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MAXIMAL INEQUALITIES AND SOME CONVERGENCE THEOREMS FOR FUZZY RANDOM VARIABLES

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Some maximal inequalities for quadratic forms of independent and linearly negative quadrant dependent fuzzy random variables are established. Strong convergence of such quadratic forms are proved based on the martingale theory. A weak law of large numbers for linearly negative quadrant dependent fuzzy random variables is stated and proved.

Keywords: fuzzy random variable, quadratic form, linearly negative quadrant dependence, law of large numbers, almost surely convergence

Classification: 60F05, 60F15

1. INTRODUCTION

In recent years, the theory of fuzzy random variables have been extensively studied in various area. A fuzzy random variable has been extended as a vague perception of a real valued random variable and subsequently redefined as a particular random set, see e. g. [23, 24, 30], and [32]. Since, in this article, we focus on convergence properties of fuzzy random variables and maximal inequalities for such random variables, let us breifly review some works related to these topics. For the first time, a strong law of large numbers for fuzzy random variables was given by Miyakoshi and Shimbo [25]. Klement et al. [20] established a strong law of large numbers for fuzzy random variables, based on embedding theorem as well as certain probability techniques in the Banach spaces. Taylor et al. [36] proved a weak law of large numbers for fuzzy random variables in separable Banach spaces. Joo et al. [19] obtained Chung's type strong law of large numbers for fuzzy random variables based on isomorphic isometric embedding theorem. Hong [17] derived a strong law of large number for level-wise independent and level-wise identically distributed fuzzy random variables. Hong and Kim [18] established a weak law of large number of independent and identically distributed fuzzy random variables. Fu and Zhang [13] obtained some strong limit theorems for fuzzy random variables with slowly varying weight. It should be mentioned that, although the concept of variance has been found very convenient in studying limit theorems, but, as the authors know, it has not been developed the limit theorems for fuzzy random variables based on the concept of variance, except the works by Korner [22] and Feng [10]. Korner [22] proved

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strong and weak laws of large numbers for fuzzy random variables. Based on a natural extension of the concept of variance, he extended the Kolmogorov's inequality to independent fuzzy random variables and obtained some limit theorems. His method is a direct application of classical methods in probability theory to fuzzy random variables. Based on this method, Ahmadzade et al. [1] established some limit theorems for independent fuzzy random variables. As everyone knows, selecting of suitable metric spaces plays important role in studying of convergence theorems. Thus, Ahmadzade et al. [2] derived several convergence theorems for fuzzy martingales based on $D_{p,q}$-metric. Furthermore, Parchami et al. [29] obtained a consistent confidence interval for fuzzy capability index. On the other hand, there are many authors who have devoted their studies to maximal inequalities and almost sure (a.s.) convergence of quadratic forms of random variables. For instance, Cuzich et al. [4] and Zhang [41] provided the sufficient and necessary conditions for the strong law of large numbers of sequence of quadratic forms. Gadidov [14] proved two Rosenthal-type inequalities of sums of products for independent and identically distributed random variables. Shanchao [34] improved the corresponding ones in Gadidov [14]. Eghbal et al. [5, 6] obtained some maximal inequalities for quadratic forms of negative superadditive dependence random variables and investigated Kolmogorov inequalities for quadratic forms of dependent uniformly bounded random variables. Quadratic forms of fuzzy random variables and their applications to linear model and regression analysis are studied by Viertl [37]. The concept of linearly negative quadrant dependent sequence was introduced and investigated by Newman [27]. Some applications for linearly negative quadrant dependent sequence have been found; see, for example, the work by Newman who established the central limit theorem for a strictly stationary process. Wang and Zhang [39] provided uniform rates of convergence in the central limit theorem for linearly negative quadrant dependent sequence. Ko et al. [21] obtained the Hoeffding-type inequality for linearly negative quadrant dependent sequence. They studied the strong convergence for weighted sums of linearly negative quadrant dependent arrays. Fu and Wu [12] studied the almost sure central limit theorem for linearly negative quadrant dependent sequences. In this article, for the first time, we investigate some well known maximal inequalities for quadratic form of independent fuzzy random variables. Moreover, we introduce the concept of linearly negative quadrant dependence for fuzzy random variables, and study some limit theorems for such random variables.

The structure of this paper is as follows. In Section 2, we recall some preliminaries of fuzzy arithmetic and fuzzy random variables. Section 3 provides the maximal inequalities for quadratic form $\tilde{T}_n = \oplus_{1\leq i<j\leq n}\tilde{X}_i \otimes \tilde{X}_j$ and randomly weighted quadratic form $\tilde{W}_n = \oplus_{1\leq i<j\leq n}Y_{ij}\tilde{X}_i \otimes \tilde{X}_j$, where $\{\tilde{X}_i, i \geq 1\}$ is a sequence of independent fuzzy random variables and $\{Y_{i,j}; 1 \leq i < j \leq n\}$ is an array of non-negative real-valued random variables such that the sequences $\{Y_{i,j}; 1 \leq i < j \leq n\}$ and $\{\tilde{X}_n; n \geq 1\}$ are independent. In Section 4, we introduce the concepts of negative dependence and linearly negative quadrant dependence and obtain some maximal inequalities for fuzzy random variables. In Section 5, a weak law of large numbers for linearly negative quadrant dependent fuzzy random variables is stated and proved. In last section, some conclusions are provided.

2. PRELIMINARIES

In this section, we provide some definitions and elementary concepts of fuzzy set theory that will be used in the next sections. For more details, the reader is referred to [11, 26, 37].

Define $E = \{\tilde{u} : R \to [0,1] | \tilde{u}$ satisfies (i) − (iii)$\}$; where (i) $\tilde{u}$ is normal, (ii) $\tilde{u}$ is fuzzy convex, and (iii) $\tilde{u}$ is upper semi-continuous. Any $\tilde{u} \in E$ is called a fuzzy number. A fuzzy number u is called non-negative if $\tilde{u}(x) = 0, \quad \forall x < 0$. The set of all non-negative fuzzy numbers is denoted by E^+. For $\tilde{u} \in E, [\tilde{u}]^r = \{x \in R | \tilde{u}(x) \geq r\}, 0 < r \leq 1$ is r-level set of u. Let $\tilde{u}, \tilde{v} \in E$, and set

$$d_\infty(\tilde{u}, \tilde{v}) = \sup_{0<r\leq 1} h([\tilde{u}]^r, [\tilde{v}]^r),$$

where h is Hausdorff metric i. e.

$$h([\tilde{u}]^r, [\tilde{v}]^r) = \max\{|\tilde{u}^-(r) - \tilde{v}^-(r)|, |\tilde{u}^+(r) - \tilde{v}^+(r)|\}.$$

The norm of u is defined by $||\tilde{u}||_\infty = d_\infty(\tilde{u}, \tilde{0})$ where $\tilde{0}$ is the fuzzy number in E whose membership function equals 1 at 0 and zero otherwise. In fact, d_∞ is a metric in E i. e. for x, y, z in E, d_∞ satisfies the following conditions [30]

i) $d_\infty(x, y) \geq 0,$

ii) $d_\infty(x, y) = 0$ iff $x = y,$

iii) $d_\infty(x, z) \leq d_\infty(x, y) + d_\infty(y, z).$

Definition 2.1. (Sadeghpour-Gildeh and Gien [33]) Let $\tilde{u}$ and $\tilde{v}$ be two fuzzy random numbers, for all $r \in [0,1]$ we use the Minkowski sum of two set and we have $([\tilde{u} \oplus \tilde{v}]^r = [\tilde{u}^-(r) + \tilde{v}^-(r), \tilde{u}^+(r) + \tilde{v}^+(r)]$; and $[\tilde{u} \ominus \tilde{v}]^r = [\tilde{u}^-(r) - \tilde{v}^+(r), \tilde{u}^+(r) - \tilde{v}^-(r)]$: If $\lambda \in \mathbb{R}^+$, we have $[\lambda \odot \tilde{u}]^r = [\lambda\tilde{u}^-(r), \lambda\tilde{u}^+(r)]$; and $[\lambda \odot \tilde{u}]^r = [\lambda\tilde{u}^+(r), \lambda\tilde{u}^-(r)]$; if $\lambda < 0$. If $\tilde{u}$ and $\tilde{v}$ be non-negative ($\tilde{u}; \tilde{v} \in E^+$), then $[\tilde{u} \otimes \tilde{v}] = [\tilde{u}^-(r).\tilde{v}^-(r), \tilde{u}^+(r).\tilde{v}^+(r)]$.

