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STRONG CONVERGENCE FOR WEIGHED SUMS OF NEGATIVELY SUPERADDITIVE DEPENDENT RANDOM VARIABLES

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In this paper, the strong law of large numbers for weighted sums of negatively superadditive dependent (NSD, in short) random variables is obtained, which generalizes and improves the corresponding one of Bai and Cheng ([2]) for independent and identically distributed random variables to the case of NSD random variables.

Keywords: NSD random variables, weighted sums, strong law of large numbers

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1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. We say that the sequence $\{X_n, n \geq 1\}$ satisfies the strong law of large numbers if there exist some increasing sequence $\{a_n,\ n \geq 1\}$ and some sequence $\{c_n,\ n \geq 1\}$ such that

$$\frac{1}{a_n} \sum_{i=1}^{n} (X_i - c_i) \to 0 \ \text{ a.s. as } n \to \infty.$$

Many authors have studied the strong law of large numbers for sequences of independent and identically distributed random variables. The following Theorems A is due to Bai and Cheng ([2]).

Theorem A. Suppose that $1 < \alpha, \beta < \infty$, $1 \leq p < 2$, and $1/p = \frac{1}{\alpha} + \frac{1}{\beta}$. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables satisfying $EX_1 = 0$, and let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of real numbers satisfying

$$\limsup_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |a_{nk}|^{\alpha} \right)^{1/\alpha} < \infty.$$

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If $E|X_1|^\beta < \infty$, then

$$n^{-1/p} \sum_{k=1}^{n} a_{nk} X_k \to 0 \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

The result of Theorem A for independent and identically distributed random variables has been extended to the case of dependent random variables. See for example, Jing and Liang ([12]) established Marcinkiewicz–Zygmund strong law of large numbers for weighted sums of negatively associated random variables. Meng and Lin ([14]) obtained the Marcinkiewicz–Zygmund strong law of large numbers for $\tilde{\rho}$-mixing random variables. Moveover, Shen ([17]) discussed the strong limit theorem for weighted sums of sequences of negatively dependent random variables. Sung ([22]) gave some sufficient conditions to prove the strong law of large numbers for weighted sums of random variables. Recently, Hu et al. ([9]) established the strong law of large numbers of partial sums for pairwise of negatively quadrant dependent sequences. Shen and Wu ([16]) investigated strong and weak convergence for asymptotically almost negatively associated random variables. Shen ([18]) established a general result on strong convergence for weighted sums of a class of random variables. Inspired by the literatures above, we will extend and improve the result of Theorem A to the case of weighted sums of negatively superadditive dependent random variables.

The definitions of negatively associated random variables and negatively superadditive dependent random variables are as follows.

Definition 1.1. A finite collection of random variables $X_1, X_2, \dots, X_n$ is said to be negatively associated (NA) if for every pair of disjoint subsets A_1, A_2 of $\{1, 2, \dots, n\}$,

$$Cov\{f(X_i : i \in A_1), g(X_j : j \in A_2)\} \le 0, \tag{1.1}$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n \ge 1\}$ is NA if every finite subcollection is NA.

The notion of NA was first introduced by Alam and Lai Saxena ([1]) and carefully studied by Joag-Dev and Proschan ([11]). Joag-Dev and Proschan ([11]) showed that many well known multivariate distributions possess the NA property. For more details about NA random variables, one can refer to Block et al. ([3]), Matuala ([13]), Budsaba et al. ([4]), Wu and Jiang ([28, 29]), Wang et al. ([23]), Yang et al. ([30]), Gerasimov et al. ([8]), and so on.

Definition 1.2. A function $\phi : R^n \to R$ is called superadditive if $\phi(\mathbf{x} \vee \mathbf{y}) + \phi(\mathbf{x} \wedge \mathbf{y}) \ge \phi(\mathbf{x}) + \phi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in R^n$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.

Next, we provide the concept of negatively superadditive dependent random variables as follows.

