \quad m_{i,2j} = \begin{cases} 0, & \text{if } q_{ij}^- = 1, \\ \infty, & \text{otherwise,} \end{cases} \tag{12}$$

for each $i \in I$ and $j \in J$.

Consider the value matrix M. Apply the branch and bound method with the jump-tracking technique on matrix M to solve the optimization problem (11). In this approach, we have to consider one modification on this method as follows:

1. If we select $\hat{x}_j$ (or $\check{x}_j$) to branch from one node to another node, then we never use $\check{x}_j$ (or $\hat{x}_j$) to branch further on the current node.

Which further details will be illustrated in Example 5.1.

Based on the concepts discussed above, we now propose an algorithm to find an optimal solution of the optimization problem (1).

Algorithm 2. An algorithm for resolution of problem (1)

Step 1. Compute the lower and upper bound $\check{x}$ and $\hat{x}$ using Lemma 2.7.
Step 2. If $d > 0$ and $\check{x} < \hat{x}$, then go to Step 3. Otherwise, use Remarks 2.8 and 2.10.
Step 3. Generate matrices Q^+ and Q^- using Definition 2.13.
Step 4. Check the consistency of the system of bipolar fuzzy relation equations (4). If it is inconsistent, then stop! Otherwise, go to Step 5.
Step 5. Compute the index sets I_j^+, I_j^-, J_i^+, and J_i^- using Definition 2.15 and Notation 2.16.

Step 6. Define the optimization problem (11) and apply Rules 1-4 to reduce it and find corresponding optimal variables (if possible). Then remove their corresponding row(s) and column(s) from the matrices Q^+ and Q^- according to Remark 3.2 and reformulate the objective function removing the determined variables by Rules 1-4 (reduced problem).
Step 7. Compute the initial upper bound U on the optimal objective value of reduced problem using Algorithm 1.
Step 8. Apply Rule 5. If there exists $k \in J$ such that $c_k(\hat{x}_k - \check{x}_k) > U$, then set $x_k^* = \check{x}_k$ and reduce problem (11) by Rules 1-4 again (if possible). Also, remove their corresponding row(s) and column(s) from the matrices Q^+ and Q^- according to Remark 3.2.
Step 9. If $Q^+ = Q^- = \emptyset$, then assign $\check{x}_j$ to x_j^* and go to Step 12.
Step 10. Generate the value matrix M using relation (12).
Step 11. Employ the modified branch and bound method with the jump-tracking technique on the matrix M to solve the optimization problem (11). In addition, the initial upper bound U will be improved by a better solution during the procedure.
Step 12. Produce the optimal solution x^* and the optimal objective value $Z'(x^*)$ of the optimization problem (11). Then x^* is an optimal solution of the optimization problem (1) with the optimal objective value $Z(x^*) = Z'(x^*) + \sum_{j\in J} c_j \check{x}_j$. End.

5. NUMERICAL EXAMPLES

We now illustrate our algorithm by two examples.

Example 5.1. Consider the following optimization problem:

$$\begin{aligned} &\text{Minimize} \quad Z(x) = x_1 + 2x_2 + 6x_3 + 2x_4 + 5x_5 + 3x_6 + 3x_7 + x_8 \\ &\text{Subject to} \quad a_i^+ \circ_{\gamma_i} x \vee a_i^- \circ_{\gamma_i} \neg x = d_i, \qquad i = 1, \dots, 10, \\ &\qquad\qquad x \in [0,1]^8. \end{aligned} \tag{13}$$