The operation $\langle \cdot, \cdot \rangle : E \times E \to [-\infty, \infty]$ is defined by

$$\langle \tilde{u}, \tilde{v} \rangle = \int_0^1 (\tilde{u}^-(r)\tilde{v}^-(r) + \tilde{u}^+(r)\tilde{v}^+(r))\,\mathrm{d}r.$$

If the indeterminacy of the form $\infty - \infty$ arises in the Lebesgue integral, then we say that $\langle \tilde{u}, \tilde{v} \rangle$ does not exist. It is easy to see that the operation $\langle \cdot, \cdot \rangle$ has following properties:

$$\begin{array}{ll}
\text{(i)} & \langle \tilde{u}, \tilde{u} \rangle \geq 0 \textit{ and } \langle \tilde{u}, \tilde{u} \rangle = 0 \Leftrightarrow u = \tilde{0}, \\
\text{(ii)} & \langle \tilde{u}, \tilde{v} \rangle = \langle \tilde{v}, \tilde{u} \rangle, \\
\text{(iii)} & \langle \tilde{u} + \tilde{v}, \tilde{w} \rangle = \langle \tilde{u}, \tilde{w} \rangle + \langle \tilde{v}, \tilde{w} \rangle, \\
\text{(iv)} & \langle \lambda\tilde{u}, \tilde{v} \rangle = \lambda\langle \tilde{u}, \tilde{v} \rangle, \\
\text{(v)} & |\langle \tilde{u}, \tilde{v} \rangle| < \sqrt{\langle \tilde{u}, \tilde{u} \rangle \langle \tilde{v}, \tilde{v} \rangle}.
\end{array}$$

For all $\tilde{u}, \tilde{v} \in E$, if $\langle \tilde{u}, \tilde{u} \rangle < \infty$ and $\langle \tilde{v}, \tilde{v} \rangle < \infty$ then the property (v) implies that $\langle \tilde{u}, \tilde{v} \rangle < \infty$. So, we can define

$$d_*(\tilde{u}, \tilde{v}) = \sqrt{\langle \tilde{u}, \tilde{u} \rangle - 2\langle \tilde{u}, \tilde{v} \rangle + \langle \tilde{v}, \tilde{v} \rangle}.$$

In fact, d_* is a metric in $\{\tilde{u} \in E | \langle \tilde{u}, \tilde{u} \rangle < \infty\}$ i.e. for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ in E, the metric d_* satisfies the following conditions.

i) $d_*(\tilde{x}, \tilde{y}) \geq 0$

ii) $d_*(\tilde{x}, \tilde{y}) = 0$ iff $\tilde{x} = \tilde{y}$,

iii) $d_*(\tilde{x}, \tilde{z}) \leq d_*(\tilde{x}, \tilde{y}) + d_*(\tilde{y}, \tilde{z})$ (subadditivity or triangle inequality).

Moreover, the norm $||\tilde{u}||_*$ of fuzzy number $\tilde{u} \in E$ is defined by $||\tilde{u}||_* = d_*(\tilde{u}, \tilde{0})$.

Definition 2.2. (Sadeghpour-Gildeh and Gien [33]) The $D_{p,q}$ distance, indexed by parameters $1 \leq p \leq \infty$, $0 \leq q \leq 1$, between two fuzzy numbers $\tilde{u}$ and $\tilde{v}$ is a nonnegative function on $E \times E$ given as follows

$$D_{p,q}(\tilde{u}, \tilde{v}) = \Big[(1-q) \int_0^1 |\tilde{u}^-(r) - \tilde{v}^-(r)|^p \, \mathrm{d}r + q \int_0^1 |\tilde{u}^+(r) - \tilde{v}^+(r)|^p \, \mathrm{d}r\Big]^{\frac{1}{p}}.$$

To prove main results we need to apply an order relation. Thus, we use notations $\prec$, $\succ$, $\preceq$ and $\succeq$ which mean [37]

$\tilde{a} \prec \tilde{b}$ if and only if $\tilde{a}^-(r) < \tilde{b}^-(r)$ and $\tilde{a}^+(r) < \tilde{b}^+(r)$ $\forall r \in [0, 1]$,

$\tilde{a} \succ \tilde{b}$ if and only if $\tilde{a}^-(r) > \tilde{b}^-(r)$ and $\tilde{a}^+(r) > \tilde{b}^+(r)$ $\forall r \in [0, 1]$,

$\tilde{a} \preceq \tilde{b}$ if and only if $\tilde{a}^-(r) \leq \tilde{b}^-(r)$ and $\tilde{a}^+(r) \leq \tilde{b}^+(r)$ $\forall r \in [0, 1]$,

$\tilde{a} \succeq \tilde{b}$ if and only if $\tilde{a}^-(r) \geq \tilde{b}^-(r)$ and $\tilde{a}^+(r) \geq \tilde{b}^+(r)$ $\forall r \in [0, 1]$.

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A fuzzy random variable (briefly: f.r.v.) is a Borel measurable function $\tilde{X} : (\Omega, \mathcal{A}) \to (E, d_\infty)$ [11]. Let $\tilde{X}$ be a f.r.v. is defined on $(\Omega, \mathcal{A}, P)$ then $[\tilde{X}]^r = [\tilde{X}^-(r), \tilde{X}^+(r)], r \in (0, 1]$, is a random closed interval, and $\tilde{X}^-(r)$ and $\tilde{X}^+(r)$ are real valued random variables. A f.r.v. $\tilde{X}$ is called integrably bounded if $E||\tilde{X}||_\infty < \infty$ and the expectation value $E\tilde{X}$ is defined as the unique fuzzy number which satisfies the property $[E\tilde{X}]^r = E[\tilde{X}]^r$, $0 < r \leq 1$ [30].

Definition 2.3. (Feng et al. [11]) Let $\tilde{X}$ and $\tilde{Y}$ be two f.r.v.'s in L_2 ($L_2 = \{\tilde{X} | \tilde{X}$ is f.r.v. and $E||\tilde{X}||_2^2 < \infty\}$). The covariance of $\tilde{X}$ and $\tilde{Y}$ is defined as

$$Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2} \int_0^1 (Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) + Cov(\tilde{X}^+(r), \tilde{Y}^+(r))) \, \mathrm{d}r.$$

Specially, the variance of $\tilde{X}$ is defined by $Var(\tilde{X}) = Cov(\tilde{X}, \tilde{X})$.

Theorem 2.4. (Feng et al. [11]) Let $\tilde{X}$ and $\tilde{Y}$ be f.r.v.'s in L_2 and $\tilde{u}, \tilde{v} \in E$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned}
&\text{(i)} \quad Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2}(E\langle \tilde{X}, \tilde{Y}\rangle - \langle E\tilde{X}, E\tilde{Y}\rangle) \\
&\text{(ii)} \quad Var(\tilde{X}) = \frac{1}{2} E d_*^2(\tilde{X}, E\tilde{X}) \\
&\text{(iii)} \quad Cov(\lambda\tilde{X} \oplus \tilde{u}, k\tilde{Y} \oplus \tilde{v}) = \lambda k Cov(\tilde{X}, \tilde{Y}) \\
&\text{(iv)} \quad Var(\lambda\tilde{X} \oplus \tilde{u}) = \lambda^2 Var(\tilde{X}); \\
&\text{(v)} \quad Var(\tilde{X} \oplus \tilde{Y}) = Var(\tilde{X}) + Var(\tilde{Y}) + 2Cov(\tilde{X}, \tilde{Y}).
\end{aligned}$$

The following Lemma which is due to Hoeffding shows the relationship between quadrant dependent and correlated real valued random variables.

Lemma 2.5. (Hoeffding [16]) Let X and Y be real valued random variables with joint distribution F and margins F_1 and F_2, respectively, then

$$\begin{aligned}
Cov(X, Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{F(x,y) - F_1(x)F_2(y)\}\,\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{P(X > x, Y > y) - P(X > x)P(Y > y)\}\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Definition 2.6. (Wu [40]) Two fuzzy random variables $\tilde{X}$ and $\tilde{Y}$ are called independent if two σ-fields $\sigma(\tilde{X}) = \sigma(\{\tilde{X}^-(r), \tilde{X}^+(r) | r \in [0,1]\})$ and $\sigma(\tilde{Y}) = \sigma(\{\tilde{Y}^-(r), \tilde{Y}^+(r) | r \in [0,1]\})$ are independent.

Definition 2.7. (Feng [9]) A fuzzy conditional expectation of $\tilde{X}$ with respect to the sub-σ field $\mathcal{B}$ of $\mathcal{A}$, denoted as $E(\tilde{X}|\mathcal{B})$, is defined as a f.r.v. which satisfies in the following conditions

i) $E(\tilde{X}|\mathcal{B})$ is $\mathcal{B}$- measurable.

ii) $\int_B E(\tilde{X}|\mathcal{B})\,\mathrm{d}P = \int_B \tilde{X}\,\mathrm{d}P$ for every $B \in \mathcal{B}$.

