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INCOMPARABILITY WITH RESPECT TO THE TRIANGULAR ORDER

Emel Aşici and Funda Karaçal

In this paper, we define the set of incomparable elements with respect to the triangular order for any t-norm on a bounded lattice. By means of the triangular order, an equivalence relation on the class of t-norms on a bounded lattice is defined and this equivalence is deeply investigated. Finally, we discuss some properties of this equivalence.

Keywords: triangular norm, T-partial order, bounded lattice

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1. INTRODUCTION

Triangular norms were originally studied in the framework of probabilistic metric spaces [20, 21, 22, 23] aiming at an extension of the triangle inequality. Later on, they turned out to be interpretations of the conjuction in many-valued logics [6], in particular in fuzzy logics, where the unit interval serves as set of truth values.

In [18], it was defined a natural order for semigroups. Similarly, in [8], a partial order defined by means of t-norms on a bounded lattice was introduced. For any elements x, y of a bounded lattice L

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L,$$

where T is a t-norm. This order $\preceq_T$ is called a t-partial order of T. Moreover, the authors have investigated connections between the natural order $\leq$ on L and the T-partial order $\preceq_T$ on L.

In [8], it was obtained that $\preceq_T$ implies the natural order $\leq$ but its converse needs not be true. It was showed that a partially ordered set is not a lattice with respect to $\preceq_T$. Some sets were determined which, under some special conditions, are lattices with respect to $\preceq_T$. For more details on t-norms on bounded lattices, we refer to [3, 9, 10, 11, 13, 15, 16, 17, 19].

In [12], by means of the T-partial order, an equivalence relation on the class of t-norms was given and the equivalence classes linked to some special t-norms were characterized. In [7], an equivalence relation on the class of the t-norms on $[0, 1]$ was defined. It was

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showed that the equivalence class of the weakest t-norm T_D on $[0,1]$ contains a t-norm which was different from T_D.

In [1], with the help of any t-norm T on $[0,1]$, it was obtained that the family $(T_\lambda)_{\lambda\in(0,1)}$ of t-norms on $[0,1]$. If T was a divisible t-norm, then it was obtained that $([0,1],\preceq_{T_\lambda})$ was a lattice.

In the present paper, we introduce the set of incomparable elements with respect to the t-order for any t-norm on a bounded lattice $(L,\leq,0,1)$. By defining such an set, the set of incomparable elements with respect to the t-order for any t-norm on $[0,1]$ is extended to a more general form. The main aim is to investigate some properties of this set. The paper is organized as follows. We shortly recall some basic notions in Section 2. In Section 3, we define the set of incomparable elements with respect to the t-order for any t-norm on a bounded lattice $(L,\leq,0,1)$ and we determine the sets of incomparable elements w.r.t. t-order of the infimum t-norm $T_\wedge$ and the weakest t-norm T_W. In Section 4, we define an equivalence on the class of t-norms on a bounded lattice $(L,\leq,0,1)$. We determine the equivalence class of the infimum t-norm $T_\wedge$ when L is a chain. Thus, we obtain that, in the case of $L=[0,1]$, all continuous t-norms are equivalent. Although, we give some examples illustrating that left-continuous t-norms need not be equivalent, in general. We show by an example that the left- continuity of any of the t-norms in the equivalence class does not imply the left-continuity for another t-norm in the equivalence class. In [1], it was shown that "T_1 and T_2 are two t-norms on $[0,1]$ such that for all $x\in[0,1]$, ${\mathcal{I}_{T_1}}^{(x)}={\mathcal{I}_{T_2}}^{(x)}$ if and only if the t-norms T_1 and T_2 are equivalent under the relation $\sim$ in (2)". In this study, by an example we show that this proposition only provides a sufficient and not a necessary condition for the relation β_L in (3).

2. NOTATIONS, DEFINITIONS AND A REVIEW OF PREVIOUS RESULTS

Definition 2.1. (Klement et al. [14]) A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0,1]$, i. e., a function $T:[0,1]^2\to[0,1]$, such that for all $x,y,z\in[0,1]$ the following four axioms are satisfied:

$(T1)$ $T(x,y)=T(y,x)$ (commutativity)

$(T2)$ $T(x,T(y,z))=T(T(x,y),z)$ (associativity)

$(T3)$ $T(x,y)\leq T(x,z)$ whenever $y\leq z$ (monotonicity)

$(T4)$ $T(x,1)=x$ (boundary condition)

Example 2.2. (Klement et al. [14]) The following are the four basic t-norms T_M,T_P,T_L,T_D given by, respectively:

$$T_M(x,y)=\min(x,y)$$

$$T_P(x,y)=x\cdot y$$

$$T_L(x,y)=\max(x+y-1,0)$$

$$T_D(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Also, t-norms on a bounded lattice $(L, \leq, 0, 1)$ are defined in similar way, and then extremal t-norms $T_\wedge$ and T_W on L is defined as follows, respectively:

$$T_\wedge(x,y) = x \wedge y,$$

$$T_W(x,y) = \begin{cases} x, & \text{if } y = 1, \\ y, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Especially we obtained that $T_W = T_D$ and $T_\wedge = T_M$ for $L = [0,1]$.