Where

$$A^+ = \begin{pmatrix} 0.4 & 1 & 0.3 & 0.2 & 0.6 & 0.41 & 0.35 & 0.42 \\ 0.11 & 0.4 & 0.36 & 0.1 & 0.15 & 0.07 & 0.1 & 0.16 \\ 0.6 & 0.1 & 0.15 & 0.24 & 0.32 & 0.2 & 0.13 & 0.4 \\ 0.9 & 0.3 & 0.2 & 0.36 & 0.25 & 0.33 & 0.45 & 0.6 \\ 0.1 & 0.08 & 0.3 & 0.12 & 0.04 & 0.13 & 0.2 & 0.1 \\ 0.6 & 0.3 & 0.7 & 0.25 & 0.6 & 0.7 & 1 & 0.1 \\ 0.55 & 0.1 & 0.2 & 0.12 & 0.3 & 0.25 & 0.1 & 1 \\ 0.3 & 1 & 1 & 0.05 & 0.25 & 0.2 & 0.08 & 0.5 \\ 1 & 0.05 & 0.4 & 0.4 & 0.25 & 0.2 & 0.06 & 0.1 \\ 0.95 & 0.25 & 0.1 & 0.4 & 0.6 & 0.15 & 0.3 & 0.45 \end{pmatrix},$$

$$A^- = \begin{pmatrix} 0.3 & 0.1 & 0.5 & 0.4 & 0.33 & 0.05 & 0.2 & 0.43 \\ 0.12 & 0.24 & 0.1 & 0.24 & 0.17 & 0.03 & 0.15 & 0.12 \\ 0.24 & 0.2 & 0.15 & 0.11 & 0.23 & 0.4 & 0.48 & 0.25 \\ 0.1 & 0.04 & 0.3 & 0.22 & 0.45 & 0.35 & 0.72 & 0.33 \\ 0.03 & 0.2 & 0.1 & 0.07 & 0.09 & 0.14 & 0.05 & 0.01 \\ 0.7 & 0.25 & 0.45 & 0.1 & 0.3 & 1 & 0.6 & 0.35 \\ 0.6 & 0.4 & 0.1 & 1 & 0.54 & 0.35 & 0.7 & 0.25 \\ 0.8 & 0.3 & 1 & 0.65 & 0.38 & 0.1 & 0.7 & 1 \\ 0.2 & 0.4 & 0.24 & 0.35 & 0.25 & 0.2 & 0.31 & 0.27 \\ 0.31 & 0.27 & 0.42 & 0.35 & 1 & 0.2 & 1 & 0.32 \end{pmatrix},$$

$\gamma = (1,1,1,1,1,0,0,0,0,0)^T$, $d = (0.45, 0.18, 0.24, 0.36, 0.15, 0.7, 0.6, 0.9, 0.4, 0.5)^T$, and $x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T$. Now, we want to solve this example by Algorithm 2.

Step 1. In this example the lower and upper bound $\check{x}$ and $\hat{x}$ are as follows:
$\check{x} = (0, 0.25, 0.1, 0.4, 0.5, 0.4, 0.5, 0.1)^T$ and $\hat{x} = (0.4, 0.45, 0.5, 1, 0.75, 1, 0.7, 0.6)^T$.

Step 2. Since $d > 0$ and $\check{x} < \hat{x}$, we go to Step 3.

Step 3. The matrices Q^+ and Q^- can be obtained as follows:

$$Q^+ = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

and

$$Q^- = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Step 4. The system of bipolar fuzzy relation equations is consistent. So, we go to Step 5.

Step 5. The index sets I_j^+ and I_j^-, for all $j \in J$, can be computed as follows:
$I_1^+ = \{3,4,9\}$, $I_2^+ = \{1,2\}$, $I_3^+ = \{2,5\}$, $I_4^+ = \{3,4,9\}$, $I_5^+ = \{1,3,10\}$, $I_6^+ = \{6\}$, $I_7^+ = \{6\}$, $I_8^+ = \{3,4,7\}$, $I_1^- = \{3,6,7\}$, $I_2^- = \{2,5\}$, $I_3^- = \{1,8\}$, $I_4^- = \{7\}$, $I_5^- = \{10\}$, $I_6^- = \{3\}$, $I_7^- = \{3,4,10\}$, and $I_8^- = \{8\}$.
Also, the index sets J_i^+ and J_i^-, for all $i \in I$, can be computed as follows:
$J_1^+ = \{2,5\}$, $J_2^+ = \{2,3\}$, $J_3^+ = \{1,4,5,8\}$, $J_4^+ = \{1,4,8\}$, $J_5^+ = \{3\}$, $J_6^+ = \{6,7\}$, $J_7^+ = \{8\}$, $J_8^+ = \emptyset$, $J_9^+ = \{1,4\}$, $J_{10}^+ = \{5\}$, $J_1^- = \{3\}$, $J_2^- = \{2\}$, $J_3^- = \{1,6,7\}$, $J_4^- = \{7\}$, $J_5^- = \{2\}$, $J_6^- = \{1\}$, $J_7^- = \{1,4\}$, $J_8^- = \{3,8\}$, $J_9^- = \emptyset$, and $J_{10}^- = \{5,7\}$.

