KYBERNETIKA — VOLUME 52 (2016), NUMBER 4, PAGES 514–530

ON THE RESOLUTION OF BIPOLAR MAX-MIN EQUATIONS

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This paper investigates bipolar max-min equations which can be viewed as a generalization of fuzzy relational equations with max-min composition. The relation between the consistency of bipolar max-min equations and the classical boolean satisfiability problem is revealed. Consequently, it is shown that the problem of determining whether a system of bipolar max-min equations is consistent or not is NP-complete. Moreover, a consistent system of bipolar max-min equations, as well as its solution set, can be fully characterized by a system of integer linear inequalities.

Keywords: bipolar max-min equations, fuzzy relational equations, satisfiability, linear inequalities

Classification: 90C70, 49M37

1. INTRODUCTION

Let $F = ([0,1], \vee, \wedge, \rightarrow, \neg)$ be the fuzzy algebra where $\neg$ is a unary operator on $[0,1]$ such that $\neg a = 1-a$ and $\vee$, $\wedge$, and $\rightarrow$ are binary operators on $[0,1]$ such that $a \vee b = \max\{a,b\}$, $a \wedge b = \min\{a,b\}$, and

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{otherwise.} \end{cases}$$

A finite system of bipolar max-min equations, first described in Freson et al. [4], is of the form

$$\bigvee_{j \in N} (a_{ij}^{+} \wedge x_j) \vee (a_{ij}^{-} \wedge \neg x_j) = b_i, \quad i \in M, \tag{1}$$

where $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$ are two index sets, and a_{ij}^{+}, a_{ij}^{-}, b_i, and x_j are all real numbers in $[0,1]$. Denote $A^{+} = (a_{ij}^{+})_{m \times n}$, $A^{-} = (a_{ij}^{-})_{m \times n}$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)^T$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, and $\neg\mathbf{x} = (1-x_1, 1-x_2, \ldots, 1-x_n)^T$, respectively. A given system of bipolar max-min equations, with $\mathbf{x}$ unknown, can be expressed in the matrix form as

$$A^{+} \circ \mathbf{x} \vee A^{-} \circ \neg\mathbf{x} = \mathbf{b} \tag{2}$$

DOI: 10.14736/kyb-2016-4-0514

where "$\circ$" denotes the max-min composite operation for matrix multiplication. Its solution set is denoted by $S(A^+, A^-, \mathbf{b})$, that is,

$$S(A^+, A^-, \mathbf{b}) = \{\mathbf{x} \in [0,1]^n | A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}\}.$$

Solving $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ is to determine its solution set $S(A^+, A^-, \mathbf{b})$. The system of bipolar max-min equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ is called consistent if $S(A^+, A^-, \mathbf{b}) \neq \emptyset$ and inconsistent otherwise.

Note that if either A^+ or A^- is a zero matrix, $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ degenerates into $A^- \circ \neg\mathbf{x} = \mathbf{b}$ or $A^+ \circ \mathbf{x} = \mathbf{b}$, respectively, i.e., a system of max-min equations which has been intensively investigated under the name of fuzzy relational equations. The consistency of $A^+ \circ \mathbf{x} = \mathbf{b}$ can be determined in polynomial time and its solution set, if not empty, can be characterized by a maximum solution and finitely many minimal solutions. The system $A^- \circ \neg\mathbf{x} = \mathbf{b}$ can be handled analogously and its solution set, if not empty, can be characterized by a minimum solution and finitely many maximal solutions. For some detailed discussion on fuzzy relational equations, see, e.g., Di Nola et al. [3], De Baets [2], Peeva and Kyosev [11], Li and Fang [7, 8], Li [6], and references therein.

The bipolar max-min equations and the associated linear optimization problem were first proposed and investigated by Freson et al. [4] with a potential application in revenue management. Since the solution set can be well characterized for each single equation of $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$, it follows that the desired solution set $S(A^+, A^-, \mathbf{b})$, whenever nonempty, can be characterized by a finite set of maximal and minimal solution pairs. Consequently, the linear optimization problem subject to a system of bipolar max-min equations can be solved by evaluating all those maximal and minimal solutions. However, this procedure is not computationally efficient since the number of maximal and minimal solution pairs could be exponentially large. Besides, the identification of these maximal and minimal solution pairs itself may not be an easy problem. In this paper, by combining the techniques developed in Li and Fang [7] and Li and Jin [9], we provide a reformulation approach to bipolar max-min equations and demonstrate that a system of bipolar max-min equations can be characterized by a system of integer linear inequalities. This implies that the bipolar max-min equation constrained optimization problems may be handled within the framework of integer and combinatorial optimization and hence demand no particular solving techniques.

The rest of this paper is organized as follows. The consistency issues of bipolar max-min equations are investigated in Section 2 via the polynomial-time reduction from the boolean satisfiability problem. It is shown that determining the consistency of bipolar max-min equations is NP-complete. An integer optimization based approach is applied in Section 3 to reformulate a system of bipolar max-min equations and characterize its solution set in a succinct manner. Some concluding remarks are presented in Section 4.

2. CONSISTENCY OF BIPOLAR MAX-MIN EQUATIONS

In this section, we apply some basic techniques originally developed for solving fuzzy relational equations, see, e.g., Li and Fang [7] and Li and Jin [9], to demonstrate that determining the consistency of a system of bipolar max-min equations is an NP-complete problem.

For a given system of bipolar max-min equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg \mathbf{x} = \mathbf{b}$, we may assume without loss of generality that $b_1 \geq b_2 \geq \cdots \geq b_m$, i. e., the equations are arranged such that the right hand side coefficients are in a decreasing order. Moreover, we may assume that $b_i > 0$ for all $i \in M$. Otherwise, any solution $\mathbf{x} \in S(A^+, A^-, \mathbf{b})$ must have $x_j = 0$ for $j \in N_0^+ = \{j \in N | a_{ij}^+ > 0, i \in M_0\}$ and $x_j = 1$ for $j \in N_0^- = \{j \in N | a_{ij}^- > 0, i \in M_0\}$ where $M_0 = \{i \in M | b_i = 0\} \neq \emptyset$. Hence, if $N_0^+ \cap N_0^- \neq \emptyset$, the system is inconsistent. Otherwise, it is a routine to delete the equations corresponding to M_0 and the columns in the coefficient matrices and the unknowns corresponding to N_0^+ and N_0^-. Any solution to the reduced system of bipolar max-min equations can be transformed into a solution to the original system by setting $x_j = 0$ for $j \in N_0^+$ and $x_j = 1$ for $j \in N_0^-$.

Before we tackle bipolar max-min equations, we should introduce some simple but fundamental results in Lemma 2.1, which, as well as their variants, play a key role in solving fuzzy relational equations of various types.

