

# CHARACTERIZING MATRICES WITH X-SIMPLE IMAGE EIGENSPACE IN MAX-MIN SEMIRING

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A matrix  $A$  is said to have  $\mathbf{X}$ -simple image eigenspace if any eigenvector  $x$  belonging to the interval  $\mathbf{X} = \{x: \underline{x} \leq x \leq \bar{x}\}$  is the unique solution of the system  $A \otimes y = x$  in  $\mathbf{X}$ . The main result of this paper is a combinatorial characterization of such matrices in the linear algebra over max-min (fuzzy) semiring.

The characterized property is related to and motivated by the general development of tropical linear algebra and interval analysis, as well as the notions of simple image set and weak robustness (or weak stability) that have been studied in max-min and max-plus algebras.

*Keywords:* max-min algebra, interval, weakly robust, weakly stable, eigenspace, simple image set

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## 1. INTRODUCTION

This paper is concerned with a problem of max-min linear algebra, which is one of the sub-areas of tropical mathematics. In a wider algebraic context, tropical mathematics (also known as idempotent mathematics) can be viewed as a mathematical theory developed over additively idempotent ( $a \oplus a = a$ ) semirings. Note that the operation of taking maximum of two numbers is the simplest and the most useful example of an idempotent addition.

Idempotent semirings can be used in a range of practical problems related to scheduling and optimization, and offer many new problems to pure mathematicians. There are several monographs [13, 14, 15, 16] and collections of papers [18, 19] on tropical mathematics and its applications. Let us also mention some connections between idempotent algebra and fuzzy sets theory [8, 9].

In the max-min algebra, sometimes also called the “fuzzy algebra” [10, 11, 23] the arithmetical operations  $a \oplus b := \max(a, b)$  and  $a \otimes b := \min(a, b)$  are defined over a linearly ordered set. As usual, the two arithmetical operations are naturally extended to matrices and vectors.

The development of linear algebra over idempotent semirings was historically motivated by multi-machine interaction processes. In these processes we have  $n$  machines which work in stages, and in the algebraic model of their interactive work, entry  $x_i^{(k)}$  of

a vector  $x^{(k)} \in \mathbb{B}(n)$  where  $i \in \{1, \dots, n\}$  and  $\mathbb{B}$  is an idempotent semiring, represents the state of machine  $i$  after some stage  $k$ , and the entry  $a_{ij}$  of a matrix  $A \in \mathbb{B}(n, n)$ , where  $i, j \in \{1, \dots, n\}$ , encodes the influence of the work of machine  $j$  in the previous stage on the work of machine  $i$  in the current stage. For simplicity, the process is assumed to be homogeneous, like in the discrete time Markov chains, so that  $A$  does not change from stage to stage. Summing up all the influence effects multiplied by the results of previous stages, we have  $x_i^{(k+1)} = \bigoplus_j a_{ij} \otimes x_j^{(k)}$ . In the case of  $\oplus = \max$  this “summation” is often interpreted as waiting till all the processes are finished and all the necessary influence constraints are satisfied.

Thus the orbit  $x, A \otimes x, \dots, A^k \otimes x$ , where  $A^k = A \otimes \dots \otimes A$ , represents the evolution of such a process. Regarding the orbits, one wishes to know the set of starting vectors from which a given objective can be achieved. One of the most natural objectives in tropical algebra, where the ultimate periodicity of the orbits often occurs, is to arrive at an eigenvector. The set of starting vectors from which one reaches an eigenvector of  $A$  after a finite number of stages, is called attraction set of  $A$  (see [2, 29]). In general, attraction set contains the set of all eigenvectors, but it can be also as big as the whole space. This leads us, in turn, to another question: in which case is attraction set precisely the same as the set of all eigenvectors? Matrices with this property are called weakly robust or weakly stable [3].

In the special case of max-min algebra which we are going to consider, it can be argued that an orbit can stabilize at a fixed point ( $A \otimes x = x$ ) [6], but not at an eigenvector with an eigenvalue different from unity. Therefore, *by eigenvectors of  $A$  we shall mean the fixed points of  $A$  (satisfying  $A \otimes x = x$ )*.

In terms of the systems  $A \otimes x = b$ , weak robustness (with eigenvectors understood as fixed points) is equivalent to the following condition: every eigenvector  $y$  belongs to the *simple image set* of  $A$ , that is, for every eigenvector  $y$ , the system  $A \otimes x = y$  has unique solution  $x = y$ .

In the present paper, we consider an interval version of this condition. Namely, we describe matrices  $A$  such that for any eigenvector  $y$  belonging to an interval  $\mathbf{X} = [\underline{x}, \bar{x}] := \{x \in \mathbb{B}(n); \underline{x} \leq x \leq \bar{x}\}$  the system  $A \otimes x = y$  has a unique solution  $x = y$  in  $\mathbf{X}$ . This is what we mean by saying that “ $A$  has  $\mathbf{X}$ -simple image eigenspace”. In Theorem 3.8, which is the main result of the paper, we show that under a certain natural condition,  $A$  has  $\mathbf{X}$ -simple image eigenspace if and only if it satisfies a nontrivial combinatorial criterion, which makes use of threshold digraphs and to which we refer as “ $\mathbf{X}$ -conformism” (see Definition 3.3).

The next section will be occupied by some definitions and notation of the max-min algebra, leading to the discussion of weak  $\mathbf{X}$ -robustness and  $\mathbf{X}$ -simple image eigenvectors. Section 3 is devoted to the main result (characterizing matrices with  $\mathbf{X}$ -simple image eigenspace), and its rather technical combinatorics.

Let us conclude with a brief overview of the works on max-min algebra to which this paper is related. The concepts of robustness in max-min algebra were introduced and studied in [25]. Following that work, some equivalent conditions and efficient algorithms were presented in [20, 23, 26]. In particular, see [26] for some polynomial procedures checking the weak robustness (weak stability) in max-min algebra.

## 2. PRELIMINARIES

### 2.1. Max-min algebra and associated digraphs

Let us denote the set of all natural numbers by  $\mathbb{N}$ . Let  $(\mathbb{B}, \leq)$  be a bounded linearly ordered set with the least element in  $\mathbb{B}$  denoted by  $O$  and the greatest one by  $I$ .

A max-min semiring is a set  $\mathbb{B}$  equipped with two binary operations  $\oplus = \max$  and  $\otimes = \min$ , called addition and multiplication, such that  $(\mathbb{B}, \oplus)$  is a commutative monoid with identity element  $O$ ,  $(\mathbb{B}, \otimes)$  is a monoid with identity element  $I$ , multiplication left and right distributes over addition and multiplication by  $O$  annihilates  $\mathbb{B}$ .

We will use the notations  $N$  and  $M$  for the sets of natural numbers not exceeding  $n$  and  $m$ , respectively, i.e.,  $N = \{1, 2, \dots, n\}$  and  $M = \{1, 2, \dots, m\}$ . The set of  $n \times m$  matrices over  $\mathbb{B}$  is denoted by  $\mathbb{B}(n, m)$ , and the set of  $n \times 1$  vectors over  $\mathbb{B}$  is denoted by  $\mathbb{B}(n)$ . If each entry of a matrix  $A \in \mathbb{B}(n, n)$  (a vector  $x \in \mathbb{B}(n)$ ) is equal to  $O$  we shall denote it as  $A = O$  ( $x = O$ ).

Let  $x = (x_1, \dots, x_n) \in \mathbb{B}(n)$  and  $y = (y_1, \dots, y_n) \in \mathbb{B}(n)$  be vectors. We write  $x \leq y$  ( $x < y$ ) if  $x_i \leq y_i$  ( $x_i < y_i$ ) holds for each  $i \in N$ .