Definition 1.3. A random vector $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is said to be negatively superadditive dependent (NSD) if

$$E\phi(X_1, X_2, \dots, X_n) \le E\phi(X_1^*, X_2^*, \dots, X_n^*), \tag{1.2}$$

where $X_1^*, X_2^*, \dots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i, and ϕ is a superadditive function such that the expectations in (1.2) exist.

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be NSD if for all $n \geq 1$, $(X_1, X_2, \dots, X_n)$ is NSD.

The concept of NSD random variables was introduced by Hu ([10]). An example of an NSD sequence which is not NA was constructed by Hu ([10]), and illustrated that NSD implies NOD (negatively orthant dependent). Christofides and Vaggelatou ([5]) indicated that the family of NSD sequence contains NA. So we can see that NSD is weaker than NA. Negatively superadditive dependent structure is an extension of negatively associated structure and sometimes more useful than it and can be used to get many important probability inequalities. A number of limit theorems and applications for NSD random variables have been found by many authors. See for example, Eghbal et al. ([6]) derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables, and Eghbal et al. ([7]) provided some Kolmogorov inequality for quadratic forms of nonnegative NSD uniformly bounded random variables, Shen et al. ([20]) obtained Kolmogorov-type inequality and the almost sure convergence for NSD sequences, Shen et al. ([21]) established some inequalities for NSD random variables, Shen et al. ([19]) gave some applications of the Rosenthal-type inequality for NSD random variables, Wang et al. ([26]) presented some results on complete convergence for weighted sums of NSD random variables and gave its application in the EV regression model, and so forth.

Finally, we will present the concept of stochastic domination, which will be used frequently in this paper.

Definition 1.4. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C, such that

$$P(|X_n| > x) \leq CP(|X| > x) \tag{1.3}$$

for all $x \geq 0$ and $n \geq 1$.

The main purpose of the paper is to study the strong law of large numbers for weighted sums of NSD random variables, which generalizes the corresponding one of Theorem A for independent and identically distributed random variables. The techniques used in the paper are the truncation method and the moment inequality for NSD random variables.

Throughout this paper, let $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$ and C denotes a positive constant which may be different in various places. Let $I(A)$ be the indicator function of the set A and $a_n = O(b_n)$ stands for $a_n \leq Cb_n$.

2. PRELIMINARY LEMMAS

The main results of this paper are dependent on the following lemmas.

Lemma 2.1. (cf. Hu, [10]) Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables, and let $\{f_n, n \geq 1\}$ be a sequence of nondecreasing functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of NSD random variables.

Lemma 2.2. (cf. Hu, [10]) Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be an NSD random vector, and let $\mathbf{X}^* = (X_1^*, X_2^*, \dots, X_n^*)$ be independent vector such that X_i^* and X_i have the same distribution for each i. Then for any nondecreasing convex function f,

$$Ef\left(\max_{1\le k\le n}\sum_{i=1}^{k} X_i\right) \le Ef\left(\max_{1\le k\le n}\sum_{i=1}^{k} X_i^*\right). \tag{2.1}$$

Lemma 2.2 is the so called comparison theorem on moments between the NSD and independent random variables. Similarly to the proof of Theorem 2 of Shao ([15]) and by using Lemma 2.2, Wang et al. ([25]) got the following Rosenthal-type maximal inequality for NSD random variables.

Lemma 2.3. (Rosenthal-type maximal inequality) Let $\{X_n, n \ge 1\}$ be a sequence of NSD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for some $p > 2$. Then there exists a positive constant C_p depending only on p such that

$$E\left(\max_{1\le i\le n}\left|\sum_{i=1}^{n} X_i\right|^p\right) \le C_p\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E{X_i}^2\right)^{p/2}\right\}, \quad n \ge 1. \tag{2.2}$$

Lemma 2.4. (cf. Shen et al., [20]) Let $\{X_n, n \ge 1\}$ be a sequence of NSD random variables. If

$$\sum_{n=1}^{\infty} Var(X_n) < \infty, \tag{2.3}$$

then $\sum\limits_{n=1}^{\infty}(X_n - EX_n)$ converges almost surely.