Lemma 2.1. For any $a, b \in [0, 1]$, it holds that $a \wedge x \leq b$ if and only if $x \leq a \to b$. Moreover, $a \wedge x = b$ has a solution if and only if $b \leq a$, in which case its solution set is the closed interval $[b, a \to b]$. Analogously, $a \wedge \neg x \leq b$ if and only if $x \geq 1 - (a \to b)$, while $a \wedge \neg x = b$ has a solution if and only if $b \leq a$, in which case its solution set is the closed interval $[1 - (a \to b), 1 - b]$.

Lemma 2.1 can be readily verified. It is actually the fact that the operators $\wedge$ and $\to$ form an adjoint pair over the unit interval. Note that the equation $a \wedge x = b$ or $a \wedge \neg x = b$ has multiple solutions only when $a = b < 1$. A direct consequence of Lemma 2.1 is that for any $a^+, a^-, b \in [0, 1]$,

$$(a^+ \wedge x) \vee (a^- \wedge \neg x) \leq b \tag{3}$$

if and only if

$$1 - (a^- \to b) \leq x \leq a^+ \to b. \tag{4}$$

Lemma 2.2. A vector $\mathbf{x} \in [0, 1]^n$ is a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \neg \mathbf{x} = \mathbf{b}$ if and only if $a_{ij}^+ \wedge x_j \leq b_i$ and $a_{ij}^- \wedge \neg x_j \leq b_i$ for all $i \in M$ and $j \in N$, and also there exists an index $j_i \in N$ for each $i \in M$ such that either $a_{ij_i}^+ \wedge x_{j_i} = b_i$ or $a_{ij_i}^- \wedge \neg x_{j_i} = b_i$.

Lemma 2.2 holds in a straightforward manner because the operator $\vee$ is non-interactive, i. e., $a \vee b \in \{a, b\}$ for any $a, b \in [0, 1]$. Lemmas 2.1 and 2.2 also indicate that if $A^+ \circ \mathbf{x} \vee A^- \circ \neg \mathbf{x} = \mathbf{b}$ is consistent, i. e., $S(A^+, A^-, \mathbf{b}) \neq \emptyset$, it necessarily holds that

$$b_i \leq \bigvee_{j \in N} a_{ij}^+ \vee a_{ij}^-, \quad \forall i \in M. \tag{5}$$

Furthermore, denote, respectively, $\check{\mathbf{x}} = (\check{x}_1, \check{x}_2, \ldots, \check{x}_n)^T$ with

$$\check{x}_j = \bigvee_{i \in M} (1 - a_{ij}^- \to b_i) = 1 - \bigwedge_{i \in M} (a_{ij}^- \to b_i) \tag{6}$$

and $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$ with

$$\hat{x}_j = \bigwedge_{i \in M} (a^+_{ij} \to b_i). \tag{7}$$

By Lemmas 2.1 and 2.2, whenever $S(A^+, A^-, \mathbf{b})$ is nonempty, it holds that $\check{\mathbf{x}} \leq \hat{\mathbf{x}}$ and $\check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}$ for any $\mathbf{x} \in S(A^+, A^-, \mathbf{b})$. This means that the vectors $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ serve the lower and upper bounds of the solutions to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$, respectively. Note that if $\check{x}_j = \hat{x}_j$ for some $j \in N$, the variable x_j in any possible solution would assume this unique value. As a consequence, the variable x_j and the equations such that either $a^+_{ij} \wedge \hat{x}_j = b_i$ or $a^-_{ij} \wedge \neg\check{x}_j = b_i$ can be omitted in further analysis, resulting in a system of bipolar max-min equations of a smaller size with strictly different lower and upper bounds on solutions. We may hereafter assume that the lower and upper bounds $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are strictly different, i. e., $\check{x}_j < \hat{x}_j$ for all $j \in N$, for the system $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ under consideration.

Nevertheless, the lower bound $\check{\mathbf{x}}$ and upper bound $\hat{\mathbf{x}}$ themselves may not necessarily be solutions to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ when $S(A^+, A^-, \mathbf{b}) \neq \emptyset$. Even if $\check{\mathbf{x}}, \hat{\mathbf{x}} \in S(A^+, A^-, \mathbf{b})$, it does not imply that every vector $\mathbf{x}$ such that $\check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}$ belongs to $S(A^+, A^-, \mathbf{b})$. Hence, a further study on the structure of $S(A^+, A^-, \mathbf{b})$ is required.

For $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$, two set-valued matrices $\tilde{Q}^+ = (\tilde{q}^+_{ij})_{m \times n}$ and $\tilde{Q}^- = (\tilde{q}^-_{ij})_{m \times n}$, called the characteristic matrices, can be constructed according to $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ such that

$$\tilde{q}^+_{ij} = \begin{cases} \{\hat{x}_j\}, & \text{if } a^+_{ij} \wedge \hat{x}_j = b_i, a^+_{ij} > b_i \\ [\check{x}_j \vee b_i, \hat{x}_j], & \text{if } a^+_{ij} \wedge \hat{x}_j = b_i, a^+_{ij} = b_i, \\ \emptyset & \text{otherwise,} \end{cases} \tag{8}$$

and

$$\tilde{q}^-_{ij} = \begin{cases} \{\check{x}_j\}, & \text{if } a^-_{ij} \wedge \neg\check{x}_j = b_i, a^-_{ij} > b_i \\ [\check{x}_j, \hat{x}_j \wedge \neg b_i], & \text{if } a^-_{ij} \wedge \neg\check{x}_j = b_i, a^-_{ij} = b_i, \\ \emptyset & \text{otherwise.} \end{cases} \tag{9}$$

By Lemmas 2.1 and 2.2, it is clear that the set $\tilde{q}^+_{ij} \cup \tilde{q}^-_{ij}$ contains all the possible values that the variable x_j may assume to meet the ith equality without violating the bound restrictions. As a consequence, $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ is consistent only if the merged characteristic matrix $\tilde{Q} = (\tilde{Q}^+, \tilde{Q}^-)$ contains at least one nonempty element in each row, while the converse is not true. However, as will be illustrated in Theorem 2.3, the matrix $\tilde{Q}$, along with $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$, does record all the critical information to characterize the solution set of $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$. Note that it is possible that $\tilde{q}^+_{ij} \cap \tilde{q}^-_{ij} \neq \emptyset$ for some $i \in M$ and $j \in N$, which means $a^+_{ij} \wedge x = a^-_{ij} \wedge \neg x = b_i$ for $x \in \tilde{q}^+_{ij} \cap \tilde{q}^-_{ij}$.