Note that $\int_B \tilde{X}\,\mathrm{d}P$ is a Aumann integral of the f.r.v. $\tilde{X}$ [3].

Proposition 2.8. (Feng [9]) The fuzzy conditional expectation has the following properties

1) $E(a \odot \tilde{X} \oplus b \odot \tilde{Y}|\mathcal{B}) = a \odot E(\tilde{X}|\mathcal{B}) \oplus b \odot E(\tilde{Y}|\mathcal{B})$ a.s.

2) $\tilde{X}$ is $\mathcal{B}$-measurable, then $E(\tilde{X}|\mathcal{B}) = \tilde{X}$ a.s.

3) $EE(\tilde{X}|\mathcal{B}) = E\tilde{X}$

4) If $\tilde{X} \preceq \tilde{Y}$ a.s. then $E(\tilde{X}|\mathcal{B}) \preceq E(\tilde{Y}|\mathcal{B})$ a.s.

5) $d_\infty(E(\tilde{X}|\mathcal{B}), E(\tilde{Y}|\mathcal{B})) \leq E(d_\infty(\tilde{X}, \tilde{Y})|\mathcal{B})$ a.s., and consequently

$$||E(\tilde{X}|\mathcal{B})||_\infty \leq E(||\tilde{X}||_\infty|\mathcal{B}) \quad a.s. \tag{1}$$

Definition 2.9. (Feng [9]) The sequence $\{\tilde{X}_n, \mathcal{B}_n\}$ of fuzzy random variables and σ-algebras is called a fuzzy martingale if we have, for each $n \geq 1$:

a) $\tilde{X}_n$ is $\mathcal{B}_n$-measurable and $E||\tilde{X}_n||_\infty < \infty$

b) $E(\tilde{X}_{n+1}|\mathcal{B}_n) = \tilde{X}_n$.

The sequence $\{\tilde{X}_n, \mathcal{B}_n\}$ is called a fuzzy sub-martingale, if property (b) is replaced by

b') $E(\tilde{X}_{n+1}|\mathcal{B}_n) \succeq \tilde{X}_n$.

For more on fuzzy martingale and related topics, see e. g. [7, 8, 9].

Definition 2.10. (Wu [40]) Let $\tilde{X}$ and $\tilde{X}_n$ be f.r.v.'s defined on the same probability space $(\Omega, \mathcal{A}, P)$. i) We say that $\{\tilde{X}_n\}$ converges to $\tilde{X}$ in probability with respect to the metric d if

$$\lim_{n\to\infty} P(\omega : d(\tilde{X}_n(\omega), \tilde{X}(\omega)) > \epsilon) = 0, \quad \forall \epsilon > 0.$$

ii) We say that $\{\tilde{X}_n\}$ converges to $\tilde{X}$ almost surely (briefly: a.s.) with respect to the metric d, if

$$P\Big(\omega : \lim_{n\to\infty} d(\tilde{X}_n(\omega), \tilde{X}(\omega)) = 0\Big) = 1.$$

3. MAXIMAL INEQUALITY OF QUADRATIC FORMS

In this section, we prove some maximal inequalities for $\tilde{T}_n = \oplus_{1\leq i<j\leq n} \tilde{X}_i \otimes \tilde{X}_j$ and $\tilde{W}_n = \oplus_{1\leq i<j\leq n} Y_{ij}\{\tilde{X}_i \otimes \tilde{X}_j\}$, where $\{\tilde{X}_i, i \geq 1\}$ is a sequence of independent fuzzy random variables and $\{Y_{i,j}; 1 \leq i < j \leq n\}$ be an array of non-negative real valued random variables such that the sequences $\{Y_{i,j}; 1 \leq i < j \leq n\}$ and $\{\tilde{X}_n; n \geq 1\}$ are independent. First, we extend the Doob's maximal inequality to f.r.v.'s.

Lemma 3.1. (Doob's maximal inequality for fuzzy random variables)

i) If $\{\tilde{X}_n, \mathcal{F}_n\}$ is a fuzzy martingale, then

$$E\Big(\max_{1\leq k\leq n} ||\tilde{X}_k||_\infty\Big)^r \leq \Big(\frac{r}{r-1}\Big)^r E||\tilde{X}_n||_\infty^r, \quad r > 1.$$

ii) If $\{\tilde{X}_n, \mathcal{F}_n | \tilde{X}_n \in E^+\}$ is a fuzzy submartingale, then

$$E\Big(\max_{1\leq k\leq n} ||\tilde{X}_k||_\infty\Big)^r \leq \Big(\frac{r}{r-1}\Big)^r E||\tilde{X}_n||_\infty^r, r > 1.$$

P r o o f. i) Definition of fuzzy martingale implies that $E(\tilde{X}_{n+1}|\mathcal{F}_n) = \tilde{X}_n$. By taking norm and invoking (1), we have $E(||\tilde{X}_{n+1}||_\infty|\mathcal{F}_n) \geq ||\tilde{X}_n||_\infty$. This shows that $\{||\tilde{X}_n||_\infty, \mathcal{F}_n\}$ is a real valued submartingale, thus we can use the ordinary Doob's maximal inequality to prove the claim.

ii) By definition of fuzzy submartingale, we have $E(\tilde{X}_{n+1}|\mathcal{F}_n) \succeq \tilde{X}_n$. Since $\tilde{X}_i \in E^+$), taking norm and using (1) imply that $E(||\tilde{X}_{n+1}||_\infty|\mathcal{F}_n) \geq ||\tilde{X}_n||_\infty$. Now, using the Doob's maximal inequality, the proof is complete. $\square$

Corollary 3.2. If $\{\tilde{X}_n, n \geq 1\}$ is a sequence of non-negative f.r.v.'s, then

$$E\Big(\max_{1\leq k\leq n} ||\tilde{S}_k||_\infty\Big)^r \leq \Big(\frac{r}{r-1}\Big)^r E||\tilde{S}_n||_\infty^r, \quad r > 1,$$

where $\tilde{S}_n = \oplus_{i=1}^n \tilde{X}_i$.

Theorem 3.3. Let $\{\tilde{X}_i | \tilde{X}_i \in E^+; i \geq 1\}$ be a sequence of independent f.r.v.'s with $E||\tilde{X}_i||_\infty < \infty$, for all $i \geq 1$ and some $1 < r \leq 2$. Then for every $\epsilon > 0$

$$P\Big(\max_{2\leq k\leq n} ||\tilde{T}_k||_\infty > \epsilon\Big) \leq C\Big(\frac{r}{\epsilon(r-1)}\Big)^r \Big(\sum_{j=2}^n E||\tilde{X}_j||_\infty^r \sum_{i=1}^{j-1} E||\tilde{X}_i||_\infty^r\Big),$$

where $\tilde{T}_k = \oplus_{1\leq i<j\leq k} \tilde{X}_i \otimes \tilde{X}_j$, and C is a positive constant depends only on r.

Proof. Since $\tilde{X}_i$ belongs to E^+ for each i, and $\tilde{T}_{n+1} = \tilde{T}_n \oplus \tilde{X}_{n+1} \otimes (\oplus_{i=1}^n \tilde{X}_i)$, so, by Definition 2.10, $\{\tilde{T}_n, \mathcal{F}_n\}$ is a fuzzy sub-martingale, i. e. $E(\tilde{T}_{n+1}|\mathcal{F}_n) \succeq \tilde{T}_n$. By invoking part(ii) of Lemma 3.1, we obtain

$$E\Big(\max_{1\leq k\leq n} ||\tilde{T}_k||_\infty\Big)^r \leq \Big(\frac{r}{r-1}\Big)^r E(||\tilde{T}_n||_\infty)^r, \quad 1 < r \leq 2. \tag{2}$$

So, by using the Markov inequality and relation (2) , we have

$$\begin{aligned} P\Big(\max_{1\leq k\leq n} ||\tilde{T}_k||_\infty > \epsilon\Big) &\leq \frac{1}{\epsilon^r} E\Big(\max_{1\leq k\leq n} ||\tilde{T}_k||_\infty\Big)^r \\ &\leq \frac{1}{\epsilon^r}\Big(\frac{r}{r-1}\Big)^r E(||\tilde{T}_n||_\infty)^r, \quad 1 < r \leq 2. \end{aligned}$$

But

$$\begin{aligned} E||\tilde{T}_n||_\infty^r &\leq E\Big(\sum_{j=2}^n ||\tilde{X}_j||_\infty ||\tilde{S}_{j-1}||_\infty\Big)^r \\ &\leq D_r \sum_{j=2}^n E||\tilde{X}_j||_\infty^r E||\tilde{S}_{j-1}||_\infty^r \\ &\leq D_r^2 \sum_{j=2}^n E||\tilde{X}_j||_\infty^r \sum_{i=1}^{j-1} E||\tilde{X}_i||_\infty^r, \quad 1 < r \leq 2. \end{aligned}$$

This completes the proof. □

Theorem 3.4. Let $\{\tilde{X}_i | \tilde{X}_i \in E^+; i \geq 1\}$ be a sequence of non-negative independent fuzzy random variables with $E||\tilde{X}_i||_\infty^r < \infty$ for all i and some $1 < r \leq 2$. If for some $1 < r \leq 2$, $\sum_{j=2}^\infty E||\tilde{X}_j||_\infty^r \sum_{i=1}^{j-1} E||\tilde{X}_i||_\infty^r < \infty$, then $\oplus_{i<j\leq n}\{\tilde{X}_i \otimes \tilde{X}_j\}$ converges a.s. with respect to the metric d_∞.