Definition 2.3. (Klement et al. [14]) A function $F : [0,1]^2 \to [0,1]$ is called continuous if for all convergent sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$, the following holds

$$F\Big(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\Big) = \lim_{n\to\infty} F(x_n, y_n).$$

Definition 2.4. (Casasnovas and Mayor [5]) A t-norm T on L is divisible if the following condition holds:

$$\forall x, y \in L \text{ with } x \leq y \text{ there is a } z \in L \text{ such that } x = T(y,z).$$

Proposition 2.5. (De Baets and Mesiar [4]) Let T be a t-norm on $[0,1]$. T is divisible if and only if T is continuous.

Definition 2.6. (Birkhoff [2]) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case we use the notation $a \parallel b$.

Definition 2.7. (Karaçal and Kesicioğlu [8]) Let $(L, \leq, 0, 1)$ be a bounded lattice, T be a t-norm on L. The order defined as following is called a t-order (triangular order) for t-norm T.

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \quad \text{for some} \quad \ell \in L. \tag{1}$$

Proposition 2.8. (Karaçal and Kesicioğlu [8]) Let $(L, \leq, 0, 1)$ be a bounded lattice, T be a t-norm on L. Then the binary relation $\preceq_T$ is a partial order on L.

Proposition 2.9. (Karaçal and Kesicioğlu [8]) Let T be a t-norm on a bounded lattice $(L, \leq, 0, 1)$. Then, if $x \preceq_T y$ necessarily we have also $x \leq y$.

Lemma 2.10. (Kesicioğlu et al. [12]) Let $(L, \leq, 0, 1)$ be a bounded lattice. For all t-norms on L and all $x \in L$ it holds that $0 \preceq_T x$, $x \preceq_T x$ and $x \preceq_T 1$.

Definition 2.11. (Kesicioğlu et al. [12]) Let T be a t-norm on $[0,1]$ and let K_T be defined by

$$K_T = \{x \in [0,1] \mid \text{for some } y \in [0,1],\ [x \leq y \text{ and } x \npreceq_T y] \text{ or } [y \leq x \text{ and } y \npreceq_T x]\}.$$

Definition 2.12. (Kesicioğlu et al. [12]) Let $(L, \leq, 0, 1)$ be a given bounded lattice. Define a relation $\sim$ on the class of all t-norms on $(L, \leq, 0, 1)$ by $T_1 \sim T_2$ if and only if the T_1-partial order coincides with the T_2-partial order, that is

$$T_1 \sim T_2 \ :\Leftrightarrow \preceq_{T_1} = \preceq_{T_2} . \tag{2}$$

3. ABOUT THE SET K_T^L ON ANY BOUNDED LATTICE

In this section, we study on the set of all incomparable elements with respect to the T partial order $\preceq_T$ with some t-norm T on a bounded lattice $(L, \leq, 0, 1)$.

Definition 3.1. Let T be a t-norm on a bounded lattice $(L, \leq, 0, 1)$ and let K_T^L be defined by

$$K_T^L = \{x \in L\backslash\{0,1\} \mid \text{for some } y \in L\backslash\{0,1\},\ [x < y \text{ and } x \npreceq_T y] \text{ or}$$
$$[y < x \text{ and } y \npreceq_T x] \text{ or } x \parallel y\}.$$

If $L = [0,1]$, then it is trivial to see that $K_T = K_T^L$.

Proposition 3.2. Let $(L, \leq, 0, 1)$ be a bounded lattice and T be a t-norm on L. If there exist two elements of L such that these are incomparable, then $K_T^L \neq \emptyset$.

This result is obvious therefore we omit its proof.

Although the set $K_T^L \neq \emptyset$, it need not be the case that elements in L are incomparable. Now, let us investigate the following example.