Lemma 2.5. (cf. Wu, [27]) Let $\{X_n, n \ge 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:

$$\begin{aligned} E|X_n|^\alpha I(|X_n| \le b) &\le C_1[E|X|^\alpha I(|X| \le b) + b^\alpha P(|X| > b)], \\ E|X_n|^\alpha I(|X_n| > b) &\le C_2 E|X|^\alpha I(|X| > b), \end{aligned} \tag{2.4}$$

where C_1 and C_2 are positive constants.

3. MAIN RESULTS AND THEIR PROOFS

In this section, we will provide some results on strong convergence for weighted sums of NSD random variables.

Theorem 3.1. Let $0 < p < 2$, $0 < \alpha, \beta < \infty$, and $1/p = \frac{1}{\alpha} + \frac{1}{\beta}$. Assume that $\{X_n,\ n \ge 1\}$ is a sequence of NSD random variables stochastically dominated by a random variable X such that $E|X|^\beta < \infty$. Let $EX_n = 0$, $n \ge 1$, if $\beta > 1$ and $\{a_{ni},\ i \ge 1,\ n \ge 1\}$ be an array of real numbers satisfying

$$\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n). \tag{3.1}$$

Then

$$n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|\to 0 \quad \text{a.s.} \quad \text{as} \quad n\to\infty. \tag{3.2}$$

P r o o f. For fixed $n\ge 1$ and $1\le i\le n$, denote

$$Y_i=X_iI(|X_i|\le n^{1/\beta})+n^{1/\beta}I(X_i>n^{1/\beta})-n^{1/\beta}I(X_i<-n^{1/\beta}),$$

$$Z_i=(X_i+n^{1/\beta})I(X_i<-n^{1/\beta})+(X_i-n^{1/\beta})I(X_i>n^{1/\beta}).$$

Meanwhile, one can see that $a_{ni}=a_n^+-a_{ni}^-$. Without loss of generality, we can assume that $a_{ni}>0$. Hence, $X_i=Y_i+Z_i$, which implies that

$$\begin{aligned}
n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right| &\le n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}Y_i\right|+n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}Z_i\right|\\
&\le n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}(Y_i-EY_i)\right|+n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}EY_i\right|\\
&\quad +n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}Z_i\right|\\
&\doteq H+I+J.
\end{aligned}\tag{3.3}$$

To prove (3.2) we need to prove $H\to 0$ a.s., $I\to 0$ and $J\to 0$ a.s. as $n\to\infty$.

Firstly, we will show that $H\to 0$ a.s.
Combining (3.1) with Hölder's inequality, we have for $1\le\gamma<\alpha$ that

$$\sum_{i=1}^{n}|a_{ni}|^\gamma\le\left(\sum_{i=1}^{n}|a_{ni}|^\alpha\right)^{\gamma/\alpha}\left(\sum_{i=1}^{n}1\right)^{1-\gamma/\alpha}\le Cn. \tag{3.4}$$

Jensen's inequality implies that for any $0<\alpha\le\gamma$,

$$\sum_{i=1}^{n}|a_{ni}|^\gamma\le\left(\sum_{i=1}^{n}|a_{ni}|^\alpha\right)^{\gamma/\alpha}\le Cn^{\gamma/\alpha}. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\sum_{i=1}^{n}|a_{ni}|^\gamma\le Cn^{(1\vee\gamma/\alpha)}. \tag{3.6}$$

By Borel–Cantelli lemma, we only need to show that for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}(Y_i-EY_i)\right|>\varepsilon n^{1/p}\right)<\infty. \tag{3.7}$$