Proof. By using Theorem 3.3, for any $\epsilon > 0$, we obtain

$$\begin{aligned} P\Big(\max_{1\le k\le m} d_\infty(\tilde{T}_{n+k},\tilde{T}_n) > \epsilon\Big) &= P\Big(\max_{1\le k\le m} \| \oplus_{n<i<j\le k} \tilde{X}_i \otimes \tilde{X}_j \|_\infty\Big) \\ &\le C\Big(\frac{r}{\epsilon(r-1)}\Big)^r \Big(\sum_{j=n+1}^{n+m} E\|\tilde{X}_j\|_\infty^r \sum_{i=1}^{j-1} E\|\tilde{X}_i\|_\infty^r\Big), \end{aligned}$$

where C is a non-negative constant depends only on r. Letting $m \to \infty$ and $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty}\lim_{m\to\infty} P\Big(\max_{1\le k\le m} d_\infty(\tilde{T}_{n+k},\tilde{T}_n) > \epsilon) \\ \le\ &\lim_{n\to\infty}\lim_{m\to\infty} C\Big(\frac{r}{\epsilon(r-1)}\Big)^r \Big(\sum_{j=n+1}^{n+m} E\|\tilde{X}_j\|_\infty^r \sum_{i=1}^{j-1} E\|\tilde{X}_i\|_\infty^r\Big), \end{aligned}$$

But, by the assumption, the right side is equal to zero, and this completes the proof. □

Example 3.5. Let $\tilde{u}$ be a fuzzy number with $\|\tilde{u}\|_\infty = 1$, and $\tilde{X}_n$ be a sequence of independent f.r.v.'s such that $P(\tilde{X}_n = n\tilde{u}) = \frac{1}{n^\alpha}$, $P(\tilde{X}_n = \tilde{0}) = 1 - \frac{1}{n^\alpha}$ for $\alpha > 3$. Then

$$\sum_{j=2}^{\infty} E\|\tilde{X}_j\|_\infty^2 \sum_{i=1}^{j-1} E\|\tilde{X}_i\|_\infty^2 = \sum_{j=2}^{\infty} \frac{1}{j^{\alpha-2}} \sum_{i=1}^{j-1} \frac{1}{i^{\alpha-2}} \le \sum_{j=1}^{\infty} \frac{1}{j^{\alpha-2}} \frac{1}{j^{\alpha-3}} < \infty.$$

So, we can use the above theorem to conclude that $\oplus_{i<j\le n}\{\tilde{X}_i \otimes \tilde{X}_j\}$ converges a.s. with respect to the metric d_∞.

In the following, we extend Theorem 3.4, to randomly weighted quadratic form of f.r.v.'s.

Theorem 3.6. Let $\{\tilde{X}_n | n \ge 1\}$ be a sequence of non-negative independent f.r.v.'s with $E\|\tilde{X}_n\|_\infty^r < \infty$ for all n and some $1 < r \le 2$. Suppose that $\{Y_{i,j}; 1 \le i < j \le n\}$ is an array of non-negative real valued random variables such that the sequences $\{Y_{i,j}; 1 \le i < j \le n\}$ and $\{\tilde{X}_n; n \ge 1\}$ are independent. If for some $1 < r \le 2$, $\sum_{j=2}^{\infty} E\|\tilde{X}_j\|_\infty^r \sum_{i=1}^{j-1} EY_{i,j}^r E\|\tilde{X}_i\|_\infty^r < \infty$, then $\tilde{W}_n = \oplus_{i<j\le n} Y_{i,j}\{\tilde{X}_i \otimes \tilde{X}_j\}$ converges a.s. with respect to metric d_∞.

Proof. It is easy to show that $\{\tilde{W}_n, \mathcal{F}_n; n \ge 1\}$ is a non-negative fuzzy submartingale. Similar to the proof of Theorem 3.3, we can obtain the maximal inequality, as follows.

$$E\Big(\max_{1\le k\le n} \|\tilde{W}_k\|_\infty\Big)^r \le \Big(\frac{r}{r-1}\Big)^r E\|\tilde{W}_n\|_\infty.$$

On the other hand, we have

$$E\|\tilde{W}_n\|_\infty^r \le E\Big(\sum_{j=2}^{n} \|\tilde{X}_j\|_\infty \sum_{i=1}^{j-1} Y_{ij} \|\tilde{X}_i\|_\infty\Big)^r$$

$$\begin{aligned}
&\le D_r \sum_{j=1}^{n} E||\tilde{X}_j||_\infty^r E\Big(\sum_{i=1}^{j-1} Y_{ij}||\tilde{X}_i||_\infty\Big)^r \\
&\le D_r^2 \sum_{j=2}^{n} E||\tilde{X}_j||_\infty^r \sum_{i=1}^{j-1} EY_{ij}^r E||\tilde{X}_i||_\infty^r.
\end{aligned}$$

Now, by applying similar the proof of Theorem 3.4, it can be shown that $\oplus_{i<j\le n} Y_{i,j}\{\tilde{X}_i \otimes \tilde{X}_j\}$ converges a.s. with respect to metric d_∞. □

Corollary 3.7. Under the assumptions of Theorem 3.6, let we have an array of non-negative real numbers $\{a_{i,j}; 1 \le i < j \le n\}$ instead of the array $\{Y_{i,j}; 1 \le i < j \le n\}$. If $\sum_{j=2}^{\infty} E||\tilde{X}_j||_\infty^r \sum_{i=1}^{j-1} a_{i,j}^r E||\tilde{X}_i||_\infty^r < \infty$, then $\oplus_{i<j\le n} a_{i,j}\{\tilde{X}_i \otimes \tilde{X}_j\}$ converges a.s. with respect to the metric d_∞.

Corollary 3.8. Let $\{\tilde{X}_n, n \ge 1\}$ be a sequence of independent f.r.v.'s with $E||\tilde{X}_n||_\infty < \infty$ for all $n \ge 1$ and some $1 < r \le 2$. If $\sum_{j=2}^{\infty} \frac{1}{b_j^r} E||\tilde{X}_j||_\infty^r \sum_{i=1}^{j-1} E||\tilde{X}_i||_\infty < \infty$. Then $\frac{1}{b_n} \oplus_{i<j\le n} \{\tilde{X}_i \otimes \tilde{X}_j\}$ converges to $\tilde{0}$ with respect to the metric d_∞, where $\{b_n; n \ge 1\}$ is a sequence of positive increasing real numbers such that $b_n \to \infty$ as $n \to \infty$.

Corollary 3.9. If for some $1 < r \le 2$, $E||\tilde{X}_i||_\infty^r = O(i^{-\alpha})$ for any $0 < \alpha \le 1$, $b_n^{-1} = O(n^{-\frac{\beta}{r}})$ such that $\beta > 2(1-\alpha)$, then $\frac{1}{b_n} \oplus_{i<j\le n} \{\tilde{X}_i \otimes \tilde{X}_j\}$ converges to $\tilde{0}$ with respect to the metric d_∞.

4. MAXIMAL INEQUALITY AND LNQD F.R.V.'S

In this section, we introduce the concept of linearly negative quadrant dependence (LNQD, for short) for f.r.v.'s. Then by invoking a maximal inequality, we obtain some convergence theorems based on the metric $D_{p,q}$. In order to introduce the LNQD f.r.v.'s, we need to introduce the concept of negatively dependent (ND, for short) for f.r.v.'s.

Definition 4.1. Two f.r.v.'s $\tilde{X}$ and $\tilde{Y}$ are said negatively dependent if for any Borel sets B_1 and B_2 and all $r \in (0,1]$,

$$P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_2) \le P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_2),$$

where, $P([\tilde{X}]^r \subset B) = P(\omega : [\tilde{X}]^r(\omega) \subset B)$.