Example 3.3. Let T be a t-norm on $[0,1]$ and the family $(T_\lambda)_{\lambda \in (0,1)}$ of t-norms be given by

$$T_\lambda(x,y) = \begin{cases} 0, & T(x,y) \leq \lambda \text{ and } x, y \neq 1, \\ T(x,y), & \text{otherwise.} \end{cases}$$

Observe that due to (Theorem 15 in [1]) the function T_λ is a t-norm. Then we have that $K_{T_\lambda} = (0,1)$, but since L is a chain all elements are comparable.

Let us show that $K_{T_\lambda} = (0,1)$. Let $x \in (0,1)$.

• Firstly, let $x \leq \lambda$ and we choose $1 \neq y > \lambda$. Then, $x < y$ and $x \npreceq_{T_\lambda} y$. Indeed; suppose that $x \preceq_{T_\lambda} y$. Then, there exists an element $\ell \in [0,1]$ such that $T_\lambda(y, \ell) = x$. Since $x \neq 0$, by the definition of T_λ, it is obtained that

$$x = T_\lambda(y, \ell) = T(y, \ell).$$

Since $x \neq y$, it is not possible $\ell = 1$. Since $\ell \neq 1$ and $y \neq 1$, again by the definition of T_λ, it is obtained that

$$x = T_\lambda(y, \ell) = T(y, \ell) > \lambda,$$

a contradiction. Since for any $x \leq \lambda$ there exists an element $y > \lambda$ such that $x < y$ but $x \npreceq_{T_\lambda} y$, $x \in K_{T_\lambda}$.

• Secondly, let $x > \lambda$ and we choose $0 \neq y \leq \lambda$. Then, $y < x$ and $y \npreceq_{T_\lambda} x$. On the contrary, we suppose that $y \preceq_{T_\lambda} x$. Then, there exists an element $k \in [0,1]$ such that $T_\lambda(x, k) = y$. Since $y \neq 0$, by the definition of T_λ, it is obtained that

$$y = T_\lambda(x, k) = T(x, k).$$

Since $x \neq y$, it is not possible that $k = 1$. Since $k \neq 1$ and $x \neq 1$, again by the definition of T_λ, it is obtained that

$$y = T_\lambda(x, k) = T(x, k) > \lambda,$$

a contradiction. Since for any $x > \lambda$ there exists an element $y \leq \lambda$ such that $y < x$ but $y \not\preceq_{T_\lambda} x$, $x \in K_{T_\lambda}$. So, it is obtained that $(0, 1) \subseteq K_{T_\lambda}$. Conversely, for any t-norm T, it is clear that $K_T \subseteq (0, 1)$. So, it is obtained that $K_{T_\lambda} = (0, 1)$. But since $L = [0, 1]$ is a chain, all elements in L are comparable according to the natural order.

Definition 3.4. Let $(L, \leq, 0, 1)$ be a bounded lattice. The set I_L is defined by

$$I_L = \{x \in L \mid \exists y \in L \text{ such that } x \parallel y\}.$$

Due to the definition of the set K_T^L, it is obtained that $I_L \subseteq K_T^L$ for any t-norm T on L.

Remark 3.5. For any t-norm T on any bounded lattice L, if $|L| = 3$, then it is obtained that $K_T^L = \emptyset$.

Proposition 3.6. Let $(L, \leq, 0, 1)$ be a bounded lattice and $|L| > 3$. For the weakest t-norm T_W on L, $K_{T_W}^L = L \backslash \{0, 1\}$.

Proposition 3.7. Let $(L, \leq, 0, 1)$ be a bounded lattice. For the infimum t-norm $T_\wedge$ on L, $K_{T_\wedge}^L = I_L$.

Remark 3.8. The converse of Proposition 3.7 is not be true. That is, T is a t-norm on L such that if $K_T^L = I_L$, then need not be $T = T_\wedge$.

Definition 3.9. Let T be a t-norm on $[0, 1]$, $c \in [0, 1]$ and let $\mathcal{I}_T{}^{(c\downarrow)}$, $\mathcal{I}_T{}^{(c\uparrow)}$ defined by

$$\mathcal{I}_T{}^{(c\downarrow)} = \{x \in (0, 1) \mid x < c \text{ and } x \not\preceq_T c\}$$

$$\mathcal{I}_T{}^{(c\uparrow)} = \{y \in (0, 1) \mid c < y \text{ and } c \not\preceq_T y\}.$$

Note that $\mathcal{I}_T{}^{(c)} = \mathcal{I}_T{}^{(c\downarrow)} \cup \mathcal{I}_T{}^{(c\uparrow)}$ for $c \in [0, 1]$.