For a matrix  $A \in \mathbb{B}(n, n)$ , its entries will be denoted by  $a_{ij}$  for  $i, j \in N$ . The symbol  $G(A) = (N, E)$  stands for a complete, arc-weighted digraph associated with  $A$ , i.e., the node set of  $G(A)$  is  $N$ , and the weight (capacity) of any arc  $(i, j)$  is  $a_{ij} \geq O$ . For given  $h \in \mathbb{B}$ , the *threshold digraph*  $G(A, h)$  is the digraph with the node set  $N$  and with the arc set  $E = \{(i, j); i, j \in N, a_{ij} \geq h\}$ . A path in the digraph  $G(A) = (N, E)$  is a sequence of nodes  $p = (i_1, \dots, i_{k+1})$  such that  $(i_j, i_{j+1}) \in E$  for  $j = 1, \dots, k$ . The number  $k$  is the length of the path  $p$  and is denoted by  $l(p)$ . If  $i_1 = i_{k+1}$ , then  $p$  is called a cycle and it is called an elementary cycle if moreover  $i_j \neq i_m$  for  $j, m = 1, \dots, k$ . Notation  $(k, l) \in p$  means that edge  $(k, l)$  is lying in  $p$ .

### 2.2. Orbits, eigenvectors and weak robustness

For  $A \in \mathbb{B}(n, n)$  and  $x \in \mathbb{B}(n)$ , the orbit  $\mathcal{O}(A, x)$  of  $x = x^{(0)}$  generated by  $A$  is the sequence

$$x^{(0)}, x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots,$$

where  $x^{(r)} = A^r \otimes x^{(0)}$  for each  $r \in \mathbb{N}$ .

The operations  $\max, \min$  are idempotent, so no new numbers are created in the process of generating of an orbit. Therefore any orbit contains only a finite number of different vectors. It follows that any orbit starts repeating itself after some time, in other words, it is ultimately periodic. The same holds for the power sequence  $(A^k; k \in \mathbb{N})$ .

We are interested in the case when the ultimate period is 1, or in other words, when the orbit is ultimately stable. Note that in this case the ultimate vector of the orbit necessarily satisfies  $A \otimes x = x$ . This is the main reason why in this paper by eigenvectors we mean fixed points. (Also observe that if  $x$  is not a fixed point but a more general eigenvector satisfying  $A \otimes x = \lambda \otimes x$ , then  $A \otimes x$  is already a fixed point due to the idempotency of multiplication.)

Formally we can define the set of fixed points (eigenvectors)  $V(A)$  and the attraction

set  $\text{attr}(A)$  as follows

$$V(A) = \{x \in \mathbb{B}(n); A \otimes x = x\}$$

and

$$\text{attr}(A) = \{x \in \mathbb{B}(n); \mathcal{O}(A, x) \cap V(A) \neq \emptyset\},$$

respectively.

The present paper is closely related to the following kind of matrices.

**Definition 2.1.** Let  $A \in \mathbb{B}(n, n)$  be a matrix. Then  $A$  is called weakly robust (or weakly stable), if  $\text{attr}(A) = V(A)$ .

Observe that in general  $V(A) \subseteq \text{attr}(A) \subseteq \mathbb{B}(n)$ . The matrices for which  $\text{attr}(A) = \mathbb{B}(n)$  are called (strongly) robust or (strongly) stable, as opposed to weakly robust (weakly stable). The following fact, which holds in max-min algebra and max-plus algebra alike, is one of the main motivations for our paper.

**Theorem 2.2.** (Plavka and Szabó [25], Plavka et al. [3]) Let  $A \in \mathbb{B}(n, n)$  be a matrix. Then  $A$  is weakly robust if and only if  $(\forall x \in \mathbb{B}(n))[A \otimes x \in V(A) \Rightarrow x \in V(A)]$ .

Let us conclude this section by recalling some information on the greatest eigenvector and constant eigenvectors in max-min algebra.

Let  $A = (a_{ij}) \in \mathbb{B}(n, n)$  be a matrix. Define the greatest eigenvector  $x^\oplus(A) = (x_1^\oplus(A), \dots, x_n^\oplus(A))^T$  corresponding to a matrix  $A$  as

$$x^\oplus(A) = \bigoplus_{x \in V(A)} x.$$

It has been proved in [30] for a more general structure (distributive lattice) that the greatest eigenvector  $x^\oplus(A)$  of  $A$  exists for each matrix  $A \in \mathbb{B}(n, n)$ . The greatest eigenvector  $x^\oplus(A)$  can be computed by the following iterative  $O(n^2 \log n)$  procedure ([5]). Let us denote  $x_i^1(A) = \bigoplus_{j \in N} a_{ij}$  for each  $i \in N$  and  $x^{k+1}(A) = A \otimes x^k(A)$  for all  $k \in \{1, 2, \dots\}$ . Then  $x^{k+1}(A) \leq x^k(A)$  and  $x^\oplus(A) = x^n(A)$ . Observe that  $x_i^\oplus(A) \leq \bigoplus_{j \in N} a_{ij}$  for all  $i$ .

Next, denote

$$m_A = \bigoplus_{i, j \in N} a_{ij}, \quad c(A) = \bigotimes_{i \in N} \bigoplus_{j \in N} a_{ij}, \quad c^*(A) = (c(A), \dots, c(A))^T \in \mathbb{B}(n).$$

It can be checked that  $A \otimes c^*(A) = c^*(A)$ , since every row of  $A$  contains an entry that is not smaller than  $c^*(A)$ . In fact, this condition is necessary and sufficient for a constant eigenvector to be an eigenvector of  $A$ . Therefore any constant vector that is smaller than  $c^*(A)$  is also an eigenvector, and  $c^*(A)$  is the largest constant eigenvector of  $A$ . However, as  $x^\oplus(A)$  is the greatest eigenvector of  $A$ , we have  $c^*(A) \leq x^\oplus(A)$ .

**2.3. Weak  $\mathbf{X}$ -robustness and  $\mathbf{X}$ -simplicity**

In this section we consider an interval extension of weak robustness and its connection to  $\mathbf{X}$ -simplicity, the main notion studied in this paper. We remind that throughout the paper,

$$\mathbf{X} = [\underline{x}, \bar{x}] = \{x \in \mathbb{B}(n) : \underline{x} \leq x \leq \bar{x}, \}, \quad \text{where } \underline{x}, \bar{x} \in \mathbb{B}(n).$$

Consider the following interval extension of weak  $\mathbf{X}$ -robustness.

**Definition 2.3.**  $A \in \mathbb{B}(n, n)$  is called weakly  $\mathbf{X}$ -robust if  $\text{attr}(A) \cap \mathbf{X} \subseteq V(A)$ .

The notion of  $\mathbf{X}$ -simplicity, which we will introduce next, is related to the concept of simple image set [1]: by definition, this is the set of vectors  $b$  such that the system  $A \otimes x = b$  has a unique solution, which is usually denoted by  $|S(A, b)| = 1$  ( $S(A, b)$  standing for the solution set of  $A \otimes x = b$ ). If the only solution of the system  $A \otimes x = b$  is  $x = b$ , then  $b$  is called a *simple image eigenvector*.

If  $\mathbf{X} = \mathbb{B}$  then the notion of weak robustness can be described in terms of simple image eigenvectors:

**Proposition 2.4.** Let  $A \in \mathbb{B}(n, n)$ . The following are equivalent:

- (i)  $A$  is weakly robust;
- (ii)  $(\forall x \in V(A))[|S(A, x)| = 1]$ ;
- (iii) Each  $x \in V(A)$  is a simple image eigenvector.