Step 6. Consider the following equivalent optimization problem:

$$\begin{aligned} \text{Minimize} \quad & Z'(x) = x_1 + 2x_2 + 6x_3 + 2x_4 + 5x_5 + 3x_6 + 3x_7 + x_8 - 7.2 \\ \text{Subject to} \quad & a_i^+ \circ_{\gamma_i} x \vee a_i^- \circ_{\gamma_i} \neg x = d_i, \qquad i = 1,\dots,10, \\ & x \in [0,1]^8, \end{aligned} \tag{14}$$

where A^+, A^-, γ, d, and x are defined before. Now, we are ready to apply Rules 1-4 to reduce problem (14).

a) Since $I_2^+ \cup I_3^+ \subseteq I_2^- \cup I_3^-$, according to Rule 1, $x_2^* = \check{x}_2 = 0.25$ and $x_3^* = \check{x}_3 = 0.1$. Also, the columns 2, 3, and the rows 1, 2, 5, and 8 can be deleted from the matrices Q^+ and Q^-.

b) Since $I_1^+ \subseteq I_4^+$, $I_1^- \supseteq I_4^-$ and $c_1(\hat{x}_1 - \check{x}_1) = 0.4 \not\geq 1.2 = c_4(\hat{x}_4 - \check{x}_4)$, we cannot use Rule 2 in this case. Also, we cannot use this rule for the variables x_6 and x_7. But the conditions of Rule 3 are satisfied, i. e., $q_{45}^+ = q_{45}^- = 0$ and $I_5^+ \setminus I_5^- = \{3\}$ is a subset of I_1^+, I_4^+, I_7^-, and I_8^+. So, we can set $x_5^* = \check{x}_5 = 0.5$. Also, the column 5 and the row 10 can be deleted from the matrices Q^+ and Q^-. Note that we can also use this rule for $i_0 = 9$ and $j_0 = 5$ and give the same results. After reduction of the parts (a) and (b), the matrices Q^+ and Q^- are converted as follows:

$$Q^+ = \begin{array}{c} \\ 3 \\ 4 \\ 6 \\ 7 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad Q^- = \begin{array}{c} \\ 3 \\ 4 \\ 6 \\ 7 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

The matrices Q^+ and Q^- cannot be reduced further applying Rules 1-4. So, we can reformulate the objective function removing the determined variables $x_2^* = 0.25, x_3^* = 0.1$, and $x_5^* = 0.5$. Hence, the reduced objective function is as follows: $x_1 + 2x_4 + 3x_6 + 3x_7 + x_8 - 3.6$.

Step 7. Compute the initial upper bound U on the optimal objective value of the reduced objective function using Algorithm 1. First, we begin with the feasible solution $x = (\check{x}_1, \hat{x}_4, \hat{x}_6, \hat{x}_7, \hat{x}_8)^T$. It is clear that $J' = \{4,6,7,8\}$ and the set $\{6,4,7,8\}$ returns from the Procedure Decreasing order (H). After using Algorithm 1, we obtain the feasible solution $x = (\check{x}_1, \hat{x}_4, \check{x}_6, \check{x}_7, \check{x}_8)^T$ with the initial upper bound $U = c_4\left(\hat{x}_4 - \check{x}_4\right) = 1.2$.