Lemma 3.10. Let T be a right continuous t-norm on $[0, 1]$. Then the set $\mathcal{I}_T{}^{(x\downarrow)}$ for $x \in [0, 1]$ is either empty or infinite.

Lemma 3.11. Let T be a left continuous t-norm on $[0, 1]$. Then the set $\mathcal{I}_T{}^{(x\uparrow)}$ for $x \in [0, 1]$ is either empty or infinite.

Corollary 3.12. Let T be a right continuous t-norm on $[0, 1]$ and the set $\mathcal{I}_T{}^{(x\downarrow)} \neq \emptyset$. Then the set $\mathcal{I}_T{}^{(x)}$ is infinite.

Corollary 3.13. Let T be a left continuous t-norm on $[0, 1]$ and the set $\mathcal{I}_T{}^{(x\uparrow)} \neq \emptyset$. Then the set $\mathcal{I}_T{}^{(x)}$ is infinite.

Lemma 3.14. Let T be a t-norm on $[0,1]$. T is continuous t-norm if and only if $\mathcal{I}_T{}^{(x\downarrow)} = \emptyset$ and $\mathcal{I}_T{}^{(x\uparrow)} = \emptyset$ for all $x \in [0,1]$.

Corollary 3.15. Let T be a t-norm on $[0,1]$. T is continuous t-norm if and only if $\mathcal{I}_T{}^{(x)} = \emptyset$ for all $x \in [0,1]$.

Definition 3.16. Let T be a t-norm on $(L, \leq, 0, 1)$ and let $\mathcal{I}_T^{L}{}^{(c)}$ for a $c \in L$ be defined by

$$\mathcal{I}_T^{L}{}^{(c)} = \{x \in L\backslash\{0,1\} \mid x \text{ is incomparable to } c \text{ according to } \preceq_T\}.$$

Proposition 3.17. Let T be a t-norm on $(L, \leq, 0, 1)$. If there exist elements x and y in L such that these are incomparable, then $\mathcal{I}_T^{L}{}^{(x)} \neq \emptyset$ and $\mathcal{I}_T^{L}{}^{(y)} \neq \emptyset$.

The converse of Proposition 3.17 is not be true. To illustrate this claim we shall give the following example.

Example 3.18. Consider the t-norm of Example 3.3. We obtain that

a_1) $\mathcal{I}_{T_\lambda}{}^{(x)} = \{y \in (0,1) \mid \ x \neq y\}$ for $x \in (0, \lambda]$

a_2) $\mathcal{I}_{T_\lambda}{}^{(x)} = \{y \in (0,\lambda] \mid \ x \neq y\}$ for $x \in (\lambda, 1)$.

Since L is a chain all elements are comparable.

Now we want to show this claim.

a_1) It is trivial that $\mathcal{I}_{T_\lambda}{}^{(x)} \subseteq (0,1)$. Conversely, $y \in (0,1)$ be arbitrary such that $x \neq y$ for $x \in (0, \lambda]$. Let us show that $y \in \mathcal{I}_{T_\lambda}{}^{(x)}$. Suppose that $y \notin \mathcal{I}_{T_\lambda}{}^{(x)}$. That is, $y < x$ and $y \preceq_{T_\lambda} x$ or $x < y$ and $x \preceq_{T_\lambda} y$.

• Let $y < x$ and $y \preceq_{T_\lambda} x$. Then, there exists an elements $k \in [0,1]$ such that $T_\lambda(x,k) = y$. Since $y \neq 0$, by the definition of T_λ, we obtain that $y = T_\lambda(x,k) = T(x,k)$. Since $x \neq 1$ and $k \neq 1$, we have that $y > \lambda$, a contradiction. So it is obtained that $y \npreceq_{T_\lambda} x$, that is $y \in \mathcal{I}_{T_\lambda}{}^{(x)}$. Similarly it can be show that $x < y$ and $x \preceq_{T_\lambda} y$. Consequently we obtained that $\mathcal{I}_{T_\lambda}{}^{(x)} = \{y \in (0,1) \mid \ x \neq y\}$ for $x \in (0, \lambda]$.

a_2) Similarly it is obtained that $\{y \in (0,\lambda] \mid \ x \neq y\}$ for $x \in (\lambda, 1)$.

Lemma 3.19. Let T be a t-norm on $(L, \leq, 0, 1)$. Then $K_T^L = \bigcup_{x \in L} \mathcal{I}_T^{L}{}^{(x)}$.

This result is obvious therefore we omit its proof.