*Proof.* We will only prove the equivalence between the first two claims (the other equivalence being evident). Suppose that there is  $x \in V(A)$  such that  $|S(A, x)| > 1$  (notice that  $|S(A, x)| \geq 1$  for each  $x$  because of  $x \in V(A)$ ). Then there is at least one solution  $y$  of the system  $A \otimes y = x$  and  $y \neq x$ . Using Theorem 2.2 we get  $A \otimes (A \otimes y) = A \otimes x = x$  and  $A \otimes y = x \neq y$ , this is a contradiction.

The converse implication trivially follows. □

This motivates us to consider an interval version of simple image eigenvectors.

**Definition 2.5.** Let  $A \in \mathbb{B}(n, n)$  and  $\mathbf{X} = [\underline{x}, \bar{x}] \subseteq \mathbb{B}(n)$  be given.

- (i) An eigenvector  $x \in V(A) \cap \mathbf{X}$  is called an  $\mathbf{X}$ -simple image eigenvector if  $x$  is the unique solution of the equation  $A \otimes y = x$  in the interval  $\mathbf{X}$ .
- (ii) Matrix  $A$  is said to have  $\mathbf{X}$ -simple image eigenspace if any  $x \in V(A) \cap \mathbf{X}$  is an  $\mathbf{X}$ -simple image eigenvector.

**Definition 2.6.** Let  $A \in \mathbb{B}(n, n)$  and  $\mathbf{X} = [\underline{x}, \bar{x}] \subseteq \mathbb{B}(n)$  be given. We say that  $\mathbf{X}$  is invariant under  $A$  if  $x \in \mathbf{X}$  implies  $A \otimes x \in \mathbf{X}$ .

**Theorem 2.7.** Let  $A \in \mathbb{B}(n, n)$  and  $\mathbf{X} = [\underline{x}, \bar{x}] \subseteq \mathbb{B}(n)$  be given.

- (i) If  $A$  is weakly  $\mathbf{X}$ -robust then  $A$  has  $\mathbf{X}$ -simple image eigenspace.
- (ii) If  $A$  has  $\mathbf{X}$ -simple image eigenspace and if  $\mathbf{X}$  is invariant under  $A$  then  $A$  is weakly  $\mathbf{X}$ -robust.

*Proof.* (i) Suppose that  $A$  is weakly  $\mathbf{X}$ -robust and  $x \in V(A) \cap \mathbf{X}$ . If the system  $A \otimes y = x$  has a solutions  $y \neq x$  in  $\mathbf{X}$ , then  $y$  is not an eigenvector but belongs to  $\text{attr}(A) \cap \mathbf{X}$ , which contradicts the weak  $\mathbf{X}$ -robustness.

(ii) Assume that  $A$  has  $\mathbf{X}$ -simple image eigenspace and  $x$  is an arbitrary element of  $\text{attr}(A) \cap \mathbf{X}$ . As  $\mathbf{X}$  is invariant under  $A$ , we have that  $A^k \otimes x \in \mathbf{X}$  for all  $k$ . Then  $A^k \otimes x \in V(A)$  for some  $k$  implies  $A^{k-1} \otimes x = A^k \otimes x \in V(A), \dots, x \in V(A)$ . □

As  $A$  is order-preserving, the invariance of  $\mathbf{X}$  under  $A$  admits the following simple characterization:

**Proposition 2.8.**  $\mathbf{X}$  is invariant under  $A$  if and only if  $A \otimes \underline{x} \geq \underline{x}$  and  $A \otimes \bar{x} \leq \bar{x}$ .

Thus the  $\mathbf{X}$ -simplicity is a necessary condition for weak  $\mathbf{X}$ -robustness. It is also sufficient if the interval  $\mathbf{X}$  is invariant under  $A$ , i. e.,  $\underline{x} \leq A \otimes \underline{x}$  and  $A \otimes \bar{x} \leq \bar{x}$ .

### 3. $\mathbf{X}$ -SIMPLE IMAGE EIGENSPACE AND $\mathbf{X}$ -CONFORMISM

The purpose of this section is to define a nontrivial combinatorial condition for matrix  $A$  (namely,  $A$  being an “ $\mathbf{X}$ -conforming matrix”) and show that this condition describes a class of matrices for which every eigenvector  $x \in V(A) \cap \mathbf{X}$  is an  $\mathbf{X}$ -simple image eigenvector (in other words, the matrices with  $\mathbf{X}$ -simple image eigenspace).

We first introduce the property of level- $\alpha$  permutation, which (as argued in Lemma 3.2) gives rise to a necessary condition for  $A$  having an  $\mathbf{X}$ -simple image eigenspace.

**Definition 3.1.** A matrix  $A = (a_{ij}) \in \mathbb{B}(n, n)$  is called a generalized level- $\alpha$  permutation matrix (abbr. level- $\alpha$  permutation) if digraph  $G(A, \alpha)$  is a set of disjoint elementary cycles containing all nodes.

Let us also define the following quantity:

$$\gamma(A, \bar{x}) = \min(c(A), \min_{i \in N} \bar{x}_i), \quad \gamma^*(A, \bar{x}) = (\gamma(A, \bar{x}), \dots, \gamma(A, \bar{x})). \tag{1}$$

Since  $\gamma^*(A, \bar{x})$  is a constant vector such that each row of  $A$  contains an entry not smaller than  $\gamma(A, \bar{x})$ , we obtain  $A \otimes \gamma^*(A, \bar{x}) = \gamma^*(A, \bar{x})$  (i. e.,  $\gamma^*(A, \bar{x}) \in V(A)$ ).

**Lemma 3.2.** Let  $A = (a_{ij}) \in \mathbb{B}(n, n)$  and  $\mathbf{X} = [\underline{x}, \bar{x}] \subseteq \mathbb{B}(n)$  be given and conditions  $\underline{x} < c^*(A)$  and  $\max_{i \in N} \underline{x}_i < \min_{i \in N} \bar{x}_i$  be fulfilled. If  $A$  has  $\mathbf{X}$ -simple image eigenspace then  $A$  is level- $\gamma(A, \bar{x})$  permutation.

*Proof.* For a contrary suppose that, under the given conditions,  $A$  is not level- $\gamma(A, \bar{x})$  permutation. We shall look for two solutions of  $A \otimes y = \gamma^*(A, \bar{x})$ . One solution is  $\gamma^*(A, \bar{x}) \in V(A) \cap \mathbf{X}$ . Since  $A$  is not level- $\gamma(A, \bar{x})$  permutation and each row of  $A$  contains at least one element  $a_{ij} \geq \gamma(A, \bar{x})$  we shall consider two cases.

Case 1.  $(\exists k \in N)[\max_{s \in N} a_{sk} < \gamma(A, \bar{x})]$ . The second solution  $y' \in \mathbf{X}$  is

$$y'_i = \begin{cases} \underline{x}_i, & \text{if } i = k \\ \gamma(A, \bar{x}), & \text{otherwise,} \end{cases}$$

since we have  $a_{sk} < \gamma(A, \bar{x})$  for all  $s$ , implying that the terms  $a_{sk} \otimes y'_k$  are unimportant and  $y'_k$  can be set to any admissible value.