Remark 4.2. If $\tilde{X}$ and $\tilde{Y}$ reduce to real valued random variables and $B_1 = (-\infty, x_1]$ or (x_2, ∞) and $B_2 = (-\infty, y_1]$ or $(y_2, +\infty)$, definition 4.1 conclude the concept of negative dependence in the case of real valued random variables. Note that, in the ordinary case, two real valued random variables X and Y are said to be negatively dependent random variables if

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y) \quad \forall\, x \text{ and } y \in \mathbb{R},$$

and consequently

$$P(X > x, Y > y) \le P(X > x)P(Y > y) \quad \forall\, x \text{ and } y \in \mathbb{R}.$$

The following examples explain in the above definition.

Example 4.3. Suppose that two f.r.v.s $\tilde{X}$ and $\tilde{Y}$ have following probability functions,

$$P(\tilde{X}=\tilde{v},\tilde{Y}=\tilde{u})=\frac{3}{10},\quad P(\tilde{X}=\tilde{v},\tilde{Y}=\tilde{v})=\frac{2}{10},$$
$$P(\tilde{X}=\tilde{u},\tilde{Y}=\tilde{v})=\frac{3}{10},\quad P(\tilde{X}=\tilde{u},\tilde{Y}=\tilde{u})=\frac{2}{10},$$

where $\tilde{u}$ and $\tilde{v}$ are fuzzy numbers with the following membership function respectively

$$\mu_{\tilde{u}}(x)=\begin{cases} x, & 0\le x<1,\\ 1, & 1\le x\le 2,\\ 3-x, & 2<x<3,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\mu_{\tilde{v}}(x)=\begin{cases} 2x-2, & 1\le x<\frac{3}{2},\\ 4-2x, & \frac{3}{2}\le x\le 2,\\ 0, & \text{otherwise.}\end{cases}$$

The membership functions of $\tilde{u}$ and $\tilde{v}$ are presented in Figure 1. Then, X and Y are

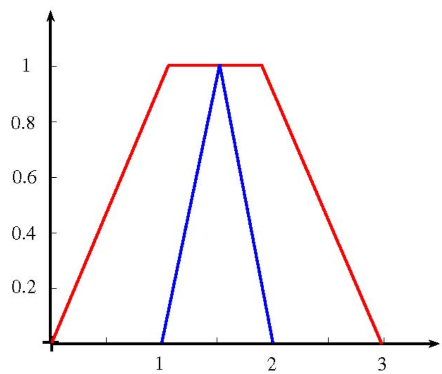

Fig. 1. The membership functions of $\tilde{u}$ and $\tilde{v}$.

ND f.r.v.s. Since, for $B_1=[x,y],(x,y],[x,y),(x,y),(x,+\infty),[x,+\infty),(-\infty,y),(-\infty,y]$ where $1\le x\le\frac{3}{2}$ and $\frac{3}{2}\le y\le 2$ also $B_2=[z,w],(z,w],[z,w),(z,w),(z,+\infty),[z,+\infty),$ $(-\infty,w),(-\infty,w]$ where $1\le z\le\frac{3}{2}$ and $\frac{3}{2}\le w\le 2$, we obtain

$$\begin{aligned} P([\tilde{X}]^r\subset B_1,[\tilde{Y}]^r\subset B_2) &= P(\tilde{X}=\tilde{v},\tilde{Y}=\tilde{v})=\frac{2}{10}\\ &< P([\tilde{X}]^r\subset B_1)P([\tilde{Y}]^r\subset B_2)=P(\tilde{X}=\tilde{v})P(\tilde{Y}=\tilde{v})\\ &= \frac{5}{10}\times\frac{5}{10}=\frac{25}{100}.\end{aligned}$$

For $B_3 = [x,y], (x,y], [x,y), (x,y), (x,+\infty), [x,+\infty), (-\infty,y), (-\infty,y]$ where $0 \le x \le 1$ and $2 \le y \le 3$ also
$B_4 = [z,w], (z,w], [z,w), (z,w), (z,+\infty), [z,+\infty), (-\infty,w), (-\infty,w]$ where $0 \le z \le 1$ and $2 \le w \le 3$, we obtain

$$\begin{aligned}
P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_3) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) = \frac{2}{10} + \frac{3}{10} = \frac{5}{10} \\
&= P([\tilde{X}]^r \subset B_1) P([\tilde{Y}]^r \subset B_3) \\
&= P(\tilde{X} = \tilde{v}) \{P(\tilde{Y} = \tilde{u}) + P(\tilde{Y} = \tilde{v})\} \\
&= \frac{5}{10} \times 1.
\end{aligned}$$

Also,

$$\begin{aligned}
&P([\tilde{X}]^r \subset B_3, [\tilde{Y}]^r \subset B_1) \\
&\quad = P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{v}) \\
&\quad = \frac{2}{10} + \frac{3}{10} = \frac{5}{10} \\
&\quad = P([\tilde{X}]^r \subset B_3) P([\tilde{Y}]^r \subset B_1) = P(\tilde{Y} = \tilde{v}) \{P(\tilde{X} = \tilde{u}) + P(\tilde{X} = \tilde{v})\} \\
&\quad = \frac{5}{10} \times 1.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
P([\tilde{X}]^r \subset B_3, [\tilde{Y}]^r \subset B_4) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{v}) \\
&\quad + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) = 1 \\
&= P([\tilde{X}]^r \subset B_3) P([\tilde{Y}]^r \subset B_4) \\
&= \{P(\tilde{Y} = \tilde{v}) + P(\tilde{Y} = \tilde{u})\} \{P(\tilde{X} = \tilde{u}) + P(\tilde{X} = \tilde{v})\} \\
&= 1 \times 1.
\end{aligned}$$

Example 4.4. If $\tilde{X}$ and $\tilde{Y}$ have following probability functions, then $\tilde{X}$ and $\tilde{Y}$ are ND f.r.v.'s,

$$\begin{aligned}
P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) = 0, P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) = \frac{1}{9}, \\
P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{w}) = \frac{2}{9}, P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{v}) = \frac{1}{9}, \\
P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) = \frac{1}{9}, P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{w}) = 0, \\
P(\tilde{X} = \tilde{w}, \tilde{Y} = \tilde{v}) = \frac{2}{9}, P(\tilde{X} = \tilde{w}, \tilde{Y} = \tilde{u}) = \frac{1}{9} \\
P(\tilde{X} = \tilde{w}, \tilde{Y} = \tilde{w}) = \frac{1}{9}.
\end{aligned}$$

where $\tilde{u}$ and $\tilde{v}$ are fuzzy numbers with the following membership function respectively (the membership functions of $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are presented in Figure 2).

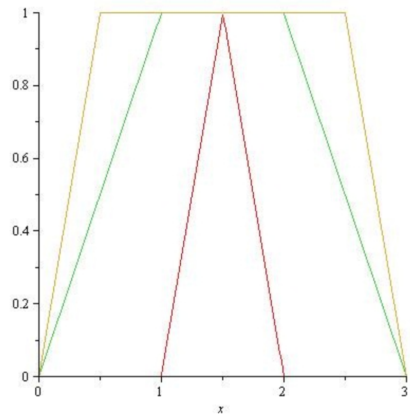

Fig. 2. The membership functions of $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ in Example 2.

$$\mu_{\tilde{u}}(x) = \begin{cases} x, & 0 \le x < 1, \\ 1, & 1 \le x \le 2, \\ 3 - x, & 2 < x < 3, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mu_{\tilde{v}}(x) = \begin{cases} 2x - 2, & 1 \le x < \frac{3}{2}, \\ 4 - 2x, & \frac{3}{2} \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

And,

$$\mu_{\tilde{w}}(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2}, \\ 1, & \frac{1}{2} \le x \le \frac{5}{2}, \\ 6 - 2x, & \frac{5}{2} < x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

Since for $B_1 = [x,y], (x,y], [x,y), (x,y), (x,+\infty), [x,+\infty), (-\infty,y), (-\infty,y]$ where $1 \le x \le \frac{3}{2}$ and $\frac{3}{2} \le y \le 2$ also for $B_2 = [z,w], (z,w], [z,w), (z,w), (z,+\infty), [z,+\infty), (-\infty,w)$, $(-\infty,w]$ where $1 \le z \le \frac{3}{2}$ and $\frac{3}{2} \le w \le 2$, we obtain

$$\begin{aligned} & P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_2) = P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) = 0 \\ < \quad & P([\tilde{X}]^r \subset B_1) P([\tilde{Y}]^r \subset B_2) = P(\tilde{X} = \tilde{v}) P(\tilde{Y} = \tilde{v}) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}. \end{aligned}$$