Theorem 2.3. A vector $\mathbf{x} \in [0, 1]^n$ is a solution to a system of bipolar max-min equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ if and only if $\check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}$ and its induced binary matrix $Q_x = (q^x_{ij})_{m \times n}$ has no zero rows where

$$q^x_{ij} = \begin{cases} 1, & \text{if } x_j \in \tilde{q}^+_{ij} \cup \tilde{q}^-_{ij}, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Proof. If $\mathbf{x} \in S(A^+, A^-, \mathbf{b}) \neq \emptyset$, it holds that $\check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}$ and also Q_x is well defined with respect to $\mathbf{x}$. Subsequently, by Lemma 2.2, there exists an index $j_i \in N$ for each $i \in M$ such that either $a^+_{ij_i} \wedge x_{j_i} = b_i$ or $a^-_{ij_i} \wedge \neg x_{j_i} = b_i$, which implies that $x_{j_i} \in \tilde{q}^+_{ij_i} \cup \tilde{q}^-_{ij_i}$ and hence $q^x_{ij_i} = 1$. Therefore, Q_x has no zero rows.

Conversely, if $\check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}$, then $a^+_{ij} \wedge x_j \leq b_i$ and $a^-_{ij} \wedge \neg x_j \leq b_i$ for each $i \in M$ and $j \in N$. Furthermore, according to the construction of Q_x, if Q_x has no zero rows, there exists an index $j_i \in N$ for each $i \in M$ such that either $a^+_{ij_i} \wedge x_{j_i} = b_i$ or $a^-_{ij_i} \wedge \neg x_{j_i} = b_i$. Hence, $\mathbf{x} \in S(A^+, A^-, \mathbf{b})$ by Lemma 2.2. □

Theorem 2.3 demonstrates the combinatorial nature of bipolar max-min equations by revealing the connection between the solutions to $A^+ \circ \mathbf{x} \vee A^- \circ \neg \mathbf{x} = \mathbf{b}$ and the characteristic matrices $\tilde{Q}^+$ and $\tilde{Q}^-$. However, to obtain such a solution, as well as to determine the solution set, is in general not easy because of the interaction of $\tilde{Q}^+$ and $\tilde{Q}^-$ in defining its induced binary matrix. Actually, a procedure used in Li and Jin [9] may be applied analogously in this context to illustrate the NP-completeness of determining the consistency of a system of bipolar max-min equations.

Theorem 2.4. The consistency problem of bipolar max-min equations is NP-complete.

Proof. It is clear that this problem is in NP. We show in this context that a boolean formula in conjunctive normal form can be viewed as a special system of bipolar max-min equations, which directly implies that determining whether a system of bipolar max-min equations is consistent or not is NP-complete.

Let $C_1, C_2, \ldots, C_m$ be a set of clauses over the boolean variables $\{y_1, y_2, \ldots, y_n\}$ and $C = \bigwedge_{i \in M} C_i$ a boolean formula in its conjunctive normal form. A clause is a disjunction of literals, while a literal is either a positive or a negative occurrence of a boolean variable, i. e., y_j or $\neg y_j$ for $j \in N$. Subsequently, define $\mathbf{b} = (1, 1, \ldots, 1)^T$ and $A^+ = (a^+_{ij})_{m \times n}$ and $A^- = (a^-_{ij})_{m \times n}$, respectively, as

$$a^+_{ij} = \begin{cases} 1, & \text{if } y_j \in C_i, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

and

$$a^-_{ij} = \begin{cases} 1, & \text{if } \neg y_j \in C_i, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Thus, a particular system of bipolar max-min equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg \mathbf{x} = \mathbf{b}$ may be formed, of which any solution $\mathbf{x} \in [0, 1]^n$ implies, by Lemma 2.2, that there exists an index $j_i \in N$ for each $i \in M$ such that either $a^+_{ij_i} \wedge x_{j_i} = 1$ or $a^-_{ij} \wedge \neg x_j = 1$, that is, either $y_{j_i} \in C_i$, $x_{j_i} = 1$ or $\neg y_{j_i} \in C_i$, $x_{j_i} = 0$. The boolean vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ with

$$y_j = \begin{cases} 1, & \text{if } x_j = 1, \\ 0, & \text{if } x_j = 0, \\ 0 \text{ or } 1, & \text{otherwise,} \end{cases} \tag{13}$$

is therefore a true assignment of $C = \bigwedge_{i\in M} C_i$. Conversely, if the boolean vector $\mathbf{y} \in \{0,1\}^n$ is a true assignment of $C = \bigwedge_{i\in M} C_i$, then $\mathbf{x} = \mathbf{y}$ is a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \neg \mathbf{x} = \mathbf{b}$ by Lemma 2.2.

As a consequence, the boolean satisfiability problem is polynomially reducible to the consistency problem of bipolar max-min equations and hence, the latter is NP-complete as well. □

Theorem 2.4 suggests a possible method to handle bipolar max-min equations within the framework of boolean satisfiability. Besides, by Theorem 2.3, we even don't need to recall the original bipolar max-min equations once we have obtained the information of the lower bound $\check{\mathbf{x}}$, the upper bound $\hat{\mathbf{x}}$, and the characteristic matrix $\tilde{Q}$. Moreover, if only the consistency issues are concerned, we may focus on the values contained in $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ to simplify the analysis.

For each $j \in N$, label the value $\hat{x}_j$ with the positive literal y_j and the value $\check{x}_j$ with the negative literal $\neg y_j$, respectively, which means that $x_j = \hat{x}_j$ implies $y_j = 1$ and $x_j = \check{x}_j$ implies $y_j = 0$, and vice versa.

Subsequently, denote, for each $i \in M$,

$$N_i^+ = \{j \in N | \hat{x}_j \in \tilde{q}_{ij}^+\} \quad \text{and} \quad N_i^- = \{j \in N | \check{x}_j \in \tilde{q}_{ij}^-\}, \tag{14}$$

and the clause

$$C_i = \bigvee_{j\in N_i^+} y_j \vee \bigvee_{j\in N_i^-} \neg y_j. \tag{15}$$

It is clear that the clause C_i is just an alternative representation of the ith row of $\tilde{Q}$ concerning only its nonempty elements. Note that N_i^+ and N_i^- are not necessarily disjoint since $\tilde{q}_{ij}^+$ and $\tilde{q}_{ij}^-$ can be simultaneously nonempty for some $j \in N$. In such a case, $a_{ij}^+ \wedge \hat{x}_j = a_{ij}^- \wedge \neg\check{x}_j = b_i$, which means that setting either $x_j = \hat{x}_j$ or $x_j = \check{x}_j$ would lead the ith equation to an equality. Consequently, the corresponding clause C_i, containing both y_j and $\neg y_j$, becomes a tautology and hence can be omitted as long as only the consistency is concerned. By this approach, it turns out that the consistency of $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ is fully determined by the satisfiability of $C = \bigwedge_{i\in M} C_i$, called its characteristic boolean formula.

Theorem 2.5. A system $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ is consistent if and only if its characteristic boolean formula $C = \bigwedge_{i\in M} C_i$ is satisfiable.