Case 2. The remaining case is when  $(\forall k \in N)[\max_{s \in N} a_{sk} \geq \gamma(A, \bar{x})]$ . Recall that we have at least one entry greater then or equal to  $\gamma(A, \bar{x})$  in each column of  $A$ . By the definition of  $C(A)$  and  $\gamma(A, \bar{x})$  we also have at least one such entry in each row. However  $A$  is not level- $\gamma(A, \bar{x})$  permutation hence there is a row with at least two such entries, i. e.,  $(\exists i, j, k \in N)[a_{ij} \geq \gamma(A, \bar{x}) \text{ and } a_{ik} \geq \gamma(A, \bar{x})]$ . Then there is  $v \in N$  such that  $(\forall i \in N)[\max_{j \in N \setminus \{v\}} a_{ij} \geq \gamma(A, \bar{x})]$  and the second solution  $y' \in \mathbf{X}$  can be defined as follows

$$y'_i = \begin{cases} \underline{x}_i, & \text{if } i = v \\ \gamma(A, \bar{x}), & \text{otherwise,} \end{cases}$$

since attainment of the maximum value in every row of  $A \otimes y$  by other terms than  $a_{sv} \otimes y_v$  makes these terms redundant, so that  $y_v$  can be replaced by any admissible value  $y'_v < y_v$ .

In both cases we obtained a contradiction with  $A$  having  $\mathbf{X}$ -simple image eigenspace. □

We now define what an  $\mathbf{X}$ -conforming matrix is. This is the key notion of the paper building upon the definition of level- $\gamma(A, \bar{x})$  permutation matrix given above, and requiring the notation introduced below in (2)-(4) and the properties assumed in (5), (6) and in Definition (3.3). See also Example 3.5 illustrating the notion.

For a given level- $\gamma(A, \bar{x})$  permutation matrix  $A = (a_{ij}) \in \mathbb{B}(n, n)$  and  $\mathbf{X} = [\underline{x}, \bar{x}] \subseteq \mathbb{B}(n)$  assume that

$$c_u = (i_1^u, \dots, i_{s_u}^u) \text{ are elementary cycles in digraph } G(A, \gamma(A, \bar{x})), \quad 1 \leq u \leq k, \quad (2)$$

$$e_{\underline{x}} = (e_1, \dots, e_n)^T, \text{ where } e_i = \max_{v \in c_u} \underline{x}_v \text{ for } i \in c_u, \quad 1 \leq u \leq k, \quad (3)$$

$$f_{\bar{x}} = (f_1, \dots, f_n)^T, \text{ where } f_i = \min_{v \in c_u} \bar{x}_v \otimes x_v^\oplus(A) \text{ for } i \in c_u, \quad 1 \leq u \leq k \quad (4)$$

and that

$$\underline{x} < c^*(A) \text{ and } \max_{i \in N} \underline{x}_i < \min_{i \in N} \bar{x}_i, \quad (5)$$

$$a_{i_j^u i_{j+1}^u} \geq \gamma(A, \bar{x}), \quad 1 \leq j \leq s_u - 1, 1 \leq u \leq k. \quad (6)$$

Notice that  $e_{i_j^u} = e_{i_{j+1}^u}$ ,  $f_{i_j^u} = f_{i_{j+1}^u}$  for nodes  $i_j^u, i_{j+1}^u$  lying in the same cycle  $c_u$  and also  $\{i_1^1, \dots, i_{s_1}^1, \dots, i_1^k, \dots, i_{s_k}^k\} = N$ .

**Definition 3.3.** Let  $A = (a_{ij}) \in \mathbb{B}(n, n)$  and  $\mathbf{X} = [\underline{x}, \bar{x}] \subseteq \mathbb{B}(n)$  be given and  $A$  be level- $\gamma(A, \bar{x})$  permutation fulfilling (5) and (6). Then, matrix  $A$  is called  $\mathbf{X}$ -conforming if

- (i)  $\underline{x}_{i_{j+1}^u} < e_{i_{j+1}^u} \Rightarrow a_{i_j^u k} < e_{i_j^u}$  for  $k \neq i_{j+1}^u, k \in N$ ,
- (ii)  $\underline{x}_{i_{j+1}^u} = e_{i_{j+1}^u} \Rightarrow a_{i_j^u k} \leq e_{i_j^u}$  for  $k \neq i_{j+1}^u, k \in N$ ,
- (iii)  $a_{i_j^u i_{j+1}^u} = \min_{(k,l) \in c_u} a_{kl} = x_{i_{j+1}^u}^\oplus(A) = f_{i_{j+1}^u} \Rightarrow \bar{x}_{i_{j+1}^u} \leq x_{i_{j+1}^u}^\oplus(A)$ ,

in terms of the notation introduced in (2)–(4) and  $1 \leq u \leq k$ .

**Remark 3.4.** Observe that  $f_i \leq \bigoplus_{j \in N} a_{ij}$  for all  $i$ , since the same holds for  $x^\oplus(A)$  (the end of Section 2.2).

**Example 3.5.** Let us consider  $\mathbb{B} = [0, 10]$ ,  $\lambda = 10$  and

$$A = \begin{pmatrix} 4 & 4 & 4 & \mathbf{5} \\ 2 & 2 & \mathbf{7} & 2 \\ 3 & \mathbf{8} & 3 & 3 \\ \mathbf{7} & 3 & 3 & 3 \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 4 \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} 7 \\ 9 \\ 6 \\ 5 \end{pmatrix}.$$

Matrix  $A$  is level-5 permutation with  $c_1 = (i_1^1, i_2^1) = (1, 4)$ ,  $c_2 = (i_2^2, i_3^2) = (2, 3)$  and  $x^\oplus(A) = (5, 7, 7, 5)^T$ . Vectors  $e_{\underline{x}}$  and  $f_{\bar{x}}$  have the following coordinates

$$e_1 = e_4 = \max(\underline{x}_1, \underline{x}_4) = 4, \quad e_2 = e_3 = \max(\underline{x}_2, \underline{x}_3) = 3,$$

$$f_1 = f_4 = \min(\bar{x}_1 \otimes x_1^\oplus, \bar{x}_4 \otimes x_4^\oplus) = 5, \quad f_2 = f_3 = \min(\bar{x}_2 \otimes x_2^\oplus, \bar{x}_3 \otimes x_3^\oplus) = 6,$$

thus  $e_{\underline{x}} = (4, 3, 3, 4)^T$  and  $f_{\bar{x}} = (5, 6, 6, 5)^T$ .

Now, we shall argue that  $A$  is  $\mathbf{X}$ -conforming,

$$\begin{aligned} i_1^1 &= 1, i_2^1 = 4; \quad \underline{x}_1 < e_1 \Rightarrow a_{4j} < e_4 \quad (\forall j \neq 1), \\ i_2^1 &= 4, i_1^1 = 1; \quad \underline{x}_4 = e_4 \Rightarrow a_{1j} \leq e_1 \quad (\forall j \neq 4), \\ i_1^2 &= 2, i_2^2 = 3; \quad \underline{x}_2 = e_2 \Rightarrow a_{3j} \leq e_3 \quad (\forall j \neq 2), \\ i_2^2 &= 3, i_1^2 = 2; \quad \underline{x}_3 < e_3 \Rightarrow a_{2j} < e_2 \quad (\forall j \neq 3) \end{aligned}$$

and

$$a_{14} = 5 = \min_{(k,l) \in c_1=(14)} a_{kl} = x_4^\oplus(A) \Rightarrow \bar{x}_4 = 5 \leq x_4^\oplus(A) = 5, \quad a_{41} = 7 > \min_{(k,l) \in c_1=(14)} a_{kl} = 5,$$

$$a_{23} = 7 = \min_{(k,l) \in c_2=(23)} a_{kl} = x_3^\oplus(A) \Rightarrow \bar{x}_3 = 6 \leq x_3^\oplus(A) = 7, \quad a_{32} = 8 > \min_{(k,l) \in c_2=(23)} a_{kl} = 7.$$

Hence matrix  $A$  is  $\mathbf{X}$ -conforming.