Proposition 3.20. Let T_1 and T_2 be two t-norms on a bounded lattice $(L, \leq, 0, 1)$. Then for all $x \in L$, $\mathcal{I}_{T_1}^{L}{}^{(x)} = \mathcal{I}_{T_2}^{L}{}^{(x)}$ if and only if the t-norms T_1 and T_2 are equivalent under $\sim$ in (2).

4. ABOUT AN EQUIVALENCE RELATION ON THE CLASS OF T-NORMS ON ANY BOUNDED LATTICE

The above introduced set K_T^L on any bounded lattice allows us to introduce the next equivalence relation on the class of all t-norms on $(L, \leq, 0, 1)$.

Definition 4.1. Let $(L, \leq, 0, 1)$ be a bounded lattice. Define a relation β_L on the class of all t-norms on $(L, \leq, 0, 1)$ by $T_1 \beta_L T_2$,

$$T_1 \beta_L T_2 :\Leftrightarrow K_{T_1}^L = K_{T_2}^L. \tag{3}$$

The next result is obvious.

Lemma 4.2. The relation β_L given in Definition 4.1 is an equivalence relation.

Definition 4.3. For a given t-norm T on a bounded lattice $(L, \leq, 0, 1)$, we denote by $\overline{T}$ the β_L equivalence class linked to T, i. e.,

$$\overline{T} = \{T' \mid T' \text{ is a t-norm on } L \text{ and } K_T^L = K_{T'}^L\}.$$

In [12], it was shown that an equivalence class of the infimum t-norm $T_\wedge$ on L under the relation $\sim$ in (2) is the set of all divisible t-norms on L. But according to the relation β_L in (3), an equivalence class of the infimum t-norm $T_\wedge$ on L is not the set of all divisible t-norms on L. To illustrate this claim we shall give the following example.

Example 4.4. Consider the bounded lattice $(L, \leq, 0, 1)$ with $L = \{0, a, b, c, 1\}$ as shown in Fig. 1.

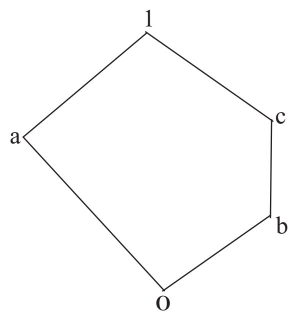

Fig. 1. The order $\leq$ on L.

We consider $T_\wedge$ and T_W t-norms on L. It is trivial that $K_{T_\wedge}^L = \{a, b, c\}$ and $K_{T_W}^L = \{a, b, c\}$. So we have that $K_{T_\wedge}^L = K_{T_W}^L$. By the definition of the relation β_L in (3), the t-norms $T_\wedge$ and T_W are equivalent, i. e., $T_\wedge \beta_L T_W$. But the weakest t-norm T_W is not divisible t-norm on L. Suppose that T_W is divisible t-norm. It is trivial $b < c$. Since T_W is divisible t-norm, there exists an element $\ell \in L$ such that $b = T(c, \ell)$. If $\ell \in \{0, a, b, c\}$, then it is obtained that $b = 0$, a contradiction. If $\ell = 1$, then we have that $b = c$, a contradiction. So, the weakest t-norm T_W is not divisible t-norm on L.

Naturally, one can think when an equivalence class of the infimum t-norm $T_\wedge$ on L under the relation β_L in (3), is the set of all divisible t-norms on L. As an answer to this question, let us investigate the following Proposition.

Proposition 4.5. If L is a chain, then an equivalence class of the infimum t-norm $T_\wedge$ on L under the relation β_L in (3), is the set of all divisible t-norms on L.

Proof. Let $T' \in \overline{T_\wedge}$. Then we have that $K^L_{T'} = K^L_{T_\wedge}$ according to the relation β_L in (3). Since L is a chain, it is obtained that $K^L_{T_\wedge} = \emptyset$ from $I_L = \emptyset$ by Proposition 3.7. So, we have that $K^L_{T'} = \emptyset$. Since L is a chain and $K^L_{T'} = \emptyset$, it is obtained that $x \le y$ and $x \preceq_{T'} y$ or $y \le x$ and $y \preceq_{T'} x$ for all $x, y \in L$. Without loss of generality, we assume that $x \le y$ and $x \preceq_{T'} y$. Then there exists an element $\ell \in L$ such that $x = T'(y, \ell)$. So, it is obtained that T' is a divisible t-norm.