Proof. The boolean formula $C = \bigwedge_{i\in M} C_i$ is properly defined for $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ as long as $\check{\mathbf{x}} \le \hat{\mathbf{x}}$. According to Theorem 2.3, whenever $S(A^+, A^-, \mathbf{b})$ is nonempty, there must be a solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ such that either $x_j = \hat{x}_j$ or $x_j = \check{x}_j$ for each $j \in N$. Consequently, the associated boolean vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$ with

$$y_j = \begin{cases} 1, & \text{if } x_j = \hat{x}_j, \\ 0, & \text{if } x_j = \check{x}_j, \end{cases} \tag{16}$$

is a true assignment of $C = \bigwedge_{i\in M} C_i$. Conversely, for any true assignment $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$, the associated vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ with

$$x_j = \begin{cases} \hat{x}_j, & \text{if } y_j = 1, \\ \check{x}_j, & \text{if } y_j = 0, \end{cases} \tag{17}$$

is a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ by Theorem 2.3. Therefore, the consistency of $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ is equivalent to the satisfiability of $C = \bigwedge_{i\in M} C_i$. □

Example 2.6. Consider the system of bipolar max-min equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ with

$$A^+ = \begin{pmatrix} 0.9 & 0.7 & 0.8 & 0.9 \\ 0.9 & 0.2 & 0.9 & 0.7 \\ 0.4 & 0.8 & 0.4 & 0.4 \\ 0.4 & 0.3 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.3 & 0.2 \end{pmatrix}, \quad A^- = \begin{pmatrix} 0.9 & 0.7 & 0.4 & 0.9 \\ 0.2 & 0.8 & 0.9 & 0.8 \\ 0.5 & 0.4 & 0.8 & 0.5 \\ 0.4 & 0.7 & 0.6 & 0.8 \\ 0.3 & 0.5 & 0.2 & 0.1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0.8 \\ 0.8 \\ 0.6 \\ 0.5 \\ 0.4 \end{pmatrix}.$$

The lower bound $\check{\mathbf{x}}$ and upper bound $\hat{\mathbf{x}}$ can be calculated, respectively, as

$$\check{\mathbf{x}} = (0.2, 0.6, 0.5, 0.5)^T, \qquad \hat{\mathbf{x}} = (0.8, 0.6, 0.8, 0.8)^T,$$

and neither of them is a solution. Subsequently, the associated characteristic matrices can be calculated as

$$\tilde{Q}^+ = \begin{pmatrix} \{0.8\} & \emptyset & \{0.8\} & \{0.8\} \\ \{0.8\} & \emptyset & \{0.8\} & \emptyset \\ \emptyset & \{0.6\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \{0.6\} & \emptyset & \emptyset \end{pmatrix}, \quad \tilde{Q}^- = \begin{pmatrix} \{0.2\} & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \{0.5\} & \{0.5\} \\ \emptyset & \{0.6\} & \emptyset & \emptyset \end{pmatrix}.$$

Notice that $\check{x}_2 = \hat{x}_2 = 0.6$ which means in any possible solution the variable x_2 can only assume the value 0.6. Moreover, the equalities hold for the third and fifth equations with $x_2 = 0.6$ such that $0.8 \wedge 0.6 = 0.6$ and $0.4 \wedge 0.6 = 0.5 \wedge (1 - 0.6) = 0.4$, respectively. Consequently, the variable x_2 can be omitted in further analysis together with the third and fifth equations. Besides, the first equation can be omitted as well for consistency checking because both $\tilde{q}^+_{11}$ and $\tilde{q}^-_{11}$ are nonempty. Therefore, the characteristic boolean formula can be constructed and simplified as

$$(y_1 \vee y_3) \wedge (\neg y_3 \vee \neg y_4)$$

which is satisfiable by assigning, e. g., $y_3 = 1$ and $y_4 = 0$. The vector $\mathbf{x} = (0.8, 0.6, 0.8, 0.5)^T$ can be constructed accordingly and is indeed a solution to the given system of bipolar max-min equations. Actually, it can be further verified that the first component of this solution may assume any value between 0.2 and 0.8.

Example 2.6 is adapted from the example in Li and Jin [9]. It illustrates that the two types of equations, min-biimplication equations and bipolar max-min equations, may share a common essential structure despite their different appearances. This issue is further addressed in Section 3.1. Besides, as a direct consequence of Theorem 2.5, the consistency of bipolar max-min equations may be solved as the classical boolean satisfiability problem using the current state-of-the-art SAT solvers, e. g., Chaff, BerkMin, SATO, and Siege.

3. SOLUTION SETS OF BIPOLAR MAX-MIN EQUATIONS

The problem of determining the solution set $S(A^+, A^-, \mathbf{b})$ of a system of bipolar max-min equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ is a little complicated since its characteristic matrix $\tilde{Q}$ may involve two types of nonempty elements, i. e., singletons and intervals. Besides, those omitted equations corresponding to the tautologies in the characteristic boolean formula should be taken into consideration as well because the components of a solution $\mathbf{x} \in S(A^+, A^-, \mathbf{b})$ may assume the values that are not contained in $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$. It turns out that a system of integer linear inequalities is sufficient to characterize $S(A^+, A^-, \mathbf{b})$ by applying the techniques developed in Li and Fang [7] and Li and Jin [9]. Moreover, if the nonempty elements in $\tilde{Q}$ are all singletons, e. g., Example 2.6, the situation is somehow easier to deal with as is illustrated analogously in Li and Jin [9] for min-biimplication equations.

3.1. Simple scenarios of bipolar max-min equations

For a system of bipolar max-min equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$, we assume that its merged characteristic matrix $\tilde{Q} = (\tilde{Q}^+, \tilde{Q}^-)$ contains only singletons as the nonempty elements. In such a case, those singletons in $\tilde{Q}$ just duplicate the values in $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$. Consequently, $\tilde{Q}$ can be reduced into a binary matrix $Q = (q_{ij})_{m\times 2n}$ such that for each $j \in N$,

$$q_{ij} = \begin{cases} 1, & \text{if } \hat{x}_j \in \tilde{q}^+_{ij}, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad q_{i(n+j)} = \begin{cases} 1, & \text{if } \check{x}_j \in \tilde{q}^-_{ij}, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Define the binary vector $\mathbf{u} = (u_1, u_2, \ldots, u_{2n})^T$ such that u_j and u_{n+j} are labeled with $\hat{x}_j$ and $\check{x}_j$, respectively, for each $j \in N$. In this manner, $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ may be reformulated into a system of integer linear inequalities $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$, where

$$G = \begin{pmatrix} 1 & & & & 1 & & & \\ & 1 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & & & & 1 \end{pmatrix}_{n\times 2n}$$

with unspecified elements being zero and $\mathbf{e}_m$ and $\mathbf{e}_n$ are the m-dimensional and n-dimensional vectors of all ones, respectively. According to Theorems 2.3 and 2.5, a binary vector $\mathbf{u} \in \{0,1\}^{2n}$ subject to $Q\mathbf{u} \geq \mathbf{e}_m$ is necessary to induce a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$. The constraints of $G\mathbf{u} \leq \mathbf{e}_n$ are routinely imposed to eliminate those unqualified binary vectors so that at most one value specified in $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ could be used for each single variable in the solution construction. Note that those values contained in $\check{\mathbf{x}}$ and $\hat{\mathbf{x}}$ but not in $\tilde{Q}$ play no role in the further analysis because their corresponding columns in Q contain only zeros.