For fixed $n \geq 1$, it easily seen that $\{a_{ni}(Y_i - EY_i), 1 \leq i \leq n\}$ are still NSD random variables by Lemma 2.1. Taking $r > 1/\min\{1/\alpha, 1/\beta, 1/2, 1/p - 1/2\}$, which implies that $r > \alpha$, $r > \beta$ and $r > 2$. It follows from Markov's inequality and Lemma 2.3 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}(Y_i - EY_i)\right| > \varepsilon n^{1/p}\right) \\
\leq\ & C\sum_{n=1}^{\infty} n^{-r/p} E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}(Y_i - EY_i)\right|^r\right) \\
\leq\ & C\sum_{n=1}^{\infty} n^{-r/p}\sum_{i=1}^{n} E|a_{ni}(Y_i - EY_i)|^r + C\sum_{i=1}^{\infty} n^{-r/p}\left(\sum_{i=1}^{n} E|a_{ni}(Y_i - EY_i)|^2\right)^{r/2} \\
\doteq\ & H_1 + H_2.
\end{aligned}$$

For H_1, we have by C_r inequality, Jensen's inequality, (3.5) and Lemma 2.5 that

$$\begin{aligned}
H_1 \leq\ & C\sum_{n=1}^{\infty} n^{-r/p}\sum_{i=1}^{n} |a_{ni}|^r E|Y_i|^r \\
\leq\ & C\sum_{n=1}^{\infty} n^{-r/p}\sum_{i=1}^{n} |a_{ni}|^r [E|X_i|^r I(|X_i| \leq n^{1/\beta}) + n^{r/\beta} P(|X_i| > n^{1/\beta})] \\
\leq\ & C\sum_{n=1}^{\infty} n^{-r/p}\sum_{i=1}^{n} |a_{ni}|^r [E|X|^r I(|X| \leq n^{1/\beta}) + n^{r/\beta} P(|X| > n^{1/\beta})] \\
\leq\ & C\sum_{n=1}^{\infty} n^{-r/\beta} E|X|^r I(|X| \leq n^{1/\beta}) + C\sum_{n=1}^{\infty} P(|X| > n^{1/\beta}) \\
\leq\ & C\sum_{n=1}^{\infty} n^{-r/\beta}\sum_{i=1}^{n} E|X|^r I((i-1)^{1/\beta} < |X| \leq i^{1/\beta}) \\
& + C\sum_{n=1}^{\infty}\sum_{i=n}^{\infty} P(i^{1/\beta} < |X| \leq (i+1)^{1/\beta}) \\
\leq\ & C\sum_{i=1}^{\infty} E|X|^r I((i-1)^{1/\beta} < |X| \leq i^{1/\beta})\sum_{n=i}^{\infty} n^{-r/\beta} \\
& + C\sum_{i=1}^{\infty} EI(i^{1/\beta} < |X| \leq (i+1)^{1/\beta})i \\
\leq\ & C\sum_{i=1}^{\infty} i^{(r-\beta)/\beta} E|X|^\beta I((i-1)^{1/\beta} < |X| \leq i^{1/\beta}) i^{-r/\beta+1} \\
& + CE|X|^\beta\sum_{i=1}^{\infty} I(i^{1/\beta} < |X| \leq (i+1)^{1/\beta}) \\
\leq\ & CE|X|^\beta < \infty. \qquad (3.8)
\end{aligned}$$

Next, we prove that $H_2 < \infty$. By C_r inequality, Jensen's inequality, (3.6) and Lemma 2.5 again, we can get that

$$\begin{aligned}\sum_{i=1}^{n} E|a_{ni}(Y_i - EY_i)|^2 &\le \sum_{i=1}^{n} a_{ni}^2 EY_i^2 \\ &\le C\sum_{i=1}^{n} a_{ni}^2 [EX_i^2 I(|X_i| \le n^{1/\beta}) + n^{2/\beta} P(|X_i| > n^{1/\beta})] \\ &\le C\sum_{i=1}^{n} a_{ni}^2 [EX^2 I(|X| \le n^{1/\beta}) + n^{2/\beta} P(|X| > n^{1/\beta})] \\ &\le Cn^{(1\vee 2/\alpha)} [EX^2 I(|X| \le n^{1/\beta}) + n^{2/\beta} P(|X| > n^{1/\beta})]. \quad (3.9)\end{aligned}$$