The purpose of Lemma 3.6 below is to establish useful inequalities for  $e_{\underline{x}}$ ,  $\gamma^*(A, \bar{x})$  and  $f_{\bar{x}}$  and to elucidate the structure of  $V(A) \cap \mathbf{X}$ .

**Lemma 3.6.** Let  $A = (a_{ij}) \in \mathbb{B}(n, n)$ ,  $\mathbf{X} = [\underline{x}, \bar{x}] \subseteq \mathbb{B}(n)$  be given and  $A$  be  $\mathbf{X}$ -conforming. Let  $c_u$ , for  $u = 1, \dots, k$  be the elementary cycles of  $G(A, \gamma(A, \bar{x}))$ .

- (i)  $\underline{x} \leq e_{\underline{x}} < \gamma^*(A, \bar{x})$ ,  $\gamma^*(A, \bar{x}) \leq f_{\bar{x}} \leq \bar{x}$  and  $e_{\underline{x}}, f_{\bar{x}} \in V(A) \cap \mathbf{X}$ ,
- (ii)  $(\forall x \in \mathbf{X} \cap V(A))[e_{\underline{x}} \leq x \leq f_{\bar{x}}]$ ,
- (iii)  $V(A) \cap \mathbf{X} = \{(x_1, \dots, x_n)^T : \exists \alpha_1, \dots, \alpha_k \text{ s.t. } \forall i \ x_i = \alpha_u \in [e_i, f_i] \text{ for } i \in c_u, 1 \leq u \leq k\}$ .

**Proof.** Let us first recall that since  $A$  is  $\mathbf{X}$ -conforming it is also a level- $\gamma(A, \bar{x})$  permutation matrix.

(i) The inequalities  $\underline{x} \leq e_{\underline{x}} < \gamma^*(A, \bar{x})$  follow from the conditions  $\underline{x} < c^*(A)$ ,  $\max_{i \in N} \underline{x}_i < \min_{i \in N} \bar{x}_i$  and the definition of  $e_{\underline{x}}$ . To obtain  $\gamma^*(A, \bar{x}) \leq f_{\bar{x}} \leq \bar{x}$ , recall that  $x^\oplus(A) \geq \gamma^*(A, \bar{x})$  as  $x^\oplus(A)$  is the largest eigenvector, and that  $\bar{x} \geq \gamma^*(A, \bar{x})$  by (1), implying  $f_{\bar{x}} \geq \gamma^*(A, \bar{x})$ .

To show that  $e_{\underline{x}}, f_{\bar{x}} \in V(A)$  we need to prove that  $A \otimes e_{\underline{x}} = e_{\underline{x}}$ ,  $A \otimes f_{\bar{x}} = f_{\bar{x}}$ . As matrix  $A$  is level- $\gamma(A, \bar{x})$  permutation, for each  $i \in N$  there is  $j \in N$  such that  $a_{ij} \geq \gamma(A, \bar{x})$ .

To prove  $A \otimes e_{\underline{x}} = e_{\underline{x}}$  observe that

$$(A \otimes e_{\underline{x}})_i = \bigoplus_{t \neq j} a_{it} \otimes e_t \oplus a_{ij} \otimes e_j = a_{ij} \otimes e_j = e_j = e_i$$

because  $i, j$  lie in the same cycle ( $e_i = e_j$ ) and  $a_{ij} \geq \gamma(A, \bar{x}) > e_i \geq a_{it}$  for all  $t \neq j$  by the definition of  $\mathbf{X}$ -conforming matrix.

To prove  $A \otimes f_{\bar{x}} = f_{\bar{x}}$  observe that

$$(A \otimes f_{\bar{x}})_i = \bigoplus_{t \in N} a_{it} \otimes f_t = a_{ij} \otimes f_j = f_j = f_i$$

because of  $i, j$  are lying in the same cycle ( $f_i = f_j$ ),  $a_{ij} = \bigoplus_{t \in N} a_{it} \geq x_i^\oplus(A) \geq \bar{x}_i \otimes x_i^\oplus(A) \geq f_i$  and  $a_{it} < \gamma(A, \bar{x}) \leq f_i = f_j$  for  $t \neq j$ .

(ii) Suppose that  $(\exists x \in \mathbf{X} \cap V(A)[e_{\underline{x}} \not\leq x])$ , i. e., there is at least one index  $i \in N$  such that  $\underline{x}_i \leq x_i < e_i$ . Since  $A$  is level- $\gamma(A, \bar{x})$  permutation and  $i \in N$ , then there is a cycle  $c = \{i_1, \dots, i_s\}$  such that  $i_1 = i \in c$ ,

$$a_{i_r i_{r+1}} \geq \gamma(A, \bar{x}) > e_i > x_i \text{ and } e_{i_r} = \max_{k=i_1, \dots, i_s} \underline{x}_k \text{ for } r = 1, \dots, s.$$

If  $s = 1$  we immediately obtain a contradiction with the definition of the vector  $e_{\underline{x}}$ . Suppose now that  $s \geq 2$ , then for the eigenvector  $x = (x_1, \dots, x_n)$  we have

$$x_{i_1} = (A \otimes x)_{i_1} = \bigoplus_{t \in N} a_{i_1 t} \otimes x_t \geq a_{i_1 i_2} \otimes x_{i_2} = x_{i_2}$$

because of  $a_{i_1 i_2} \geq \gamma(A, \bar{x}) > x_{i_1}$  and we obtain  $x_{i_1} \geq x_{i_2}$ ,

$$x_{i_2} = (A \otimes x)_{i_2} = \bigoplus_{t \in N} a_{i_2 t} \otimes x_t \geq a_{i_2 i_3} \otimes x_{i_3} = x_{i_3}$$

because of  $a_{i_2 i_3} \geq \gamma(A, \bar{x}) > x_{i_1} \geq x_{i_2}$  and we obtain  $x_{i_2} \geq x_{i_3}$ . Proceeding in the same way for  $x_{i_3}, \dots, x_{i_s}$  we have  $x_{i_1} = \dots = x_{i_s} < e_{i_1} = \dots = e_{i_s}$ . However, this implies  $x_{i_k} < \underline{x}_{i_k}$  for some  $i_k$ , which is a contradiction.

Suppose that  $(\exists x \in \mathbf{X} \cap V(A)[x \not\leq f_{\bar{x}}])$ , i. e., there is at least one index  $i \in N$  such that  $f_i < x_i \leq \bar{x}_i$ . Since  $x$  is an eigenvector of  $A$  the equality  $\bigoplus_{t \in N} a_{vt} \geq x_v$  holds for each  $v \in N$ . Moreover using part (ii) of Definition 3.3 and that  $A$  is  $\mathbf{X}$ -conforming, we have  $a_{it} \leq e_i \leq f_i < x_i$  for each  $(i, t) \notin \{(i_1, i_2), \dots, (i_s, i_1)\}$ . Let  $i = i_1 \in c = \{i_1, \dots, i_s\}$ . Then

$$x_{i_1} = (A \otimes x)_{i_1} = \bigoplus_{t \in N} a_{i_1 t} \otimes x_t = a_{i_1 i_2} \otimes x_{i_2} \Rightarrow x_{i_2} \geq x_{i_1},$$

$$x_{i_2} = (A \otimes x)_{i_2} = \bigoplus_{t \in N} a_{i_2 t} \otimes x_t = a_{i_2 i_3} \otimes x_{i_3} \Rightarrow x_{i_3} \geq x_{i_2},$$

since  $a_{i_1 t} < x_{i_1}$  for  $t \neq i_2$  and  $a_{i_2 t} \leq e_{i_2} \leq f_{i_2} = f_{i_1} < x_{i_1}$  for  $t \neq i_3$ . Proceeding in the same way for  $x_{i_3}, \dots, x_{i_s}$  we obtain that

$$x_{i_1} = \dots = x_{i_s} > f_{i_1} = \dots = f_{i_s} = \min_{k=i_1, \dots, i_s} (\bar{x}_k, x_k^\oplus(A)) \text{ for } r = 1, \dots, s.$$

This implies that  $x_{i_l} > \bar{x}_{i_l}$  or  $x_{i_l} > x_{i_l}^\oplus(A)$  for some  $i_l$ : in both cases, a contradiction.