For $B_3 = [x,y], (x,y], [x,y), (x,y), (x,+\infty), [x,+\infty), (-\infty,y), (-\infty,y]$ where $\frac{1}{2} \le x \le 1$ and $2 \le y \le \frac{5}{2}$ also for $B_4 = [z,w], (z,w], [z,w), (z,w), (z,+\infty), [z,+\infty), (-\infty,w)$, $(-\infty,w]$ where $\frac{1}{2} \le z \le 1$ and $2 \le w \le \frac{5}{2}$, we obtain

$$P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_4) \quad = \quad P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) = 0 + \frac{1}{9} = \frac{1}{9}$$

$$\begin{aligned} &< P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_4) \\ &= P(\tilde{X} = \tilde{v})\{P(\tilde{Y} = \tilde{u}) + P(\tilde{Y} = \tilde{v})\} \\ &= \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}. \end{aligned}$$

$$\begin{aligned} P([\tilde{X}]^r \subset B_3, [\tilde{Y}]^r \subset B_4) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) \\ &\quad + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) \\ &= 0 + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{9}{27} \\ &< P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_3) \\ &= \{P(\tilde{X} = \tilde{v}) + P(\tilde{X} = \tilde{u})\} \times \{P(\tilde{Y} = \tilde{u}) + P(\tilde{Y} = \tilde{v})\} \\ &= \frac{5}{9} \times \frac{2}{3} = \frac{10}{27}. \end{aligned}$$

For $B_5 = [x, y], (x, y], [x, y), (x, y), (x, +\infty), [x, +\infty), (-\infty, y), (-\infty, y]$ where $0 \leq x \leq \frac{1}{2}$ and $\frac{5}{2} \leq y \leq 3$ also for $B_6 = [z, w], (z, w], [z, w), (z, w), (z, +\infty), [z, +\infty), (-\infty, w), (-\infty, w]$ where $0 \leq z \leq \frac{1}{2}$ and $\frac{5}{2} \leq w \leq 3$, we obtain

$$\begin{aligned} P([\tilde{X}]^r \subset B_1, [\tilde{Y}]^r \subset B_6) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) \\ &\quad + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{w}) \\ &= 0 + \frac{1}{9} + \frac{2}{9} = \frac{3}{9} \\ &= P([\tilde{X}]^r \subset B_1)P([\tilde{Y}]^r \subset B_6) \\ &= \{P(\tilde{X} = \tilde{v})\} \times \{P(\tilde{Y} = \tilde{u}) + P(\tilde{Y} = \tilde{v}) + P(\tilde{Y} = \tilde{w})\} \\ &= \frac{1}{3} = \frac{3}{9}. \end{aligned}$$

$$\begin{aligned} P([\tilde{X}]^r \subset B_3, [\tilde{Y}]^r \subset B_6) &= P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) \\ &\quad + P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{w}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{v}) \\ &\quad + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{w}) \\ &= 0 + \frac{1}{9} + \frac{2}{9} + \frac{1}{9} + \frac{1}{9} + 0 = \frac{5}{9} \\ &< P([\tilde{X}]^r \subset B_3)P([\tilde{Y}]^r \subset B_6) = \{P(\tilde{X} = \tilde{v}) + P(\tilde{X} = \tilde{u})\} \\ &\quad \times \{P(\tilde{Y} = \tilde{u}) + P(\tilde{Y} = \tilde{v}) + P(\tilde{X} = \tilde{w})\} \\ &= \frac{2}{3} \times 1 = \frac{6}{9}. \end{aligned}$$

$$\begin{aligned} P([\tilde{X}]^r \subset B_5, [\tilde{Y}]^r \subset B_6) &= 1 \\ &= P([\tilde{X}]^r \subset B_5)P([\tilde{Y}]^r \subset B_6) = 1 \times 1. \end{aligned}$$

Proposition 4.5. Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\tilde{X}$ and $\tilde{Y}$ be ND f.r.v.'s. Then, $\tilde{X}^-(r)$ and $\tilde{Y}^-(r)$ as well as $\tilde{X}^+(r)$ and $\tilde{Y}^+(r)$ are ND real valued random variables.

P r o o f. For all $x, y \in \mathbb{R}$ and $r \in (0, 1]$, we have

$$\begin{aligned} P(\tilde{X}^-(r) > x, \tilde{Y}^-(r) > y) &= P([\tilde{X}]^r \subset (x, \infty), [\tilde{Y}]^r \subset (y, \infty)) \\ &\leq P([\tilde{X}]^r \subset (x, \infty))P([\tilde{Y}]^r \subset (y, \infty)) \\ &= P(\tilde{X}^-(r) > x)P(\tilde{Y}^-(r) > y). \end{aligned}$$

Also,

$$\begin{aligned} & P(\tilde{X}^-(r) \leq x, \tilde{Y}^-(r) \leq y) = P(\tilde{X}^-(r) \leq x) - P(\tilde{X}^-(r) \leq x, \tilde{Y}^-(r) > y) \\ = \; & P(\tilde{X}^-(r) \leq x) - [P(\tilde{Y}^-(r) > y) - P(\tilde{X}^-(r) > x, \tilde{Y}^-(r) > y)] \\ = \; & 1 - P(\tilde{X}^-(r) > x) - P(\tilde{Y}^-(r) > y) + P(\tilde{X}^-(r) > x, \tilde{Y}^-(r) > y) \\ \leq \; & 1 - P(\tilde{X}^-(r) > x) - P(\tilde{Y}^-(r) > y) + P(\tilde{X}^-(r) > x)P(\tilde{Y}^-(r) > y) \\ = \; & 1 - P(\tilde{X}^-(r) > x) - P(\tilde{Y}^-(r) > y)[1 - P(\tilde{X}^-(r) > x)] \\ = \; & [1 - P(\tilde{Y}^-(r) > y)][1 - P(\tilde{X}^-(r) > x)] \\ = \; & P(\tilde{X}^-(r) \leq x)P(\tilde{Y}^-(r) \leq y). \end{aligned}$$

A similar proof can be stated for $\tilde{X}^+(r)$ and $\tilde{Y}^+(r)$. □

The converse of Proposition 4.5 is not necessarily correct.

Example 4.6. Let $\tilde{X}$ and $\tilde{Y}$ have the following probability mass

$$P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) = 0, \qquad P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{u}) = \frac{1}{2},$$
$$P(\tilde{X} = \tilde{u}, \tilde{Y} = -\tilde{u}) = \frac{1}{2}, \qquad P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) = 0,$$

where,

$$\mu_{\tilde{u}}(x) = \begin{cases} x, & 0 < x \leq 1, \\ 2 - x, & 1 < x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\tilde{X}$ and $\tilde{Y}$ are not ND f.r.v.'s, since

$$\begin{aligned} & P\Big([\tilde{X}]^r \subset \Big[\frac{1}{2}, \frac{3}{2}\Big], [\tilde{Y}]^r \subset \Big[-\frac{3}{2}, -\frac{1}{2}\Big]\Big) = \frac{1}{2} \\ > \; & P\Big([\tilde{Y}]^r \subset \Big[-\frac{3}{2}, -\frac{1}{2}\Big]\Big)P\Big([\tilde{X}]^r \subset \Big[\frac{1}{2}, \frac{3}{2}\Big]\Big) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

But, $\tilde{X}^+(r)$ and $\tilde{Y}^+(r)$ are ND random variables also $\tilde{X}^-(r)$ and $\tilde{Y}^-(r)$ are ND random variables. Since, for $x \in (-\infty, 0)$ and $y \in (-\infty, 0)$

$$\begin{aligned} P(\tilde{X}^-(r) \leq x, \tilde{Y}^-(r) \leq y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) = 0 \\ &< P(\tilde{X}^-(r) \leq x)P(\tilde{Y}^-(r) \leq y) \\ &= P(\tilde{X} = -\tilde{u})P(\tilde{Y} = -\tilde{u}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

Also, for $x \in [0, \infty)$ and $y \in (-\infty, 0)$

$$\begin{aligned} P(\tilde{X}^-(r) \le x, \tilde{Y}^-(r) \le y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) + P(\tilde{X} = \tilde{u}, \tilde{Y} = -\tilde{u}) \\ &= 0 + \frac{1}{2} \\ &= P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) \\ &= \{P(\tilde{X} = -\tilde{u}) + P(\tilde{X} = \tilde{u})\} \times P(\tilde{Y} = -\tilde{u}) \\ &= \left(\frac{1}{2} + \frac{1}{2}\right) \times \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Also, for $y \in [0, \infty)$ and $x \in (-\infty, 0)$

$$\begin{aligned} P(\tilde{X}^-(r) \le x, \tilde{Y}^-(r) \le y) &= P(\tilde{X} = -\tilde{u}, \tilde{Y} = -\tilde{u}) + P(\tilde{X} = -\tilde{u}, \tilde{Y} = \tilde{u}) \\ &= 0 + \frac{1}{2} \\ &= P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) \\ &= \{P(\tilde{Y} = -\tilde{u}) + P(\tilde{Y} = \tilde{u})\} \times P(\tilde{X} = -\tilde{u}) \\ &= \left(\frac{1}{2} + \frac{1}{2}\right) \times \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Finally, for $y \in [0, \infty)$ and $x \in [0, \infty)$

$$\begin{aligned} P(\tilde{X}^-(r) \le x, \tilde{Y}^-(r) \le y) &= 1 \\ &= P(\tilde{X}^-(r) \le x) P(\tilde{Y}^-(r) \le y) = 1 \times 1 = 1. \end{aligned}$$