Step 8. Since $c_6\left(\hat{x}_6 - \check{x}_6\right) = 1.8 > 1.2 = U$, according to Rule 5, $x_6^* = \check{x}_6 = 0.4$. Moreover, the row 3 and the column 6 can be deleted from the matrices Q^+ and Q^-. So, the matrices Q^+ and Q^- can be simplified as follows:

$$Q^+ = \begin{array}{c} \\ 4 \\ 6 \\ 7 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad Q^- = \begin{array}{c} \\ 4 \\ 6 \\ 7 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Since the matrices Q^+ and Q^- cannot be reduced further applying Rules 1-4, we go to Step 9.

Step 9. Since the matrices $Q^+ \neq \emptyset$ and $Q^- \neq \emptyset$, we go to Step 10.

Step 10. For the reduced matrices Q^+ and Q^-, the value matrix M can be generated as follows:

$$M = \begin{array}{c} \\ 4 \\ 6 \\ 7 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 7 & 8 \end{array} \\ \begin{pmatrix} 0.4 & \infty & 1.2 & \infty & \infty & 0 & 0.5 & \infty \\ \infty & 0 & \infty & \infty & 0.6 & \infty & \infty & \infty \\ \infty & 0 & \infty & 0 & \infty & \infty & 0.5 & \infty \\ 0.4 & \infty & 1.2 & \infty & \infty & \infty & \infty & \infty \end{pmatrix} \end{array}.$$

Step 11. Now, we are ready to use the modified branch and bound method with the jump-tracking technique on the matrix M. The details of steps within Step 11 is summarized in Figure 2. On each node, the branching process should be stopped when its objective value is larger than the current upper bound. We begin from the first equation. $\{\hat{x}_1, \hat{x}_4, \check{x}_7, \hat{x}_8\}$ are four candidates for satisfying the first equation. If we select $\hat{x}_1$ (node 1), then the objective value is 0.4. Note that we never use $\check{x}_1$ to branch further on node 1. If we select $\hat{x}_4$ (node 2), then the objective value is 1.2. Also, we never use $\check{x}_4$ to branch further on node 2. If we select $\check{x}_7$ (node 3), then the objective value is 0. Also, we never use $\hat{x}_7$ to branch further on node 3. If we select $\hat{x}_8$ (node 4), then the objective value is 0.5. Also, we never use $\check{x}_8$ to branch further on node 4. We can see four branches generated from node 0 in Figure 2.

Consider the index set J_k as follows:
$J_k = \{j \in J \mid x_j$ has been selected along the branches from node 0 to node $k\}$. Along the branches to each node k, we need to check whether the selected variables x_j for each $j \in J_k$ together with $\check{x}_j$ for each $j \in J \setminus J_k$ satisfy all equations or not. Therefore, if it satisfies, then we do not branch further on this node. Furthermore, the jump-tracking technique requires to branch on the node with least objective value.

Since $\hat{x}_4$ with $\check{x}_1, \check{x}_7$, and $\check{x}_8$ satisfy all equations, we do not branch further on node 2. Now, we have three nodes to select for the next branching process. By the jump-tracking technique, we select node 3 to branch further because of least objective value there.

Consider node 3. $\{\check{x}_1, \hat{x}_7\}$ are two candidates for satisfying the second equation. But we cannot use $\hat{x}_7$ to branch further on node 3 because we select $\check{x}_7$ along node 0 to node 3. Therefore, $\{\check{x}_1\}$ is the only candidate for satisfying the second equation here. If we select $\check{x}_1$ (node 5), then the objective value remains 0. Now, we have three nodes (nodes 1, 4, and 5) to select for the next branching process. By the jump-tracking technique, we select node 5 to branch further.