Conversely, let T' be a divisible t-norm on L. Now, we will show that $T' \in \overline{T_\wedge}$, that is $K^L_{T'} = K^L_{T_\wedge}$. It is obtained that $K^L_{T_\wedge} = \emptyset$ from $I_L = \emptyset$ by Proposition 3.7. Let us show that $K^L_{T'} = \emptyset$. Suppose that $K^L_{T'} \neq \emptyset$ and we choose $x \in K^L_{T'}$. Since L is a chain, for some $y \in L \setminus \{0, 1\}$, ($x < y$ and $x \not\preceq_{T'} y$) or ($y < x$ and $y \not\preceq_{T'} x$) by the definition of K^L_T. Firstly, let $x < y$. Since T' be a divisible t-norm, there exists $m \in L$ such that $x = T'(y, m)$. So, it is obtained that $x \preceq_{T'} y$, a contradiction. Similarly, if $y < x$, then it can be easily verified that $y \preceq_{T'} x$, a contradiction. So, we have that $K^L_{T'} = \emptyset$. Thus, we have that $K^L_{T'} = K^L_{T_\wedge}$. This shows that $T' \beta_L T_\wedge$, when T' is a divisible t-norm on L. $\square$

Corollary 4.6. The equivalence class of the minimum t-norm T_M on $[0, 1]$ according to the relation β_L in (3), is the set of all divisible t-norms on $[0, 1]$.

Corollary 4.7. The equivalence class of the minimum t-norm T_M on $[0, 1]$ according to the relation β_L in (3), is the set of all continuous t-norms on $[0, 1]$.

Remark 4.8. In Corollary 4.7, we have shown that any two continuous t-norms on $[0, 1] = L$ are equivalent under the relation β_L in (3). Naturally, one can think whether any two left-continuous t-norms are in the same equivalence class, i.e, any two left-continuous t-norms are equivalent under the relation β_L in (3). To illustrate that two left-continuous t-norms may not be equivalent under the relation β_L in (3) we shall give the following example.

Example 4.9. Consider the t-norms on $[0, 1]$ defined as follows:

$$T^{nM}(x, y) = \begin{cases} 0, & \text{if } x + y \le 1, \\ \min(x, y), & \text{otherwise,} \end{cases}$$

and

$$T_4(x, y) = \begin{cases} \min(x, y), & \text{if } \max(x, y) \in (\frac{3}{4}, 1], \\ \frac{1}{4}, & \text{if } x, y \in (\frac{1}{4}, \frac{3}{4}], \\ 0, & \text{otherwise.} \end{cases}$$

T^{nM} and T_4 are left continuous t-norms [16]. But since $K_{T^{nM}} = (0,1)$ and $K_{T_4} = (0, \frac{3}{4}]$, the t-norms T^{nM} and T_4 are not equivalent under β_L in (3).

In [1], it has been shown that $K_{T^{nM}} = (0,1)$. Now, we will show that $K_{T_4} = (0, \frac{3}{4}]$.

• First, choose arbitrary $x \in (0, \frac{3}{4}]$. Let us show that $x \in K_{T_4}$.

(i) Let $x \in (0, \frac{3}{4})$ and $y = \frac{x}{3}$. In this case, $y < x$ but $y \not\preceq_{T_4} x$. Suppose that $y \preceq_{T_4} x$. Then, for some $\ell \in [0,1]$,

$$T_4(x, \ell) = y = \frac{x}{3}.$$

Thus, it follows $y \neq \frac{1}{4}$ from $x \neq \frac{3}{4}$. Since $y \neq 0$ and $y \neq \frac{1}{4}$, it is obtained that

$$T_4(x, \ell) \neq 0 \text{ and } T_4(x, \ell) \neq \frac{1}{4}.$$

By the definition of T_4, we have that $\max(x, \ell) \in (\frac{3}{4}, 1]$. Again by the definition of T_4, $T_4(x, \ell) = \min(x, \ell) = \frac{x}{3}$. Since $\frac{x}{3} \neq x$, it is obtained that $\frac{x}{3} = \ell$. Whence we have $\max(x, \frac{x}{3}) \notin (\frac{3}{4}, 1]$, a contradiction. Since for any $x \in (0, \frac{3}{4})$ there exists an element $y = \frac{x}{3} \in (0,1)$ such that $\frac{x}{3} < x$ but $\frac{x}{3} \not\preceq_{T_4} x$, $x \in K_{T_4}$.