Theorem 3.1. Let $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ be a system of bipolar max-min equations such that all the nonempty elements in its merged characteristic matrix $\tilde{Q}$ are singletons. It is consistent if and only if its characteristic system of linear inequalities $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ is consistent.

Proof. For any given solution $\mathbf{x} \in S(A^+, A^-, \mathbf{b}) \neq \emptyset$, there exists, by Lemma 2.2, an index $j_i \in N$ for each $i \in M$ such that either $a^+_{ij_i} \wedge x_{j_i} = b_i$ or $a^-_{ij_i} \wedge \neg x_{j_i} = b_i$. Under the assumption that all nonempty elements in $\tilde{Q}$ are singletons, this implies that either $x_{j_i} = \hat{x}_{j_i}$, $q_{i,j_i} = 1$ or $x_{j_i} = \check{x}_{j_i}$, $q_{i,n+j_i} = 1$, but not both. Hence, the binary vector $\mathbf{u} = (u_1, u_2, \ldots, u_{2n})^T$ with

$$u_j = \begin{cases} 1, & \text{if } x_j = \hat{x}_j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad u_{n+j} = \begin{cases} 1, & \text{if } x_j = \check{x}_j, \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

for $j \in N$, satisfies both $Q\mathbf{u} \geq \mathbf{e}_m$ and $G\mathbf{u} \leq \mathbf{e}_n$. Conversely, if $\mathbf{u} \in \{0,1\}^{2n}$ satisfies both $Q\mathbf{u} \geq \mathbf{e}_m$ and $G\mathbf{u} \leq \mathbf{e}_n$, we may define two disjoint index sets

$$\text{supp}^+(\mathbf{u}) = \{j \in N | u_j = 1\} \quad \text{and} \quad \text{supp}^-(\mathbf{u}) = \{j \in N | u_{n+j} = 1\}. \tag{20}$$

Consequently, the induced vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ with

$$x_j = \begin{cases} \hat{x}_j, & \text{if } j \in \text{supp}^+(\mathbf{u}), \\ \check{x}_j, & \text{if } j \in \text{supp}^-(\mathbf{u}), \\ \check{x}_j \text{ or } \hat{x}_j, & \text{otherwise,} \end{cases} \tag{21}$$

is a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ by Theorem 2.3. Therefore, $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ is consistent if and only if the system $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ is consistent whenever all the nonempty elements of $\tilde{Q}$ are singletons. □

As illustrated in the proof of Theorem 3.1, each solution $\mathbf{u} \in \{0,1\}^{2n}$ to $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ induces a subset of $S(A^+, A^-, \mathbf{b})$ bounded by $\check{\mathbf{v}} = (\check{v}_1, \check{v}_2, \ldots, \check{v}_n)^T$ and $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)^T$ where for $j \in N$,

$$\check{v}_j = \begin{cases} \hat{x}_j, & \text{if } j \in \text{supp}^+(\mathbf{u}), \\ \check{x}_j, & \text{otherwise,} \end{cases} \quad \text{and} \quad \hat{v}_j = \begin{cases} \check{x}_j, & \text{if } j \in \text{supp}^-(\mathbf{u}), \\ \hat{x}_j, & \text{otherwise.} \end{cases} \tag{22}$$

According to Theorem 2.3, any vector $\mathbf{x} \in [0,1]^n$ such that $\check{\mathbf{v}} \leq \mathbf{x} \leq \hat{\mathbf{v}}$ is a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$. It is clear that the union of all such subsets of solutions forms the solution set $S(A^+, A^-, \mathbf{b})$. However, in order to remove the redundancy in such a representation of $S(A^+, A^-, \mathbf{b})$, we need figure out the minimal solutions of $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$. By a minimal solution $\check{\mathbf{u}}$ to $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$, we mean any solution $\mathbf{u}$ such that $\mathbf{u} \leq \check{\mathbf{u}}$ would imply $\mathbf{u} = \check{\mathbf{u}}$. Of course, obtaining all minimal solutions of a system of integer linear inequalities itself is a computationally difficult problem and requires some sophisticated enumeration techniques. Denote

$$\check{S}(Q,G) = \{\check{\mathbf{u}}^k | k = 1,2,\ldots,|\check{S}(Q,G)|\} \tag{23}$$

the set of all minimal solutions of $Q\mathbf{u} \geq \mathbf{u}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ and denote $\check{\mathbf{v}}^k$ and $\hat{\mathbf{v}}^k$ the lower and upper bounds induced by $\check{\mathbf{u}}^k \in \check{S}(Q,G)$, respectively. It is clear that the solutions induced by a pair of $\check{\mathbf{v}}^k$ and $\hat{\mathbf{v}}^k$ may have the number of unfixed components as many as possible, all of which form as well the solution set $S(A^+,A^-,\mathbf{b})$.

Theorem 3.2. Let $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ be a consistent system of bipolar max-min equations. If all the nonempty elements in its merged characteristic matrix $\tilde{Q}$ are singletons, then

$$S(A^+,A^-,\mathbf{b}) = \bigcup_{\check{\mathbf{u}}^k \in \check{S}(Q,G)} \left\{\mathbf{x} \in [0,1]^n | \check{\mathbf{v}}^k \leq \mathbf{x} \leq \hat{\mathbf{v}}^k\right\}. \tag{24}$$

Proof. It suffices to show that any solution $\mathbf{x} \in S(A^+,A^-,\mathbf{b})$ can be induced by some minimal solution $\check{\mathbf{u}} \in \check{S}(Q,G)$. For any $\mathbf{x} \in S(A^+,A^-,\mathbf{b})$, by Theorem 3.1, a binary vector $\mathbf{u} \in \{0,1\}^{2n}$ can be constructed that satisfies $Q\mathbf{u} \geq \mathbf{e}_m$ and $G\mathbf{u} \leq \mathbf{e}_n$ as well as a minimal solution $\check{\mathbf{u}} \in \check{S}(Q,G)$ such that $\check{\mathbf{u}} \leq \mathbf{u}$. Subsequently, the pair of $\check{\mathbf{v}}$ and $\hat{\mathbf{v}}$ may be constructed with respect to $\check{\mathbf{u}}$. It holds that $x_j = \check{v}_j$ if $j \in \mathrm{supp}^-(\check{\mathbf{u}})$ and $x_j = \hat{v}_j$ if $j \in \mathrm{supp}^+(\check{\mathbf{u}})$ and hence, $\check{\mathbf{v}} \leq \mathbf{x} \leq \hat{\mathbf{v}}$. Therefore, $S(A^+,A^-,\mathbf{b})$ may be determined by the solutions induced via $\check{S}(Q,G)$. □