(iii) By part (ii) each  $x \in V(A) \cap \mathbf{X}$  satisfies  $e_{\underline{x}} \leq x \leq f_{\bar{x}}$ , and it remains to show that

$$x = (x_1, \dots, x_n)^T; \quad x_i = \alpha_u \text{ for } i \in c_u \text{ and } 1 \leq u \leq k.$$

As  $A$  is  $\mathbf{X}$ -conforming,  $A$  is also a level- $\gamma(A, \bar{x})$  fulfilling (5) and (6) and  $a_{rs} \leq e_r < \gamma(A, \bar{x})$  for  $(r, s) \notin \cup_{u=1}^k \{(i_1^u, i_2^u), \dots, (i_{s_u-1}^u, i_{s_u}^u), (i_{s_u}^u, i_1^u)\}$ .

Suppose that  $x \in V(A) \cap \mathbf{X}$ . Then  $e_{\underline{x}} \leq x \leq f_{\bar{x}}$  by (ii), and without loss of generality let us assume  $u = 1$ , that is,  $c_1 = (i_1^1, \dots, i_{s_1}^1)$ .

We shall consider two cases.

Case 1.  $x_{i_1^1} = e_{i_1^1}$ . Then we have that

$$x_{i_1^1} (= e_{i_1^1}) = (A \otimes x)_{i_1^1} = \bigoplus_{t \in N} a_{i_1^1 t} \otimes x_t = \bigoplus_{t \neq i_2^1} a_{i_1^1 t} \otimes x_t \oplus a_{i_1^1 i_2^1} \otimes x_{i_2^1} \geq a_{i_1^1 i_2^1} \otimes x_{i_2^1}.$$

Hence we have that  $x_{i_1^1} \geq x_{i_2^1}$  because of  $a_{i_1^1 i_2^1} \geq \gamma(A, \bar{x}) > e_{i_1^1} = x_{i_1^1}$ . For the index  $i_2^1$  we get

$$x_{i_2^1} = (A \otimes x)_{i_2^1} = \bigoplus_{t \in N} a_{i_2^1 t} \otimes x_t = \bigoplus_{t \neq i_3^1} a_{i_2^1 t} \otimes x_t \oplus a_{i_2^1 i_3^1} \otimes x_{i_3^1} \geq a_{i_2^1 i_3^1} \otimes x_{i_3^1}.$$

Hence we have that  $x_{i_2^1} \geq x_{i_3^1}$  because of  $a_{i_2^1 i_3^1} \geq \gamma(A, \bar{x}) > e_{i_1^1} = x_{i_1^1} \geq x_{i_2^1}$ . Proceeding in the same way we get  $x_{i_1^1} = \dots = x_{i_{s_1}^1} = e_{i_1^1} \in [e_{i_1^1}, f_{i_1^1}]$ .

Case 2.  $x_{i_1^1} > e_{i_1^1}$ . Then we get

$$x_{i_1^1} = \bigoplus_{t \in N} a_{i_1^1 t} \otimes x_t = \bigoplus_{t \neq i_2^1} a_{i_1^1 t} \otimes x_t \oplus a_{i_1^1 i_2^1} \otimes x_{i_2^1} = a_{i_1^1 i_2^1} \otimes x_{i_2^1}$$

because inequalities  $x_{i_1^1} > e_{i_1^1} \geq a_{i_1^1 t}$  holds for each  $t \neq i_2^1$  since  $A$  is  $\mathbf{X}$ -conforming. Thus, we obtain  $x_{i_1^1} = a_{i_1^1 i_2^1} \otimes x_{i_2^1}$  and hence  $x_{i_1^1} \leq x_{i_2^1}$ . Similarly for  $x_{i_2^1}$  we get

$$x_{i_2^1} = \bigoplus_{t \in N} a_{i_2^1 t} \otimes x_t = \bigoplus_{t \neq i_3^1} a_{i_2^1 t} \otimes x_t \oplus a_{i_2^1 i_3^1} \otimes x_{i_3^1} = a_{i_2^1 i_3^1} \otimes x_{i_3^1}$$

because of  $x_{i_2^1} \geq x_{i_1^1} > e_{i_1^1} \geq a_{i_2^1 t}$  for each  $t \neq i_3^1$  and we obtain  $x_{i_2^1} = a_{i_2^1 i_3^1} \otimes x_{i_3^1}$ . Hence  $x_{i_2^1} \leq x_{i_3^1}$ . Proceeding in the same way we get  $x_{i_1^1} = \dots = x_{i_{s_1}^1} = \alpha_1 \in [e_{i_1^1}, f_{i_1^1}]$ .  $\square$

**Remark 3.7.** By Lemma 3.6 (iii) it follows that the structure of each eigenvector  $x \in V(A) \cap \mathbf{X}$  of a given  $\mathbf{X}$ -conforming matrix  $A$  depends on elementary cycles in  $G(A, \gamma(A, \bar{x}))$  and all entries of  $x$  corresponding to the same cycle have an equal value.

The main result of the paper, given in the next theorem, describes an equivalent condition for  $A$  having an  $\mathbf{X}$ -simple image eigenspace and which immediately yields an  $O(n^2 \log n)$  algorithm for checking  $\mathbf{X}$ -simplicity of image eigenspace. The proof of the theorem technically works with the properties of  $\mathbf{X}$ -conformism on the detail level of vector coordinates.

**Theorem 3.8.** Let  $A = (a_{ij}) \in \mathbb{B}(n, n)$ ,  $\mathbf{X} = [\underline{x}, \bar{x}] \subseteq \mathbb{B}(n)$  be given fulfilling (5). Then  $A$  has  $\mathbf{X}$ -simple image eigenspace if and only if  $A$  is an  $\mathbf{X}$ -conforming matrix.

*Proof.* The “only if” part: As the matrix  $A$  has  $\mathbf{X}$ -simple image eigenspace, it is also a level- $\gamma(A, \bar{x})$  permutation fulfilling (5) and (6).

For the sake of a contradiction suppose that  $(\exists u \in \{1, \dots, l\})(\exists i_r^u \in c_u = (i_1^u, \dots, i_{s_u}^u))$  such that

$$\underline{x}_{i_{r+1}^u} < e_{i_{r+1}^u} \text{ and } a_{i_r^u k} \geq e_{i_r^u} \text{ for some } k \neq i_{r+1}^u, k \in N \tag{7}$$

or

$$\underline{x}_{i_{r+1}^u} = e_{i_{r+1}^u} \text{ and } a_{i_r^u k} > e_{i_r^u} \text{ for some } k \neq i_{r+1}^u, k \in N \tag{8}$$

or

$$a_{i_r^u i_{r+1}^u} = \min_{(k,l) \in c_u} a_{kl} = x_{i_{r+1}^u}^\oplus(A) = f_{i_{r+1}^u} \text{ and } \bar{x}_{i_{r+1}^u} > x_{i_{r+1}^u}^\oplus(A). \tag{9}$$

We shall consider three cases and for each case we shall construct an eigenvector  $e' \in V(A) \cap \mathbf{X}$  with  $|S(A, e') \cap \mathbf{X}| \geq 2$ .