(ii) Let $x = \frac{3}{4}$ and $\frac{1}{4} < y < \frac{3}{4}$. In this case, $y < x$ but $y \not\preceq_{T_4} x$. Suppose that $y \preceq_{T_4} x$. Then, there exists an element $m \in [0,1]$,

$$T_4(x, m) = y.$$

By the definition of T_4, it is obtained that $\max(x, m) \in (\frac{3}{4}, 1]$. Again by the definition of T_4, $T_4(x, m) = \min(x, m) = y$. Since $x \neq y$, it is obtained that $y = m$. It is obtained that $\max(x, y) \notin (\frac{3}{4}, 1]$, a contradiction. Since for $x = \frac{3}{4}$ there exists an element $\frac{1}{4} < y < \frac{3}{4}$ such that $y \not\preceq_{T_4} \frac{3}{4}$, $x \in K_{T_4}$. So, it is obtained that $(0, \frac{3}{4}] \subseteq K_{T_4}$.

• On the contrary let $x \in K_{T_4}$ be arbitrary. We will show that $x \in (0, \frac{3}{4}]$. Suppose that $x \notin (0, \frac{3}{4}]$. Since $x \in K_{T_4}$, there exists an element $y \in (0,1)$ such that $x < y$ and $x \not\preceq_{T_4} y$ or $y < x$ and $y \not\preceq_{T_4} x$. Without loss of generality, we assume that $x < y$ and $x \not\preceq_{T_4} y$. Since $x < y$, it must be $\min(x, y) = x$. Since $\frac{3}{4} < x < y$, it must be $\max(x, y) \in (\frac{3}{4}, 1]$. By the definition of T_4, we obtain that

$$x = \min(x, y) = T_4(x, y).$$

Then, it holds that $x \preceq_{T_4} y$, a contradiction. So, we have $x \in (0, \frac{3}{4}]$. Thus, it is obtained that $K_{T_4} \subseteq (0, \frac{3}{4}]$. Therefore it is obtained that $K_{T_4} = (0, \frac{3}{4}]$. Consequently, since $K_{T^{nM}} \neq K_{T_4}$, the t-norms T^{nM} and T_4 are not equivalent under β_L in (3).

Remark 4.10. One may ask whether any t-norm equivalent to a left continuous t-norm needs to be left-continuous, too. The following example shows that also this need not.

Example 4.11. Let $T^\star$ be a function on $[0,1]$ defined by

$$T^\star(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x, y = \frac{1}{2}, \\ T^{nM}(x, y), & \text{otherwise.} \end{cases}$$

The function $T^\star$ is a t-norm by [14]. We will show that this t-norm is equivalent to the left-continuous t-norm T^{nM}, but $T^\star$ is not left continuous t-norm.

To see that $T^\star \beta T^{nM}$, we must show $K_{T^\star} = K_{T^{nM}}$. In [1], it has been shown that $K_{T^{nM}} = (0,1)$. Now, we will show that $K_{T^\star} = (0,1)$.

• First, choose arbitrary $x \in (0,1)$. Let us show that $x \in K_{T^\star}$

(i) Let $x < \frac{1}{2}$ and $y = 1-x$. In this case $x < y$ and $x \not\preceq_{T^\star} y$. Suppose that $x \preceq_{T^\star} y$. Then, there exists an element $\ell \in [0,1]$, it is obtained $T^\star(y,\ell) = x$. Since $x \neq \frac{1}{2}$, we have

$$T^\star(y,\ell) = T^{nM}(y,\ell) = x.$$

Since $x \neq 0$, by the definition of T^{nM}, we have that $T^{nM}(y,\ell) = T^{nM}(1-x,\ell) = \min(1-x,\ell) = x$ and $\ell > x$ from $1-x+\ell > 1$. Since $x \neq 1-x$, it is obtained that $x = \ell$, a contradiction. Since for $x \in (0,1)$ there exists an element $y = 1-x$ such that $x < y$ but $x \not\preceq_{T^\star} y$, $x \in K_{T^\star}$.

(ii) Secondly, let $x > \frac{1}{2}$ and $y = 1-x$. Similarly it can be shown that $y < x$ but $y \not\preceq_{T^\star} x$. So, we have that $x \in K_{T^\star}$.

(iii) The last one, let $x = \frac{1}{2}$. It is shown that easily $y \not\preceq_{T^\star} \frac{1}{2}$ for $0 < y < \frac{1}{2}$. So, we have that $x \in K_{T^\star}$. Consequently it is obtained that $(0,1) \subseteq K_{T^\star}$.

• Conversely, for any t-norm T, it is clear that $K_T \subseteq (0,1)$. So, it is obtained that $K_{T^\star} = (0,1)$. This means that T^{nM} and $T^\star$ are equivalent under β_L in (3).