It follows by Markov's inequality and the fact $E|X|^\beta < \infty$ that

$$\begin{aligned}&EX^2 I(|X| \le n^{1/\beta}) + n^{2/\beta} P(|X| > n^{1/\beta}) \\ \le &\begin{cases} CEX^2 I(|X| \le n^{1/\beta}) + EX^2 I(|X| > n^{1/\beta}), & \text{for } \beta \ge 2, \\ Cn^{(2-\beta)/\beta} E|X|^\beta I(|X| \le n^{1/\beta}) + n^{-1+2/\beta} E|X|^\beta I(|X| > n^{1/\beta}), & \text{for } \beta < 2, \end{cases} \\ \le &\begin{cases} CEX^2, & \text{for } \beta \ge 2, \\ Cn^{-1+2/\beta} EX^\beta, & \text{for } \beta < 2. \end{cases}\end{aligned} \quad (3.10)$$

If we denote $\delta = \max\{(-1+2/p), 2/\alpha, 2/\beta, 1\}$, then we can get by (3.9) and (3.10) that

$$\sum_{i=1}^{n} E|a_{ni}(Y_i - EY_i)|^2 \le Cn^\delta. \quad (3.11)$$

It is easily seen that

$$\begin{aligned}\Big(-\frac{1}{p} + \frac{\delta}{2}\Big) r &= \max\Big\{-\frac{1}{2}, -\frac{1}{\beta}, -\frac{1}{\alpha}, -\frac{1}{p} + \frac{1}{2}\Big\} r \\ &= -\min\Big\{\frac{1}{2}, \frac{1}{\beta}, \frac{1}{\alpha}, \frac{1}{p} - \frac{1}{2}\Big\} r < -1. \end{aligned} \quad (3.12)$$

Therefore, we have by (3.11) and (3.12) that

$$H_2 \le C\sum_{i=1}^{\infty} n^{(-1/p+\delta/2)r} < \infty, \quad (3.13)$$

which together with $H_1 < \infty$ yields (3.7).

On the other hand, we will prove that

$$I \doteq n^{-1/p} \max_{1\le j\le n} \Big|\sum_{i=1}^{j} a_{ni} EY_i\Big| \to 0, \quad \text{as } n \to \infty. \quad (3.14)$$

If $0 < \beta \leq 1$, then we have by Lemma 2.5 and (3.6) that

$$\begin{aligned}
I &\leq n^{-1/p}\sum_{i=1}^{n}|a_{ni}EY_i| \\
&\leq n^{-1/p}\sum_{i=1}^{n}|a_{ni}|[E|X_i|I(|X_i| \leq n^{1/\beta}) + n^{1/\beta}P(|X_i| > n^{1/\beta})] \\
&\leq Cn^{-1/p}\sum_{i=1}^{n}|a_{ni}|[E|X|I(|X| \leq n^{1/\beta}) + n^{1/\beta}P(|X| > n^{1/\beta})] \\
&\leq Cn^{-1/p}\sum_{i=1}^{n}|a_{ni}|[n^{(1-\beta)/\beta}E|X|^{\beta}I(|X| \leq n^{1/\beta}) + n^{1/\beta-1}E|X|^{\beta}I(|X| > n^{1/\beta})] \\
&= Cn^{-1/\alpha-1}E|X|^{\beta}\sum_{i=1}^{n}|a_{ni}| \\
&\leq Cn^{-1/\alpha-1+(1\vee 1/\alpha)} \to 0, \quad \text{as } n \to \infty. \qquad (3.15)
\end{aligned}$$