Consider node 5. $\{\check{x}_1, \check{x}_4, \hat{x}_8\}$ are three candidates for satisfying the third equation. If we select $\check{x}_1$ (node 6), $\check{x}_4$ (node 7), and $\hat{x}_8$ (node 8), then the objective value is 0, 0, and 0.5, respectively. Along the branches to each node 6, 7, and 8, all equations cannot be satisfied. Now, we select node 6 for the next branching process. $\{\hat{x}_4\}$ is the only candidate for satisfying the fourth equation. If we select $\hat{x}_4$ (node 9), then the objective value is updated to 1.2. Note that we cannot branch further on node 7 because we do not have any candidate for satisfying the fourth equation. So, we have three nodes (nodes 1, 4, and 8) to select for the next branching process. Considering the jump-tracking technique, we select node 1 to branch further.

Consider node 1. $\{\hat{x}_7\}$ is the only candidate for satisfying the second equation. If we select $\hat{x}_7$ (node 10), then the objective value is 1. Since $\hat{x}_1$ and $\hat{x}_7$ with $\check{x}_4$ and $\check{x}_8$ satisfy all equations, we do not branch further on this node (node 10). Hence, $\hat{x}_1, \check{x}_4, \hat{x}_7$, and $\check{x}_8$ is a solution with the objective value 1, which is better than the initial upper bound 1.2. We update the current upper bound as 1. Continuing this process, a tree with 18 nodes is generated as Figure 2. Hence, the optimal solution of problem (14) is as $x_1^* = 0.4$, $x_2^* = 0.25$, $x_3^* = 0.1$, $x_4^* = 0.4$, $x_5^* = 0.5$, $x_6^* = 0.4$, $x_7^* = 0.7$, and $x_8^* = 0.1$ with the objective value $Z'(x^*) = 1$.

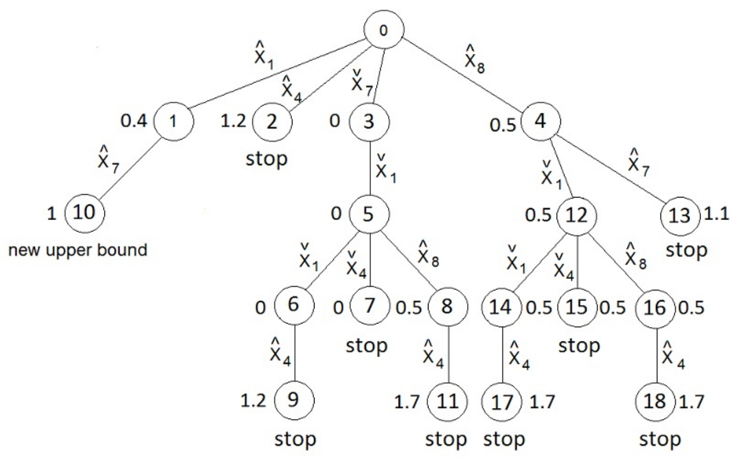

Fig. 2. Modified branch and bound method.

Step 12. The final optimal solution x^* of problem (13) is given as follows:
$x^* = (0.4, 0.25, 0.1, 0.4, 0.5, 0.4, 0.7, 0.1)^T$ with the optimal objective value $Z(x^*) = Z'(x^*) + 7.2 = 8.2$.

Example 5.2. Consider the following optimization problem:

$$\begin{aligned} &\text{Minimize} \quad Z(x) = x_1 + 2x_2 + 7x_3 + 4x_4 + 3x_5 + 4x_6 \\ &\text{Subject to} \quad a_i^+ \circ_{\gamma_i} x \vee a_i^- \circ_{\gamma_i} \neg x = d_i, \quad i = 1, \dots, 7, \\ &\qquad x \in [0,1]^6. \end{aligned} \tag{15}$$