Proposition 4.7. Let $\tilde{X}$ and $\tilde{Y}$ be two f.r.v.'s, then

$$\begin{aligned} Cov(\tilde{X}, \tilde{Y}) = & \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (x, \infty), [\tilde{Y}]^r \subset (y, \infty)) \\ & - P([\tilde{X}]^r \subset (x, \infty)) P([\tilde{Y}]^r \subset (y, \infty)) \, \mathrm{d}x \mathrm{d}y \mathrm{d}r \\ & + \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (-\infty, w), [\tilde{Y}]^r \subset (-\infty, z)) \\ & - P([\tilde{X}]^r \subset (-\infty, w)) P([\tilde{Y}]^r \subset (-\infty, z)) \, \mathrm{d}w \mathrm{d}z \mathrm{d}r. \end{aligned}$$

P r o o f. By Definition 2.3, we have

$$Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2} \int_0^1 \{Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) + Cov(\tilde{X}^+(r), \tilde{Y}^+(r))\} \, \mathrm{d}r.$$

But, by Lemma 2.5,

$$\begin{aligned} Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) = & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\tilde{X}^-(r) > x, \tilde{Y}^-(r) > y) \, \mathrm{d}x \mathrm{d}y \\ & - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\tilde{X}^-(r) > x) P(\tilde{Y}^-(r) > y) \, \mathrm{d}x \mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (x,\infty), [\tilde{Y}]^r \subset (y,\infty))\, \mathrm{d}x\mathrm{d}y \\
&\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (x,\infty))P([\tilde{Y}]^r \subset (y,\infty))\, \mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

and

$$
\begin{aligned}
Cov(\tilde{X}^+(r), \tilde{Y}^+(r)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\tilde{X}^+(r) \le w, \tilde{Y}^+(r) \le z)\, \mathrm{d}w\mathrm{d}z \\
&\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\tilde{X}^+(r) \le w)P(\tilde{Y}^+(r) \le z)\, \mathrm{d}w\mathrm{d}z \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (-\infty,w], [\tilde{Y}]^r \subset (-\infty,z])\, \mathrm{d}w\mathrm{d}z \\
&\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P([\tilde{X}]^r \subset (-\infty,w])P([\tilde{Y}]^r \subset (-\infty,z])\, \mathrm{d}w\mathrm{d}z.
\end{aligned}
$$

These complete the proof. □

Corollary 4.8. Let $\tilde{X}$ and $\tilde{Y}$ be two negatively dependent fuzzy random variables, then i) $Cov(\tilde{X}, \tilde{Y}) \le 0$, ii) $Var(\tilde{X} \oplus \tilde{Y}) \le Var(\tilde{X}) + Var(\tilde{Y})$, iii) $E\langle \tilde{X}, \tilde{Y}\rangle \le \langle E\tilde{X}, E\tilde{Y}\rangle$.

Proof. The proofs are straightforward. □

Example 4.9. If $\tilde{X}$ and $\tilde{Y}$ have following probability mass, then $Cov(\tilde{X}, \tilde{Y}) \le 0$,

$$
\begin{aligned}
P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{u}) &= \frac{1}{2}, P(\tilde{X} = \tilde{v}, \tilde{Y} = \tilde{v}) = 0, \\
P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{v}) &= \frac{1}{2}, P(\tilde{X} = \tilde{u}, \tilde{Y} = \tilde{u}) = 0,
\end{aligned}
$$

where $\tilde{u}$ and $\tilde{v}$ are fuzzy numbers with the following membership function respectively

$$
\mu_{\tilde{u}}(x) = \begin{cases} x, & 0 \le x < 1, \\ 1, & 1 \le x \le 2, \\ 3 - x, & 2 < x < 3, \\ 0, & \text{otherwise}, \end{cases}
$$

and

$$
\mu_{\tilde{v}}(x) = \begin{cases} 2x - 2, & 1 \le x < \frac{3}{2}, \\ 4 - 2x, & \frac{3}{2} \le x \le 2, \\ 0, & \text{otherwise}. \end{cases}
$$

By using a similar method of Example 4.3, it can be seen that $\tilde{X}$ and $\tilde{Y}$ are ND. Now, we want to show that $Cov(\tilde{X}, \tilde{Y}) \le 0$. By invoking Definition , we must calculate $Cov(\tilde{X}^+(r), \tilde{Y}^+(r))$ and $Cov(\tilde{X}^-(r), \tilde{Y}^-(r))$. It is easy to see that

$$
\tilde{u}^-(r) = r, \tilde{u}^+(r) = 3 - r, \tilde{v}^-(r) = \frac{r}{2} + 1, \tilde{v}^+(r) = 2 - \frac{r}{2}.
$$

Thus,

$$\begin{aligned} Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) &= r \times \frac{r+2}{2} - \Big(\frac{r}{2} + \frac{r+2}{4}\Big)^2, \\ Cov(\tilde{X}^+(r), \tilde{Y}^+(r)) &= (3-r)\Big(\frac{r+2}{2}\Big) - \Big(\frac{r}{2} + \frac{r+2}{4}\Big)^2. \end{aligned}$$

By using Proposition 4.7, we obtain

$$Cov(\tilde{X}, \tilde{Y}) = \frac{1}{2}\int_0^1 \{Cov(\tilde{X}^-(r), \tilde{Y}^-(r)) + Cov(\tilde{X}^+(r), \tilde{Y}^+(r))\}\, \mathrm{d}r = -7.34.$$

Definition 4.10. A sequence $\{\tilde{X}_n, n \geq 1\}$ of f.r.v.'s is said to be LNQD if for any disjoint subsets $A, B \in \mathcal{Z}^+$ and positive $r_j's$, $\oplus_{k\in A} r_k \tilde{X}_k$ and $\oplus_{j\in B} r_j \tilde{X}_j$ are ND f.r.v.'s.

Lemma 4.11. Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\{\tilde{X}_n, n \geq 1\}$ be a sequence of LNQD f.r.v.'s. Then $\sum_{j\in B} r_j \tilde{X}_j^-(r)$ and $\sum_{k\in A} r_k \tilde{X}_k^-(r)$ as well as $\sum_{j\in B} r_j \tilde{X}_j^+(r)$ and $\sum_{k\in A} r_k \tilde{X}_k^+(r)$ are ND real valued random variables and consequently the sequences $\{\tilde{X}_n^-(r), n \geq 1\}$ and $\{\tilde{X}_n^+(r), n \geq 1\}$ are sequences of LNQD real valued random variables, for all $r \in (0,1]$.

Proof. The proof can be done similar to that of Proposition 4.5. □

Theorem 4.12. Let $\{\tilde{X}_n, n \geq 1\}$ be a LNQD f.r.v.'s sequence. Then for $1 < p < 2$, there exists a positive constant c such that

$$E[\max_{1\leq i\leq n} D^p_{p,q}(\tilde{S}_i, E\tilde{S}_i)] \leq c n^{p-1} \sum_{i=1}^n E[D^p_{p,q}(\tilde{X}_i, E\tilde{X}_i)].$$

Proof. By using Fubini's theorem [15, p.65], Lemma 4.11, Lemma 2.8. of [38], and l_p standard inequality $(\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big)^p \leq \frac{1}{n}\sum_{i=1}^n x_i^p, \quad \forall x_i \in \mathbb{R})$, we obtain