If $\beta > 1$, then we have by $EX_n = 0$, Lemma 2.5 and (3.6) that

$$\begin{aligned}
I &\leq n^{-1/p}\sum_{i=1}^{n}|a_{ni}EY_i| \\
&\leq n^{-1/p}\sum_{i=1}^{n}|a_{ni}|[E|X_i|I(|X_i| > n^{1/\beta}) + n^{1/\beta}P(|X_i| > n^{1/\beta})] \\
&\leq Cn^{-1/p}\sum_{i=1}^{n}|a_{ni}|E|X|I(|X| > n^{1/\beta}) \\
&\leq Cn^{-1/p}\sum_{i=1}^{n}|a_{ni}|n^{1/\beta-1}E|X|^{\beta}I(|X| > n^{1/\beta}) \\
&\leq Cn^{-1/\alpha-1}E|X|^{\beta}\sum_{i=1}^{n}|a_{ni}| \\
&\leq Cn^{-1/\alpha-1+(1\vee 1/\alpha)} \to 0, \quad \text{as } n \to \infty. \qquad (3.16)
\end{aligned}$$

Hence, (3.14) follows from (3.15) and (3.16) immediately.

Finally, we prove $J \to 0$ a.s. as $n \to \infty$.
The condition $E|X|^{\beta} < \infty$ yields that

$$\sum_{n=1}^{\infty}P(Z_n \neq 0) = \sum_{n=1}^{\infty}P(|X_n| > n^{1/\beta}) \leq C\sum_{n=1}^{\infty}P(|X| > n^{1/\beta}) \leq CE|X|^{\beta} < \infty, \quad (3.17)$$

which implies that $P(Z_n \neq 0, \text{i.o.}) = 0$ by Borel–Cantelli lemma. Hence, we have by (3.1) that

$$\begin{aligned} J &\doteq n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}Z_i\right| \le n^{-1/p}\sum_{i=1}^{n}|a_{ni}Z_i| \\ &\le n^{-1/p}\Big(\max_{1\le i\le n}|a_{ni}|^{\alpha}\Big)^{1/\alpha}\sum_{i=1}^{n}|Z_i| \le n^{-1/p}(\sum_{i=1}^{n}|a_{ni}|^{\alpha})^{1/\alpha}\sum_{i=1}^{n}|Z_i| \\ &\le Cn^{-1/\beta}\sum_{i=1}^{n}|Z_i| \to 0, \quad \text{a.s.,} \quad \text{as } n\to\infty. \end{aligned} \tag{3.18}$$

Therefore, the desired result (3.2) follows from (3.7), (3.14) and (3.18) immediately. This completes the proof of the Theorem. □

Taking $a_{ni} \equiv 1$ in Theorem 3.1, then (3.1) is always valid for any $\alpha > 0$. Hence, for any $0 < p < \min(\beta, 2)$, letting $\alpha = p\beta/(\beta - p) > 0$, we can obtain the following corollary.

Corollary 3.2. Let $\{X_n, n \ge 1\}$ be a sequence of NSD identically distributed random variables with $E|X_1|^{\beta} < \infty$. If $\beta > 1$, further assume that $EX_1 = 0$, then for any $0 < p < \min(\beta, 2)$,

$$n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| \to 0 \ \text{ a.s. as } \ n\to\infty. \tag{3.19}$$

Remark 3.3. Theorem 3.1 generalizes and improves Theorem A of Bai and Cheng ([2]) for independent and identically distributed random variables to the case of NSD random variables, since Theorem 3.1 removes the identically distributed condition and expands the ranges of α, β, and p, respectively.

Remark 3.4. Comparing Theorem 3.1 with Theorem 2.1 of Wang et al. ([24]), we have the following improvements:

(i) the moment condition $E|X|^{\beta} < \infty$ in Theorem 3.1 is weaker than (2.1) in Wang et al ([24]);

(ii) the condition (3.1) in Theorem 3.1 is weaker than $\sum_{i=1}^{n}|a_{ni}|^{\alpha} = O(n^{\delta})$ for some $0 < \alpha < 2$ and $0 < \delta < 1$ in Wang et al ([24]).