Denote  $d_u = \max_{a_{tv} < \gamma(A, \bar{x}); t \in c_u, v \in N} a_{tv} (= a_{i_p^u v})$ . The first two cases (7) and (8) will be treated simultaneously.

Case 1.  $\underline{x}_{i_{r+1}^u} < e_{i_{r+1}^u}$  and  $a_{i_r^u k} \geq e_{i_r^u} (\Leftrightarrow \underline{x}_{i_{r+1}^u} < e_{i_{r+1}^u} = e_{i_r^u} \leq a_{i_r^u k})$ .

Case 2.  $\underline{x}_{i_r+1}^u = e_{i_r+1}^u$  and  $a_{i_r k}^u > e_{i_r}^u (\Leftrightarrow \underline{x}_{i_r+1}^u = e_{i_r+1}^u = e_{i_r}^u < a_{i_r k}^u)$ .

Define vector  $e'$  as follows

$$e'_i = \begin{cases} d_u, & \text{if } i \in c_u \\ \gamma(A, \bar{x}), & \text{otherwise.} \end{cases}$$

We shall show that  $A \otimes e' = e'$ . Since the matrix  $A$  is level- $\gamma(A, \bar{x})$  permutation, the equalities  $(A \otimes e')_i = (A \otimes \gamma^*(A, \bar{x}))_i = \gamma(A, \bar{x}) = e'_i$  hold for  $i \notin c_u$ , so it suffices to show that

$$(A \otimes e')_{i_t^u} = e'_{i_t^u} \text{ for } t = 1, \dots, s_u.$$

Using the definition of  $d_u$  and  $e'$  we obtain

$$(A \otimes e')_{i_t^u} = \bigoplus_{j \neq i_{t+1}^u} a_{i_t^u j} \otimes e'_j \oplus a_{i_t^u i_{t+1}^u} \otimes e'_{i_{t+1}^u} \leq d_u \oplus a_{i_t^u i_{t+1}^u} \otimes d_u = d_u = e'_{i_t^u}$$

and

$$(A \otimes e')_{i_t^u} = \bigoplus_{j \in N} a_{i_t^u j} \otimes e'_j \geq a_{i_t^u i_{t+1}^u} \otimes e'_{i_{t+1}^u} = a_{i_t^u i_{t+1}^u} \otimes d_u = d_u = e'_{i_t^u}.$$

In particular, we have

$$a_{i_t^u i_{t+1}^u} \otimes e'_{i_{t+1}^u} = e'_{i_t^u}, \quad t = 1, \dots, s_u.$$

To obtain a contradiction we shall show that the system  $A \otimes y = e'$  has at least two solutions, which will be denoted by  $y', y''$ . We have  $a_{i_r k}^u \geq e_{i_r}^u$  and  $\underline{x}_{i_r+1}^u < e_{i_r+1}^u$  (case 1), or  $a_{i_r k}^u > e_{i_r}^u$  and  $\underline{x}_{i_r+1}^u = e_{i_r+1}^u$  (case 2), and  $a_{i_p v}^u = d_u$  for some indices  $p$  and  $v$ . Define

$$y' = e', \quad y''_i = \begin{cases} \underline{x}_i, & \text{if } i = i_{p+1}^u \\ e'_i, & \text{otherwise.} \end{cases}$$

We need to show that  $y''_{i_{p+1}^u} > y'_{i_{p+1}^u}$ , to make sure that  $y'$  and  $y''$  are actually different in this position, and  $y' \geq y''$ . Next we also need to show that  $A \otimes y'' \geq e'$ , hence  $A \otimes y'' = e'$ .

To see the difference, observe that if  $p = r$ ,  $a_{i_r k}^u \geq e_{i_r}^u$  and  $\underline{x}_{i_r+1}^u < e_{i_r+1}^u$  then

$$y''_{i_r+1} = \underline{x}_{i_r+1}^u < e_{i_r+1}^u = e_{i_r}^u \leq a_{i_r k}^u \leq d_u = e'_{i_r+1},$$

if  $p = r$ ,  $a_{i_r k}^u > e_{i_r}^u$  and  $\underline{x}_{i_r+1}^u = e_{i_r+1}^u$  then

$$y''_{i_r+1} = \underline{x}_{i_r+1}^u = e_{i_r+1}^u = e_{i_r}^u < a_{i_r k}^u \leq d_u = e'_{i_r+1},$$

and if  $p \neq r$  then

$$y''_{i_{p+1}^u} = \underline{x}_{i_{p+1}^u} \leq e_{i_{p+1}^u} = e_{i_p}^u \leq a_{i_p k}^u < d_u = e'_{i_{p+1}^u}.$$

By the definition of  $d_u$  and  $e'$  and since  $A$  is a level- $\gamma(A, \bar{x})$  permutation matrix, we have  $(A \otimes y'')_{i_t^u} = a_{i_t^u i_{t+1}^u} e'_{i_{t+1}^u} = e'_{i_t^u}$  for each  $t \neq p$ . For  $t = p$  we obtain the following inequalities

$$(A \otimes y'')_{i_p^u} = \bigoplus_{j \neq i_{p+1}^u} a_{i_p^u j} \otimes y''_j \oplus a_{i_p^u i_{p+1}^u} \otimes y''_{i_{p+1}^u} \geq a_{i_p^u v} \otimes e'_v \geq d_u = e'_{i_p^u},$$

where  $k \neq i_{r+1}^u$  and  $e'_k \geq d_u$ . This implies  $A \otimes y'' \geq e'$  hence  $A \otimes y'' = e'$ .

Case 3. Suppose that (9) holds true and we will show that the system  $A \otimes y = f_{\bar{x}}$  has at least two solutions:  $y' = f_{\bar{x}}$  and  $y'' = (y''_1, \dots, y''_n)^T$ , where

$$y''_i = \begin{cases} \bar{x}_i & \text{if } i = i_{r+1}^u \\ f_i & \text{otherwise.} \end{cases}$$

Observe that the vectors  $y', y''$  are different in the  $i_{r+1}^u$ th position:

$$y''_{i_{r+1}^u} = \bar{x}_{i_{r+1}^u} > x_{i_{r+1}^u} \oplus (A) = f_{i_{r+1}^u} = y'_{i_{r+1}^u}.$$

Since  $A$  is level- $\gamma(A, \bar{x})$  permutation and  $f_{\bar{x}} \geq \gamma^*(A, \bar{x})$  we have  $(A \otimes y'')_{i_t^u} = a_{i_t^u i_{t+1}^u} \otimes f_{i_{t+1}^u} = f_{i_t^u}$  for each  $t \neq r$ . As for the case of  $r$ , recalling that  $a_{i_r^u i_{r+1}^u} = f_{i_{r+1}^u}$  and  $f_{i_{r+1}^u} < \bar{x}_{i_{r+1}^u}$  we obtain the following equalities

$$\begin{aligned} (A \otimes y'')_{i_r^u} &= \bigoplus_{k \neq i_{r+1}^u} a_{i_r^u k} \otimes y''_k \oplus a_{i_r^u i_{r+1}^u} \otimes y''_{i_{r+1}^u} = \bigoplus_{k \neq i_{r+1}^u} a_{i_r^u k} \otimes f_k \oplus a_{i_r^u i_{r+1}^u} \otimes \bar{x}_{i_{r+1}^u} \\ &= \bigoplus_{k \neq i_{r+1}^u} a_{i_r^u k} \otimes f_k \oplus (a_{i_r^u i_{r+1}^u} \otimes f_{i_{r+1}^u}) \otimes \bar{x}_{i_{r+1}^u} = \bigoplus_{k \in N} a_{i_r^u k} \otimes f_k = f_{i_r^u}. \end{aligned}$$

Here we have used the equality  $a_{i_r^u i_{r+1}^u} = f_{i_{r+1}^u}$  and the inequality  $f_{i_{r+1}^u} < \bar{x}_{i_{r+1}^u}$ , both following from the conditions describing Case 3.