$$\begin{aligned} & E[\max_{1\leq i\leq n} D^p_{p,q}(\tilde{S}_i, E\tilde{S}_i)] \\ = \; & E[\max_{1\leq i\leq n} (\int_0^1 (1-q)(\tilde{S}_i^-(r) - E\tilde{S}_i^-(r))^p\, \mathrm{d}r \\ & + \int_0^1 q(\tilde{S}_i^+(r) - E\tilde{S}_i^+(r))^p\, \mathrm{d}r)] \\ \leq \; & E[\int_0^1 (1-q) \max_{1\leq i\leq n} (\tilde{S}_i^-(r) - E\tilde{S}_i^-(r))^p\, \mathrm{d}r] \\ & + E[\int_0^1 q \max_{1\leq i\leq n} (\tilde{S}_i^+(r) - E\tilde{S}_i^+(r))^p\, \mathrm{d}r] \\ = \; & \int_0^1 (1-q) E[\max_{1\leq i\leq n} (\tilde{S}_i^-(r) - E\tilde{S}_i^-(r))^p]\, \mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 qE[\max_{1\le i\le n}(\tilde{S}_i^+(r)-E\tilde{S}_i^+(r))^p]\,\mathrm{d}r\\
\le\ & \int_0^1 c(1-q)\Big(\sum_{i=1}^n (E[|\tilde{X}_i^-(r)-E\tilde{X}_i^-(r)|^p])^{\frac{1}{p}}\Big)^p\,\mathrm{d}r\\
&+\int_0^1 cq\Big(\sum_{i=1}^n (E[|\tilde{X}_i^+(r)-E\tilde{X}_i^+(r)|^p]\Big)^{\frac{1}{p}})^p\,\mathrm{d}r\\
\le\ & cn^{p-1}\sum_{i=1}^n\int_0^1 qE[|\tilde{X}_i^+(r)-E\tilde{X}_i^+(r)|^p]dr\\
&+cn^{p-1}\sum_{i=1}^n\int_0^1 (1-q)E[|\tilde{X}_i^-(r)-E\tilde{X}_i^-(r)|^p]\,\mathrm{d}r\\
=\ & cn^{p-1}\sum_{i=1}^n E[\int_0^1 q|\tilde{X}_i^+(r)-E\tilde{X}_i^+(r)|^p\,\mathrm{d}r]\\
&+cn^{p-1}\sum_{i=1}^n E[\int_0^1 (1-q)|\tilde{X}_i^-(r)-E\tilde{X}_i^-(r)|^p\,\mathrm{d}r]\\
=\ & cn^{p-1}\sum_{i=1}^n E[D_{p,q}^p(\tilde{X}_i,E\tilde{X}_i)].
\end{aligned}
$$

This completes the proof. □

Theorem 4.13. Let $\{\tilde{X}_n, n\ge 1\}$ be LNQD f.r.v.'s sequences. Then for $p\ge 2$, there exists a positive constant c such that

$$E[\max_{1\le i\le n} D_{p,q}^p(\tilde{S}_i,E\tilde{S}_i)]\le cn^{\frac{p}{2}-1}\sum_{i=1}^n E[D_{p,q}^p(\tilde{X}_i,E\tilde{X}_i)].$$

Proof. By invoking Fubini's theorem, Lemma 4.11, Lemma 2.7. of [38], and l_p standard inequality, we have

$$
\begin{aligned}
&E[\max_{1\le i\le n} D_{p,q}^p(\tilde{S}_i,E\tilde{S}_i)]\\
=\ & E\max_{1\le i\le n}[\int_0^1 (1-q)(\tilde{S}_i^-(r)-E\tilde{S}_i^-(r))^p\,\mathrm{d}r+\int_0^1 q(\tilde{S}_i^+(r)-E\tilde{S}_i^+(r))^p\,\mathrm{d}r]\\
\le\ & E[\int_0^1 \max_{1\le i\le n}(\tilde{S}_i^-(r)-E\tilde{S}_i^-(r))^p\,\mathrm{d}r]\\
&+E[\int_0^1 q\max_{1\le i\le n}(\tilde{S}_i^+(r)-E\tilde{S}_i^+(r))^p\,\mathrm{d}r]\\
=\ & \int_0^1 (1-q)E[\max_{1\le i\le n}(\tilde{S}_i^-(r)-E\tilde{S}_i^-(r))^p]dr+\int_0^1 qE[\max_{1\le i\le n}(\tilde{S}_i^+(r)-E\tilde{S}_i^+(r))^p]\,\mathrm{d}r\\
\le\ & \int_0^1 c(1-q)\Big(\sum_{i=1}^n (E[|\tilde{X}_i^-(r)-E\tilde{X}_i^-(r)|^p])^{\frac{2}{p}}\Big)^{\frac{p}{2}}\,\mathrm{d}r
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^1 cq\Big(\sum_{i=1}^n (E[|\tilde{X}_i^+(r)-E\tilde{X}_i^+(r)|^p])^{\frac{2}{p}}\Big)^{\frac{p}{2}}\,\mathrm{d}r\\
\le\ & cn^{\frac{p}{2}-1}\sum_{i=1}^n\int_0^1 qE[|\tilde{X}_i^+(r)-E\tilde{X}_i^+(r)|^p]\,\mathrm{d}r\\
&+cn^{\frac{p}{2}-1}\sum_{i=1}^n\int_0^1 (1-q)E[|\tilde{X}_i^-(r)-E\tilde{X}_i^-(r)|^p]\,\mathrm{d}r\\
=\ & cn^{\frac{p}{2}-1}\sum_{i=1}^n E[\int_0^1 q|\tilde{X}_i^+(r)-E\tilde{X}_i^+(r)|^p\,\mathrm{d}r]\\
&+cn^{\frac{p}{2}-1}\sum_{i=1}^n E[\int_0^1 (1-q)|\tilde{X}_i^-(r)-E\tilde{X}_i^-(r)|^p\,\mathrm{d}r]\\
=\ & cn^{\frac{p}{2}-1}\sum_{i=1}^n E[D_{p,q}^p(\tilde{X}_i,E\tilde{X}_i)].
\end{aligned}
$$

And the proof is complete. □

5. WEAK LAW OF LARGE NUMBERS FOR LNQD F.R.V.'S

In this section, by invoking theorems of previous section, we established weak law of large numbers for LNQD f.r.v.'s.

Theorem 5.1. Let $\{\tilde{X}_n, n\ge 1\}$ be a sequence of LNQD f.r.v.'s.

i) If $\sum_{i=1}^n ED_{p,q}^p(\tilde{X}_i,E\tilde{X}_i)=o(n)$ for $1<p<2$, then

$$n^{-1}D_{p,q}(\tilde{S}_i,E\tilde{S}_i)\to 0 \ \text{ in } \ \text{probability},$$

i. e. with respect to the metric $D_{p,q}$, $\{\tilde{X}_n, n\ge 1\}$ obeys the weak law of large numbers.

ii) $\sum_{i=1}^n ED_{p,q}^p(\tilde{X}_i,E\tilde{X}_i)=o(n^{\frac{p}{2}+1})$ for $p\ge 2$, then

$$n^{-1}D_{p,q}(\tilde{S}_i,E\tilde{S}_i)\to 0 \ \text{ in } \ \text{probability},$$

i. e. with respect to the metric $D_{p,q}$, $\{\tilde{X}_n, n\ge 1\}$ obeys the weak law of large numbers.

Proof. i) It is obvious that $\max_{1\le i\le n} D_{p,q}(\tilde{S}_i,E\tilde{S}_i)$ dominate $D_{p,q}(\tilde{S}_i,E\tilde{S}_i)$. By using Theorem 4.12 and the assumption $\sum_{i=1}^n ED_{p,q}^p(\tilde{X}_i,E\tilde{X}_i)=o(n)$ for $1<p<2$, we obtain

$$n^{-1}\max_{1\le i\le n} D_{p,q}(\tilde{S}_i,E\tilde{S}_i)\to 0 \ \text{ in } \ l_p,$$

l_p convergence of $n^{-1}\max_{1\le i\le n} D_{p,q}(\tilde{S}_i,E\tilde{S}_i)$ implies l_p convergence of $n^{-1}D_{p,q}(\tilde{S}_n,E\tilde{S}_n)$. We know that l_p convergence concludes convergence in probability, thus

$$n^{-1}D_{p,q}(\tilde{S}_n,E\tilde{S}_n)\to 0 \text{ in } \ \text{probability}.$$

ii) It is clear that $D_{p,q}(\tilde{S}_n, E\tilde{S}_n)$ is dominated by $\max_{1\leq i\leq n} D_{p,q}(\tilde{S}_i, E\tilde{S}_i)$. By invoking Theorem 4.13 and the assumption $\sum_{i=1}^{n} ED^p_{p,q}(\tilde{X}_i, E\tilde{X}_i) = o(n^{\frac{p}{2}+1})$ for $p \geq 2$, we obtain

$$n^{-1} \max_{1\leq i\leq n} D_{p,q}(\tilde{S}_i, E\tilde{S}_i) \to 0 \ \text{ in } \ l_p,$$

l_p convergence of $n^{-1} \max_{1\leq i\leq n} D_{p,q}(\tilde{S}_i, E\tilde{S}_i)$ implies l_p convergence of $n^{-1} D_{p,q}(\tilde{S}_n, E\tilde{S}_n)$. We know that l_p convergence concludes convergence in probability, thus

$$n^{-1} D_{p,q}(\tilde{S}_n, E\tilde{S}_n) \to 0 \ \text{ in } \ \text{probability}.$$

□

6. CONCLUSION

The study of quadratic form of dependent f.r.v.'s, specially weak and strong laws of large numbers, for such random variables is a potential work for future research. Moreover, investigation of the almost surely convergence theorems and strong law of large numbers of LNQD f.r.v.'s may be some more topics for research.

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