Where

$$A^+ = \begin{pmatrix} 0.9 & 0.5 & 0.21 & 0.3 & 0.6 & 0.4 \\ 0.25 & 0.3 & 0.26 & 0.27 & 0.14 & 0.45 \\ 0.03 & 0.15 & 0.02 & 0.18 & 0.24 & 0.07 \\ 0.12 & 0.35 & 0.24 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.28 & 0.4 & 0.75 & 0.52 & 0.45 \\ 0.04 & 0.75 & 0.36 & 1 & 0.53 & 1 \\ 0.9 & 0.6 & 0.9 & 0.8 & 1 & 0.3 \end{pmatrix},$$

$$A^- = \begin{pmatrix} 0.25 & 0.42 & 0.35 & 0.15 & 0.4 & 0.3 \\ 0.3 & 0.2 & 0.14 & 0.36 & 0.25 & 0.07 \\ 0.08 & 0.2 & 0.15 & 0.04 & 0.1 & 0.36 \\ 0.24 & 0.4 & 0.45 & 0.3 & 0.6 & 0.72 \\ 0.2 & 0.5 & 0.35 & 0.75 & 0.42 & 1 \\ 1 & 0.35 & 0.75 & 0.27 & 1 & 0.3 \\ 0.5 & 0.9 & 0.24 & 0.4 & 0.32 & 0.51 \end{pmatrix},$$

$\gamma = (1,1,1,1,0,0,0)^T$, $d = (0.45, 0.27, 0.18, 0.36, 0.6, 0.75, 0.9)^T$, and $x = (x_1, x_2, x_3, x_4, x_5, x_6)^T$. Now, we want to solve this example by Algorithm 2.

Step 1. The lower and upper bound $\check{x}$ and $\hat{x}$ are as follows:
$\check{x} = (0.25, 0.1, 0.2, 0.25, 0.4, 0.5)^T$ and $\hat{x} = (0.5, 0.9, 1, 0.75, 0.75, 0.6)^T$.

Step 2. Since $d > 0$ and $\check{x} < \hat{x}$, we go to Step 3.

Step 3. The matrices Q^+ and Q^- are as follows:

$$Q^+ = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad Q^- = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Step 4. The system of bipolar fuzzy relation equations is consistent. So, we go to Step 5.

Step 5. The index sets I_j^+ and I_j^-, for all $j \in J$, are as follows:
$I_1^+ = \{1\}$, $I_2^+ = \{1,2\}$, $I_3^+ = \{7\}$, $I_4^+ = \{5,6\}$, $I_5^+ = \{1,3\}$, $I_6^+ = \{2\}$, $I_1^- = \{6\}$,

$I_2^- = \{3,4\}$, $I_3^- = \{4\}$, $I_4^- = \{2,5\}$, $I_5^- = \{4\}$, and $I_6^- = \{3,4\}$.
Also, the index sets J_i^+ and J_i^-, for all $i \in I$, are as follows:
$J_1^+ = \{1,2,5\}$, $J_2^+ = \{2,6\}$, $J_3^+ = \{5\}$, $J_4^+ = \emptyset$, $J_5^+ = \{4\}$, $J_6^+ = \{4\}$, $J_7^+ = \{3\}$, $J_1^- = \emptyset$, $J_2^- = \{4\}$, $J_3^- = \{2,6\}$, $J_4^- = \{2,3,5,6\}$, $J_5^- = \{4\}$, $J_6^- = \{1\}$, and $J_7^- = \emptyset$.

Step 6. Consider the following equivalent optimization problem:

$$\begin{aligned} \text{Minimize} \quad & Z'(x) = x_1 + 2x_2 + 7x_3 + 4x_4 + 3x_5 + 4x_6 - 6.05 \\ \text{Subject to} \quad & a_i^+ \circ_{\gamma_i} x \vee a_i^- \circ_{\gamma_i} \neg x = d_i, \qquad i = 1, \dots, 7, \\ & x \in [0,1]^6, \end{aligned} \tag{16}$$

where A^+, A^-, γ, d, and x are defined before. Now, we are ready to apply Rules 1-4 to reduce problem (16)

a) Since $I_2^+ \subseteq I_5^+ \cup I_6^+$, $I_2^- \supseteq I_5^- \cup I_6^-$, and $c_2(\hat{x}_2 - \check{x}_2) = 1.6 > 1.45 = c_5(\hat{x}_5 - \check{x}_5) + c_6(\hat{x}_6 - \check{x}_6)$, according to Rule 2, $x_2^* = \check{x}_2 = 0.1$. Also, the rows 3, 4, and the column 2 can be removed from the matrices Q^+ and Q^-.