The “if” part: Suppose that  $A$  is an  $\mathbf{X}$ -conforming matrix and we shall show that  $(\forall x \in V(A) \cap \mathbf{X})[|S(A, x) \cap \mathbf{X}| = 1]$ . For the contrary suppose that  $(\exists x \in V(A) \cap \mathbf{X})[|S(A, x) \cap \mathbf{X}| > 1]$ . By Lemma 3.6 (iii)  $x = (x_1, \dots, x_n)^T$ ,  $x_i = \alpha_u \in [e_i, f_i]$  for  $i \in c_u$ ,  $1 \leq u \leq k$  and there is a solution  $y' \neq x$  of the system  $A \otimes y = x$ . Then there is  $j \in N$  such that  $x_j \neq y'_j$ . We shall consider three possibilities: (i)  $y'_j < e_j$ , (ii)  $f_j < y'_j$ , (iii)  $y'_j \in [e_j, f_j]$ .

(i)  $y'_j < e_j$ . Since  $A$  is level- $\gamma(A, \bar{x})$  permutation there is  $p \in N$  such that  $a_{pj} \geq \gamma(A, \bar{x})$ , so that we can substitute  $p$  for  $i_j$  and  $j$  for  $i_{j+1}$  in Definition 3.3. As  $\underline{x}_j \leq y'_j < e_j$  by condition (i) of that definition we have that  $\underline{x}_j < e_j \Rightarrow a_{pt} < e_p = e_j$  for  $t \neq j$  and we obtain

$$x_p = (A \otimes y')_p = \bigoplus_{t \neq j} a_{pt} \otimes y'_t \oplus a_{pj} \otimes y'_j < e_p \leq x_p,$$

which is a contradiction.

(ii)  $f_j < y'_j$ . As  $A$  is level- $\gamma(A, \bar{x})$  permutation there is  $p \in N$  such that  $a_{pj} \geq \gamma(A, \bar{x})$ , and by Remark 3.4 we have  $a_{pj} = \bigoplus_{k \in N} a_{pk} \geq f_p (= f_j)$ . We consider two possibilities:

1.  $a_{pj} > f_p = f_j$ . Then we obtain the following

$$x_p = (A \otimes y')_p = \bigoplus_{k \neq j} a_{pk} \otimes y'_k \oplus a_{pj} \otimes y'_j \geq a_{pj} \otimes y'_j > f_p \geq x_p,$$

and this is a contradiction.

2.  $a_{pj} = f_p = f_j$ . At first we shall prove the following claim.

**Claim.** If  $A$  is level- $\gamma(A, \bar{x})$  permutation and  $x_r^\oplus(A) = a_{rs}$ ,  $(r, s) \in c_u$  then  $a_{rs} = \min_{(k,l) \in c_u} a_{kl}$ .

**Proof of Claim.** For the contrary, assume that  $A$  is level- $\gamma(A, \bar{x})$  permutation,  $a_{rs} = x_r^\oplus(A)$  and  $a_{rs} > \min_{(k,l) \in c_u} a_{kl} = a_{\alpha\beta}$ . Using

$$a_{\alpha t} < c(A) \leq a_{\alpha\beta} < a_{rs} = x_r^\oplus(A) = x_\beta^\oplus(A), \quad t \neq \beta,$$

we obtain

$$x_r^\oplus(A) = x_\alpha^\oplus(A) = (A \otimes x^\oplus(A))_\alpha = a_{\alpha\beta} \otimes x_\beta^\oplus(A) = a_{\alpha\beta} < a_{rs} = x_r^\oplus(A).$$

This is a contradiction. Note that the equalities  $x_\alpha^\oplus(A) = x_\beta^\oplus(A) = x_r^\oplus(A)$  follow from Lemma 3.6 (since  $x^\oplus(A)$  is an eigenvector).

Now we will continue to analyze “(ii), Case 2”. The assumption  $a_{pj} = f_p = f_j$  implies that  $a_{pj} = f_p \leq \min(\bar{x}_p, x_p^\oplus(A)) \leq x_j^\oplus(A)$ . Moreover from the fact that  $x^\oplus(A)$  is an eigenvector we obtain  $a_{pj} \geq x_j^\oplus(A) = x_p^\oplus(A)$ . Hence  $a_{pj} = x_j^\oplus(A) = x_p^\oplus(A)$  and by Claim we obtain  $a_{pj} = \min_{(k,l) \in c_u} a_{kl}$ . Then by Definition 3.3 we get

$$a_{pj} = f_j = x_j^\oplus(A) \Rightarrow \bar{x}_j \leq x_j^\oplus(A).$$

We conclude the proof by the following contradiction

$$\bar{x}_j \leq x_j^\oplus(A) = f_j < y'_j \leq \bar{x}_j.$$

(iii)  $y'_j \in [e_j, f_j]$ . As we also assumed  $x_j \neq y'_j$ , we shall analyze two possibilities:  $x_j < y'_j$  and  $x_j > y'_j$ .

Let  $x_j > y'_j$ . Using Remark 3.4 and Lemma 3.6, we obtain  $a_{pj} = \bigoplus_{k \in N} a_{pk} \geq f_p \geq x_p = x_j$ . By Definition 3.3 we have that  $a_{pk} \leq e_p (= e_j \leq y'_j < x_j = x_p)$  for  $k \neq j$ . These inequalities imply

$$x_p = (A \otimes y')_p = \bigoplus_{k \neq j} a_{pk} \otimes y'_k \oplus a_{pj} \otimes y'_j = a_{pj} \otimes y'_j = y'_j < x_p,$$

which is a contradiction.

Let  $x_j < y'_j$ . Using Remark 3.4, Lemma 3.6 (i) and the conditions  $y'_j \in [e_j, f_j]$  and  $x_j < y'_j$ , we obtain that

$$a_{pj} = \bigoplus_{k \neq j} a_{pk} \geq f_p (= f_j \geq y'_j > x_j = x_p).$$

This implies that

$$x_p = (A \otimes y')_p = \bigoplus_{k \neq j} a_{pk} \otimes y'_k \oplus a_{pj} \otimes y'_j \geq a_{pj} \otimes y'_j > x_p \otimes x_p = x_p,$$

which is a contradiction. □

**Remark 3.9.** Theorem 3.8 implies that in the case when  $\underline{x} < c^*(A)$  and  $\max_{i \in N} \underline{x}_i < \min_{i \in N} \bar{x}_i$ , the complexity of checking that a given matrix  $A$  has  $\mathbf{X}$ -simple image eigenspace requires  $O(n^2 \log n)$  arithmetic operations. It is based on the fact that we need  $O(n^2)$  operations to check parts (i) and (ii) of Definition 3.3 and  $O(n^2 \log n)$  operations to compute the greatest eigenvector  $x^\oplus(A)$  ([5]).

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