Theorem 3.5. Let $1 < r < 2$ and $\{X_n, n \ge 1\}$ be a sequence of mean zero NSD random variables, which is stochastically dominated by a random variable X. Let $\{a_n, n \ge 1\}$ be a sequence of positive constants satisfying $A_n \doteq \sum_{k=1}^{n} a_k \uparrow \infty$. Denote $c_n = A_n/a_n$ for each $n \ge 1$. Assume that

$$E|X|^r < \infty, \tag{3.20}$$

$$N(n) \doteq Card\{i : c_i \le n\} = O(n^r), \quad n \ge 1, \tag{3.21}$$

then

$$A_n^{-1}\sum_{k=1}^{n} a_k X_k \to 0 \ \text{ a.s., as } \ n\to\infty. \tag{3.22}$$

P r o o f. Let $N(0) = 0$ and denote

$$X_n^{(c_n)} = -c_n I(X_n < -c_n) + X_n I(|X_n| \le c_n) + c_n I(X_n > c_n), \quad n \ge 1.$$

It follows from (3.20) and (3.21) that

$$\begin{aligned}
\sum_{i=1}^{\infty} P\left(X_i \ne X_i^{(c_i)}\right) &= \sum_{i=1}^{\infty} P\left(|X_i| > c_i\right) = \sum_{j=1}^{\infty} \sum_{c_i \le j < c_i+1} P\left(|X_i| > c_i\right) \\
&\le C \sum_{j=1}^{\infty} \sum_{j-1 < c_i \le j} P\left(|X| > j-1\right) \\
&= C \sum_{j=1}^{\infty} (N(j) - N(j-1)) P\left(|X| > j-1\right) \\
&= C \sum_{j=1}^{\infty} (N(j) - N(j-1)) \sum_{n=j}^{\infty} P\left(n-1 < |X| \le n\right) \\
&= C \sum_{n=1}^{\infty} \sum_{j=1}^{n} (N(j) - N(j-1)) P\left(n-1 < |X| \le n\right) \\
&\le C \sum_{n=1}^{\infty} n^r P\left(n-1 < |X| \le n\right) \le CE|X|^r < \infty.
\end{aligned}$$

By the inequality above and Borel–Cantelli lemma, we can get $P(X_i \ne X_i^{(c_i)}, i.o.) = 0$. Therefore, in order to prove (3.22), we only need to prove

$$A_n^{-1} \sum_{i=1}^{n} a_i X_i^{(c_i)} \to 0 \quad \text{a.s.}, \quad n \to \infty. \tag{3.23}$$

By C_r inequality, Lemma 2.5, (3.20) and (3.21) again,

$$\begin{aligned}
\sum_{k=1}^{\infty} Var\left(\frac{a_k X_k^{(c_k)}}{A_k}\right) &\le \sum_{k=1}^{\infty} {c_k}^{-2} E(X_k^{(c_k)})^2 \\
&\le 3 \sum_{k=1}^{\infty} {c_k}^{-2} E\left[{c_k}^2 I(|X_k| > c_k) + X_k^2 I(|X_k| \le c_k)\right] \\
&\le C \sum_{k=1}^{\infty} P(|X| > c_k) + C \sum_{k=1}^{\infty} {c_k}^{-2} EX^2 I(|X| \le c_k) \\
&\le C + C \sum_{j=1}^{\infty} \sum_{j-1 < c_k \le j} {c_k}^{-2} EX^2 I(|X| \le c_k) \\
&\le C + C \sum_{j=1}^{\infty} \sum_{j-1 < c_k \le j} {c_k}^{-2} EX^2 I(|X| \le j)
\end{aligned}$$