Example 3.3. (Example 2.6 continued) For the system of bipolar max-min equations considered in Example 2.6, after removing the variable x_2 and the third and fifth equations, the corresponding characteristic system of linear inequalities $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ can be written as

$$\begin{array}{c} \begin{matrix} \hat{x}_1 & \hat{x}_3 & \hat{x}_4 & \check{x}_1 & \check{x}_3 & \check{x}_4 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix} \end{array} \begin{pmatrix} u_1 \\ u_3 \\ u_4 \\ u_5 \\ u_7 \\ u_8 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

The total three minimal solutions can be identified, respectively, as

$$\check{\mathbf{u}}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \check{\mathbf{u}}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \check{\mathbf{u}}^3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

as well as their corresponding induced pairs of solutions, the variable x_2 included,

$$\check{\mathbf{v}}^1 = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.5 \\ 0.5 \end{pmatrix}, \quad \check{\mathbf{v}}^2 = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.5 \\ 0.5 \end{pmatrix}, \quad \check{\mathbf{v}}^3 = \begin{pmatrix} 0.2 \\ 0.6 \\ 0.8 \\ 0.5 \end{pmatrix}$$

$$\hat{\mathbf{v}}^1 = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.5 \\ 0.8 \end{pmatrix}, \quad \hat{\mathbf{v}}^2 = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.8 \\ 0.5 \end{pmatrix}, \quad \hat{\mathbf{v}}^3 = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.8 \\ 0.5 \end{pmatrix}$$

Therefore, the solution set is

$$S(A^+, A^-, \mathbf{b}) = \bigcup_{k=1,2,3} \{\mathbf{x} \in [0,1]^4 | \check{\mathbf{v}}^k \le \mathbf{x} \le \hat{\mathbf{v}}^k\}.$$

By Example 3.3, we once again point out that the min-biimplication equations discussed in Li and Jin [9], without referring to their particular logical interpretations, may be viewed as a special scenario of bipolar max-min equations after the hidden essential nature has been revealed.

3.2. General scenarios of bipolar max-min equations

When the merged characteristic matrix $\tilde{Q}$ of a system of bipolar max-min equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ contains interval elements, it requires more binary variables to label the critical values at those interval endpoints. However, the method developed in Li and Fang [7] can be naturally extended to handle these general scenarios of bipolar max-min equations.

Note that the nonempty elements in each column of $\tilde{Q}^+$ share a common value at the right endpoints and hence it suffices to consider the values at the left endpoints. Denote r_j^+ the number of different values in the set $\{\check{x}_j \vee b_i | a_{ij}^+ \wedge \hat{x}_j = b_i, i \in M\}$ for each $j \in N$, and $\check{v}_{jk}$, $k \in K_j^+ = \{1, 2, \dots, r_j^+\}$, these different values. Arranging all these values in a sequence, we obtain a vector $\check{\mathbf{v}}^+ = (\check{v}_{11}, \dots, \check{v}_{1r_1^+}, \dots, \check{v}_{n1}, \dots, \check{v}_{nr_n^+})^T$. For each $j \in N$ and $k \in K_j^+$, the position of $\check{v}_{jk}$ in $\check{\mathbf{v}}^+$ is $k' = \sigma_j^+(k) = \sum_{s=1}^{j-1} r_s^+ + k$. Subsequently, $\tilde{Q}^+$ can be represented by a binary matrix $Q^+ = (q_{ij}^+)_{m \times n^+}$ where $n^+ = \sum_{j \in N} r_j^+$ and for each $j \in N$ and $k \in K_j^+$,

$$q_{ik'}^+ = \begin{cases} 1, & \text{if } k' = \sigma_j^+(k), \check{v}_{jk} \in \tilde{q}_{ij}^+, \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

Besides, an accompanied binary matrix $G^+ = (g_{ij}^+)_{n \times n^+}$ should be constructed such that for each $j \in N$ and $k \in K_j^+$,

$$g_{jk'}^+ = \begin{cases} 1, & \text{if } k' = \sigma_j^+(k), \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

The matrices Q^+ and G^+ are called in Li and Fang [7] the augmented characteristic matrix and the inner-variable incompatibility matrix, respectively.

It is clear that $\tilde{Q}^-$ can be handled in an analogous manner. Since the nonempty elements in each column of $\tilde{Q}^-$ share a common value at the left endpoints, only the values at the right endpoints of these intervals are of concern. Denote r_j^- the number of different values in the set $\{\hat{x}_j \wedge \neg b_i | a_{ij}^- \wedge \neg \check{x}_j = b_i, i \in M\}$ for each $j \in N$, and $\hat{v}_{jk}$, $k \in K_j^- = \{1, 2, \ldots, r_j^-\}$, these different values. Analogously, we obtain a vector $\hat{\mathbf{v}}^- = (\hat{v}_{11}, \ldots, \hat{v}_{1r_1^-}, \ldots, \hat{v}_{n1}, \ldots, \hat{v}_{nr_n^-})^T$ by arranging all these values in a sequence. For each $j \in N$ and $k \in K_j^-$, the position of $\hat{v}_{jk}$ in $\hat{\mathbf{v}}^-$ is $k' = \sigma_j^-(k) = \sum_{s=1}^{j-1} r_s^- + k$. Subsequently, the augmented characteristic matrix $Q^- = (q_{ij}^-)_{m \times n^-}$ can be defined with respect to $\tilde{Q}^-$ where $n^- = \sum_{j \in N} r_j^-$ and for each $j \in N$ and $k \in K_j^-$,

$$q_{ik'}^- = \begin{cases} 1, & \text{if } k' = \sigma_j^-(k), \hat{v}_{jk} \in \tilde{q}_{ij}^-, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

The corresponding inner-variable incompatibility matrix $G^- = (g_{ij}^-)_{n \times n^-}$ can be constructed as well such that for each $j \in N$ and $k \in K_j^-$,

$$g_{jk'}^- = \begin{cases} 1, & \text{if } k' = \sigma_j^-(k), \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

Let $\mathbf{u} \in \{0,1\}^{n^+ + n^-}$ be a binary vector such that each component is labeled with a value in $\check{\mathbf{v}}^+$ or $\hat{\mathbf{v}}^-$. A system of integer linear inequalities $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ can be defined for $A^+ \circ \mathbf{x} \vee A^- \circ \neg \mathbf{x} = \mathbf{b}$ where $Q = (Q^+, Q^-)$ and $G = (G^+, G^-)$ are the merged augmented characteristic matrix and the merged inner-variable incompatibility matrix, respectively. Analogous to Theorems 3.1 and 3.2, this system of integer linear inequalities fully characterizes the solution set of $A^+ \circ \mathbf{x} \vee A^- \circ \neg \mathbf{x} = \mathbf{b}$.