b) Since $I_7^+ = \{3\}$ and $I_7^- = \emptyset$, then we can set $x_3^* = \hat{x}_3 = 1$ with regard to Rule 4. Therefore, the row 7 and the column 3 can be deleted from the matrices Q^+ and Q^-. After deletion in the parts (a) and (b), the matrices Q^+ and Q^- can be updated as follows:

$$Q^+ = \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad Q^- = \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Note that $I_1^+ \subseteq I_5^+$, $I_1^- \supseteq I_5^-$, and $c_1(\hat{x}_1 - \check{x}_1) = 0.25 \not> 1.05 = c_5(\hat{x}_5 - \check{x}_5)$. So, we cannot use Rule 2 in this case. The matrices Q^+ and Q^- cannot be reduced further. So, we can reformulate the objective function removing the determined variables $x_2^* = 0.1$ and $x_3^* = 1$. Hence, the reduced objective function is as follows: $x_1 + 4x_4 + 3x_5 + 4x_6 - 4.45$.

Step 7. Compute the initial upper bound U on the optimal objective value of the reduced objective function using Algorithm 1. First, we begin with the feasible solution $x = (\check{x}_1, \hat{x}_4, \hat{x}_5, \hat{x}_6)^T$. After using Algorithm 1, we obtain the feasible solution $x = (\check{x}_1, \check{x}_4, \hat{x}_5, \check{x}_6)^T$ with the initial upper bound $U = c_5(\hat{x}_5 - \check{x}_5) = 1.05$.

Step 8. Since $c_4(\hat{x}_4 - \check{x}_4) = 2 > 1.05 = U$, according to Rule 5, $x_4^* = \check{x}_4 = 0.25$. So, the rows 2, 5, and the column 4 can be deleted from the matrices Q^+ and Q^-. After deletion, the matrices Q^+ and Q^- can be updated as follows:

$$Q^+ = \begin{array}{c} \\ 1 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad Q^- = \begin{array}{c} \\ 1 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}.$$

Now, we can reduce the optimization problem (16) by Rules 1-4 again.

a) Since $I_6^+ \subseteq I_6^-$, then we can set $x_6^* = \check{x}_6 = 0.5$ and the column 6 can be deleted from the matrices Q^+ and Q^- with regard to Rule 1.
b) Since $J_6^- = \{1\}$ and $J_6^+ = \emptyset$, with regard to Rule 4, we set $x_1^* = \check{x}_1 = 0.25$. Also, the row 6 and the column 1 can be deleted from the matrices Q^+ and Q^-. After reduction of the matrices Q^+ and Q^-, we can use Rule 4 again. So, we can set $x_5^* = \hat{x}_5 = 0.75$ and go to Step 9.

Step 9. Since $Q^+ = \emptyset$ and $Q^- = \emptyset$, we go to Step 12.

Step 12. The final optimal solution x^* of problem (15) is given as follows:
$x^* = (0.25, 0.1, 1, 0.25, 0.75, 0.5)^T$ with the optimal objective value $Z(x^*) = Z'(x^*) + 6.05 = 12.7$.

6. CONCLUSIONS

The linear optimization subject to the bipolar fuzzy relation equation constraints with the max-parametric hamacher operators was studied. The structure of its feasible domain was determined by a finite number of maximal and minimal solution pairs. A necessary and sufficient condition was given for solution existence. The problem was converted to an equivalent programming problem. Some simplification procedures were presented for reduction of the problem. An algorithm was designed to find an upper bound for its optimal objective value. We modified the branch and bound method to solve the problem with regard to the upper bound. An algorithm was designed to solve the original problem with regard to the simplification procedures, the above algorithm, and the modified branch and bound method.

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