$$
\begin{aligned}
&\leq C + C\sum_{j=2}^{\infty}(N(j)-N(j-1))(j-1)^{-2}\sum_{k=1}^{j}EX^2I(k-1<|X|\leq k)\\
&\leq C + C\sum_{k=2}^{\infty}\sum_{j=k}^{\infty}(N(j)-N(j-1))(j-1)^{-2}EX^2I(k-1<|X|\leq k)\\
&\leq C + C\sum_{k=2}^{\infty}\sum_{j=k}^{\infty}N(j)((j-1)^{-2}-j^{-2})EX^2I(k-1<|X|\leq k)\\
&\leq C + C\sum_{k=2}^{\infty}\sum_{j=k}^{\infty}j^{r-3}EX^2I(k-1<|X|\leq k)\\
&\leq C + C\sum_{k=2}^{\infty}k^{r-2}E|X|^rk^{2-r}I(k-1<|X|\leq k)\\
&= C + C\sum_{k=2}^{\infty}E|X|^rI(k-1<|X|\leq k)\\
&\leq C + CE|X|^r<\infty.
\end{aligned}
$$

Hence, by the inequality above, Lemma 2.4 and Kronecker's lemma, we have

$$
A_n^{-1}\sum_{i=1}^{n}a_i\left(X_i^{(c_i)}-EX_i^{(c_i)}\right)\to 0 \ \text{ a.s.}, \quad n\to\infty. \tag{3.24}
$$

In order to prove (3.22), it suffices to prove that

$$
A_n^{-1}\sum_{i=1}^{n}a_iEX_i^{(c_i)}\to 0, \quad n\to\infty. \tag{3.25}
$$

By (3.21), it easily seen that $c_n\to\infty$ as $n\to\infty$. Notice that $EX_n=0$ for each $n\geq 1$, we have

$$
|EX_nI(|X_n|\leq c_n)|=|EX_nI(|X_n|>c_n)|.
$$

It follows by Lemma 2.5, (3.20) and (3.21) that,

$$
\begin{aligned}
\sum_{k=1}^{\infty}\left|\frac{a_kEX_k^{(c_k)}}{A_k}\right| &\leq \sum_{k=1}^{\infty}[P(|X_k|>c_k)+{c_k}^{-1}|EX_kI(|X_k|\leq c_k)|]\\
&= \sum_{k=1}^{\infty}[P(|X_k|>c_k)+{c_k}^{-1}|EX_kI(|X_k|>c_k)|]\\
&\leq \sum_{k=1}^{\infty}P(|X_k|>c_k)+\sum_{k=1}^{\infty}{c_k}^{-1}E|X_k|I(|X_k|>c_k)\\
&\leq C\sum_{k=1}^{\infty}P(|X|>c_k)+C\sum_{k=1}^{\infty}{c_k}^{-1}E|X|I(|X|>c_k)
\end{aligned}
$$

$$\begin{aligned}
&\le C + C\sum_{k=1}^{\infty}\sum_{c_k\le j<c_k+1} {c_k}^{-1}E|X|I(|X|>c_k)\\
&\le C + C\sum_{k=1}^{\infty}\sum_{j-1<c_k\le j} {c_k}^{-1}E|X|I(|X|>j-1)\\
&\le C + C\sum_{j=2}^{\infty}(N(j)-N(j-1))(j-1)^{-1}\sum_{k=j-1}^{\infty} EXI(k<|X|\le k+1)\\
&\le C + C\sum_{k=1}^{\infty}\sum_{j=2}^{k+1} N(j)((j-1)^{-1}-j^{-1})EXI(k<|X|\le k+1)\\
&\le C + C\sum_{k=1}^{\infty}\sum_{j=2}^{k+1} j^{r-2}EXI(k<|X|\le k+1)\\
&\le C + C\sum_{k=1}^{\infty} k^{r-1}E|X|I(k<|X|\le k+1)\\
&\le C + C\sum_{k=1}^{\infty} E|X|^rI(k<|X|\le k+1)\le C+CE|X|^r<\infty.
\end{aligned}$$

By Kronecker's lemma, we can get(3.24) immediately. The proof is complete. □

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