Theorem 3.4. Let $A^+ \circ \mathbf{x} \vee A^- \circ \neg \mathbf{x} = \mathbf{b}$ be a system of bipolar max-min equations with Q and G being its merged augmented characteristic matrix and inner-variable incompatibility matrix, respectively. It is consistent if and only if its characteristic system of linear inequalities $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ is consistent.

Proof. For any given solution $\mathbf{x} \in S(A^+, A^-, \mathbf{b}) \neq \emptyset$, define for each $j \in N$ and $k \in K_j^+$,

$$u_{jk} = \begin{cases} 1, & \text{if } k = \operatorname{argmax}_{k \in K_j^+} \{\check{v}_{jk} \leq x_j\}, \\ 0, & \text{otherwise,} \end{cases} \tag{29}$$

and for each $j \in N$ and $k \in K_j^-$,

$$u_{jk} = \begin{cases} 1, & \text{if } k = \operatorname{argmin}_{k \in K_j^-} \{\hat{v}_{jk} \geq x_j\}, \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

Because it is possible that $\max\{\hat{v}_{jk}|k \in K_j^-\} \geq \min\{\check{v}_{jk}|k \in K_j^+\}$ for some $j \in N$, a slight modification is occasionally needed such that the obtained binary vector $\mathbf{u} \in \{0,1\}^{n^+ + n^-}$ satisfies $G\mathbf{u} \leq \mathbf{e}_n$. Besides, by Theorem 2.3, it holds that for each $i \in M$ either

$$\sum_{j\in N} \sum_{k\in K_j^+, k'=\sigma_j^+(k)} q_{ik'}^+ u_{jk} \geq 1$$

or

$$\sum_{j\in N} \sum_{k\in K_j^-, k'=\sigma_j^-(k)} q_{ik'}^- u_{jk} \geq 1$$

i. e., $Q\mathbf{u} \geq \mathbf{e}_m$. Conversely, if $\mathbf{u} \in \{0,1\}^{n^+ + n^-}$ satisfies both $Q\mathbf{u} \geq \mathbf{e}_m$ and $G\mathbf{u} \leq \mathbf{e}_n$, we may define two disjoint index sets

$$\text{supp}^+(\mathbf{u}) = \{j \in N | \textstyle\sum_{k\in K_j^+} u_{jk} = 1\} \tag{31}$$

and

$$\text{supp}^-(\mathbf{u}) = \{j \in N | \textstyle\sum_{k\in K_j^-} u_{jk} = 1\}. \tag{32}$$

Consequently, the induced vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ with

$$x_j = \begin{cases} \sum_{k\in K_j^+} \check{v}_{jk} u_{jk}, & \text{if } j \in \text{supp}^+(\mathbf{u}), \\ \sum_{k\in K_j^-} \hat{v}_{jk} u_{jk}, & \text{if } j \in \text{supp}^-(\mathbf{u}), \\ \check{x}_j \text{ or } \hat{x}_j, & \text{otherwise}, \end{cases} \tag{33}$$

is a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ by Theorem 2.3. Therefore, $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ is consistent if and only if the system $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ is consistent. □

Note that although it has a same matrix representation with its counterpart in Section 3.1, the system $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ addressed here usually has a larger size. The proof of Theorem 3.4 indicates that each solution $\mathbf{u} \in \{0,1\}^{n^+ + n^-}$ to $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$ induces a subset of $S(A^+, A^-, \mathbf{b})$ bounded by $\check{\mathbf{v}} = (\check{v}_1, \check{v}_2, \ldots, \check{v}_n)^T$ and $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)^T$ where for $j \in N$,

$$\check{v}_j = \begin{cases} \sum_{k\in K_j^+} \check{v}_{jk} u_{jk}, & \text{if } j \in \text{supp}^+(\mathbf{u}), \\ \check{x}_j, & \text{otherwise}, \end{cases} \tag{34}$$

and

$$\hat{v}_j = \begin{cases} \sum_{k\in K_j^-} \hat{v}_{jk} u_{jk}, & \text{if } j \in \text{supp}^-(\mathbf{u}), \\ \hat{x}_j, & \text{otherwise}. \end{cases} \tag{35}$$

According to Theorem 2.3, any vector $\mathbf{x} \in [0,1]^n$ such that $\check{\mathbf{v}} \leq \mathbf{x} \leq \hat{\mathbf{v}}$ is a solution to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$. Consequently, we may focus on the minimal solutions of $Q\mathbf{u} \geq \mathbf{e}_m$,

$G\mathbf{u} \leq \mathbf{e}_n$ in order to obtain a compact representation of $S(A^+, A^-, \mathbf{b})$. Analogously, denote

$$\check{S}(Q,G) = \{\check{\mathbf{u}}^k | k = 1,2,\ldots,|\check{S}(Q,G)|\} \tag{36}$$

the set of all minimal solutions of $Q\mathbf{u} \geq \mathbf{e}_m$, $G\mathbf{u} \leq \mathbf{e}_n$. For each $\check{\mathbf{u}}^k \in \check{S}(Q,G)$, it induces a pair of lower and upper bound solutions $\check{\mathbf{v}}^k$ and $\hat{\mathbf{v}}^k$. The solution set $S(A^+, A^-, \mathbf{b})$ is then readily determined by these pairs of solutions induced via $\check{S}(Q,G)$.

Theorem 3.5. Let $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ be a consistent system of bipolar max-min equations with Q and G being its merged augmented characteristic matrix and inner-variable incompatibility matrix, respectively. The solution set $S(A^+, A^-, \mathbf{b})$ is determined by

$$S(A^+, A^-, \mathbf{b}) = \bigcup_{\check{\mathbf{u}}^k \in \check{S}(Q,G)} \{\mathbf{x} \in [0,1]^n | \check{\mathbf{v}}^k \leq \mathbf{x} \leq \hat{\mathbf{v}}^k\}. \tag{37}$$

Proof. For any $\mathbf{x} \in S(A^+, A^-, \mathbf{b})$, a binary vector $\mathbf{u} \in \{0,1\}^{n^+ + n^-}$ can be constructed as in Theorem 3.4 which satisfies $Q\mathbf{u} \geq \mathbf{e}_m$ and $G\mathbf{u} \leq \mathbf{e}_n$. A minimal solution $\check{\mathbf{u}} \in \check{S}(Q,G)$ can be obtained such that $\check{\mathbf{u}} \leq \mathbf{u}$, which induces a pair of lower and upper bound solutions $\check{\mathbf{v}}$ and $\hat{\mathbf{v}}$. It holds that $x_j \leq \sum_{k \in K_j^-} \hat{v}_{jk} u_{jk}$ if $j \in \operatorname{supp}^-(\check{\mathbf{u}})$ and $x_j \geq \sum_{k \in K_j^+} \check{v}_{jk} u_{jk}$ if $j \in \operatorname{supp}^+(\check{\mathbf{u}})$ and hence, $\check{\mathbf{v}} \leq \mathbf{x} \leq \hat{\mathbf{v}}$. Therefore, $S(A^+, A^-, \mathbf{b})$ may be determined by the solutions induced via $\check{S}(Q,G)$. □

Example 3.6. Consider the system of bipolar max-min equations $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$ with

$$A^+ = \begin{pmatrix} 0.9 & 0.7 & 0.9 \\ 0.4 & 0.6 & 0.5 \\ 0.4 & 0.2 & 0.3 \end{pmatrix}, \qquad A^- = \begin{pmatrix} 0.9 & 0.7 & 0.9 \\ 0.5 & 0.4 & 0.8 \\ 0.4 & 0.2 & 0.3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix}.$$

The lower bound $\check{\mathbf{x}}$ and upper bound $\hat{\mathbf{x}}$ are, respectively,

$$\check{\mathbf{x}} = (0.2, 0, 0.4)^T, \qquad \hat{\mathbf{x}} = (0.8, 1.0, 0.8)^T,$$

and both of them are indeed solutions to $A^+ \circ \mathbf{x} \vee A^- \circ \neg\mathbf{x} = \mathbf{b}$. However, it can be verified that $\mathbf{x} = (0.6, 0.6, 0.6)^T$ is not a solution but satisfies $\check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}$. Subsequently, the characteristic matrices can be calculated as

$$\tilde{Q}^+ = \begin{pmatrix} \{0.8\} & \emptyset & \{0.8\} \\ \emptyset & [0.6, 1.0] & \emptyset \\ [0.4, 0.8] & \emptyset & \emptyset \end{pmatrix}, \qquad \tilde{Q}^- = \begin{pmatrix} \{0.2\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{0.4\} \\ [0.2, 0.6] & \emptyset & \emptyset \end{pmatrix}.$$

Accordingly, denote

$$\check{\mathbf{v}}^+ = (\check{v}_{11}, \check{v}_{12}, \check{v}_{21}, \check{v}_{31})^T = (0.8, 0.4, 0.6, 0.8)^T$$

and

$$\hat{\mathbf{v}}^- = (\hat{v}_{11}, \hat{v}_{12}, \hat{v}_{31})^T = (0.2, 0.6, 0.4)^T,$$

respectively. The corresponding characteristic system of linear inequalities becomes

$$\begin{array}{c} \begin{matrix} \check{v}_{11} & \check{v}_{12} & \check{v}_{21} & \check{v}_{31} & \hat{v}_{11} & \hat{v}_{12} & \hat{v}_{31} \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix} \end{array} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

There are six minimal solutions in total, i. e.,

$$\check{\mathbf{u}}^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \check{\mathbf{u}}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \check{\mathbf{u}}^3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\check{\mathbf{u}}^4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \check{\mathbf{u}}^5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \check{\mathbf{u}}^6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The corresponding induced pairs of solutions are, respectively,

$$\check{\mathbf{v}}^1 = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix}, \quad \check{\mathbf{v}}^2 = \begin{pmatrix} 0.8 \\ 0 \\ 0.4 \end{pmatrix}, \quad \check{\mathbf{v}}^3 = \begin{pmatrix} 0.4 \\ 0.6 \\ 0.8 \end{pmatrix},$$

$$\check{\mathbf{v}}^4 = \begin{pmatrix} 0.2 \\ 0.6 \\ 0.4 \end{pmatrix}, \quad \check{\mathbf{v}}^5 = \begin{pmatrix} 0.2 \\ 0 \\ 0.4 \end{pmatrix}, \quad \check{\mathbf{v}}^6 = \begin{pmatrix} 0.2 \\ 0.6 \\ 0.8 \end{pmatrix},$$

$$\hat{\mathbf{v}}^1 = \begin{pmatrix} 0.8 \\ 1.0 \\ 0.8 \end{pmatrix}, \quad \hat{\mathbf{v}}^2 = \begin{pmatrix} 0.8 \\ 1.0 \\ 0.4 \end{pmatrix}, \quad \hat{\mathbf{v}}^3 = \begin{pmatrix} 0.8 \\ 1.0 \\ 0.8 \end{pmatrix},$$

$$\hat{\mathbf{v}}^4 = \begin{pmatrix} 0.2 \\ 1.0 \\ 0.8 \end{pmatrix}, \quad \hat{\mathbf{v}}^5 = \begin{pmatrix} 0.2 \\ 1.0 \\ 0.4 \end{pmatrix}, \quad \hat{\mathbf{v}}^6 = \begin{pmatrix} 0.6 \\ 1.0 \\ 0.8 \end{pmatrix}.$$

Therefore, the solution set is

$$S(A^+, A^-, \mathbf{b}) = \bigcup_{k \in \{1,2,\ldots,6\}} \left\{ \mathbf{x} \in [0,1]^3 | \check{\mathbf{v}}^k \leq \mathbf{x} \leq \hat{\mathbf{v}}^k \right\}.$$

According to Theorems 3.2 and 3.5, we need enumerate all the minimal solutions of a system of integer linear inequalities, of which the number could be exponentially large, in order to express the solution set of bipolar max-min equations in a compact form. This is not surprising because as illustrated in this section a system of bipolar max-min equations may be viewed as a disguised or generalized form of a boolean formula depending on the pattern of its characteristic matrix. The practically well performed enumeration techniques are separate research issues and beyond the scope of this paper. The reader may refer to Johnson et al. [5], Palopoli et al. [10], and Crama and Hammer [1] for the discussion on these issues.

4. CONCLUDING REMARKS

The system of bipolar max-min equations, originally described by Freson et al. [4], has been investigated in this paper as a generalization of fuzzy relational equations with max-min composition. It is demonstrated that determining the consistency of a system of bipolar max-min equations is NP-complete while a compact representation of its solution set requires the enumeration of all minimal solutions of a system of integer linear inequalities. It is clear that the techniques presented in this paper work for bipolar max-T equations in an analogous manner where T is a continuous triangular norm.

When the linear optimization problem is considered subject to a consistent system of bipolar max-min equations, it is inevitably an NP-hard problem because the max-min equation constrained linear optimization problem is already NP-hard as illustrated by Li and Fang [7]. However, as also observed by Freson et al. [4], for such a linear optimization problem there exists an optimal solution whose components assume only the values specified in the lower and upper bounds and the endpoints of intervals in the characteristic matrices. Consequently, by applying Theorems 3.1 and 3.4, the bipolar max-min equation constrained linear optimization problem can be reformulated into a linear integer optimization problem and then handled taking advantage of the well developed techniques in combinatorial optimization and integer optimization.

ACKNOWLEDGEMENT

The first author was partially supported by the National Natural Science Foundation of China under Grant 61203131 and the Tsinghua University Initiative Scientific Research Program 2014z21017. The second author was partially supported by the National Natural Science Foundation of China under Grant 11301479 and Zhejiang Provincial Natural Science Foundation of China under Grant LQ13A010001.

(Received June 26, 2015)

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