Proposition 3.20 gives a sufficient and necessary condition for the t-norms T_1 and T_2 to be equivalent under the relation $\sim$ in (2). But the following Proposition only provides a sufficient and not a necessary condition for the relation β_L in (3).

Proposition 4.12. Let T_1 and T_2 be two t-norms on $(L, \leq, 0, 1)$. If for all $x \in L$, ${\mathcal{I}^L_{T_1}}^{(x)} = {\mathcal{I}^L_{T_2}}^{(x)}$, then the t-norms T_1 and T_2 are equivalent under β_L in (3).

Proof. Let T_1 and T_2 be two t-norms on $(L, \leq, 0, 1)$ and ${\mathcal{I}^L_{T_1}}^{(x)} = {\mathcal{I}^L_{T_2}}^{(x)}$ for all $x \in L$. By Lemma 3.19,

$$K^L_{T_1} = \bigcup_{x \in L} {\mathcal{I}^L_{T_1}}^{(x)} \text{ and } K^L_{T_2} = \bigcup_{x \in L} {\mathcal{I}^L_{T_2}}^{(x)}.$$

Since ${\mathcal{I}^L_{T_1}}^{(x)} = {\mathcal{I}^L_{T_2}}^{(x)}$ for all $x \in L$, it is obtained that

$$K^L_{T_1} = \bigcup_{x \in L} {\mathcal{I}^L_{T_1}}^{(x)} = \bigcup_{x \in L} {\mathcal{I}^L_{T_2}}^{(x)} = K^L_{T_2}.$$

Then, we have that $K^L_{T_1} = K^L_{T_2}$. Whence, by the definition of the relation β_L in (3), it holds that $T_1 \beta_L T_2$. Consequently, the t-norms T_1 and T_2 are equivalent under β_L in (3). □

Remark 4.13. The converse of Proposition 4.12 is not be true. Here is an example illustrating the case that need not be true.

Example 4.14. Consider the t-norm $T : [0,1]^2 \to [0,1]$ defined by

$$T(x,y) = \begin{cases} \frac{xy}{2}, & \text{if } (x,y) \in [0,1)^2, \\ \min(x,y), & \text{otherwise,} \end{cases}$$

and the t-norm T_D on $[0,1]$. Then, $K_T = K_{T_D}$ by [12]. The t-norms T and T_D are equivalent under β_L in (3) and we obtained that,

(i) a_1) $\mathcal{I}_T{}^{(x)} = \{y \in (0,1) \mid y \in [\frac{x}{2}, 2x] \text{ and } x \neq y\}$ for $x \in (0, \frac{1}{2})$,

a_2) $\mathcal{I}_T{}^{(x)} = \{y \in (0,1) \mid y \in [\frac{x}{2}, 1) \text{ and } x \neq y\}$ for $x \in [\frac{1}{2}, 1)$,

(ii) $\mathcal{I}_{T_D}{}^{(x)} = \{y \in (0,1) \mid x \neq y\}$ for $x \in (0,1)$.

Now, we want to show this claims.

(i) a_1) Let $y \in \mathcal{I}_T{}^{(x)}$ be arbitrary for $x \in (0, \frac{1}{2})$. By Lemma 2.10, it must be $x \neq y$. So, we will show that $y \in [\frac{x}{2}, 2x]$. Suppose that $y \notin [\frac{x}{2}, 2x]$. Then, it is obtained that $y < \frac{x}{2}$ or $2x < y$. First we assume that $y < \frac{x}{2}$. Since $y = x.\frac{2y}{x}.\frac{1}{2}$ and $x \neq 1$, $\frac{2y}{x} \neq 1$, by the definition of T, we obtain that $y = x.\frac{2y}{x}.\frac{1}{2} = T(x, \frac{2y}{x})$. Then, it holds that $y \preceq_T x$, a contradiction. So, this means that $y \geq \frac{x}{2}$. Similarly let $2x < y$. Since $x = y.\frac{2x}{y}.\frac{1}{2}$ and $y \neq 1$, $\frac{2x}{y} \neq 1$, by the definition of T, we obtain that $x = y \cdot \frac{2x}{y} \cdot \frac{1}{2} = T(y, \frac{2x}{y})$. Then, it holds that $x \preceq_T y$, a contradiction. So, it is obtained that $y \leq 2x$. We have that $y \in [\frac{x}{2}, 2x]$. Therefore, $\mathcal{I}_T{}^{(x)} \subseteq \{y \in (0,1) \mid y \in [\frac{x}{2}, 2x] \text{ and } x \neq y\}$ for $x \in (0, \frac{1}{2})$.

Conversely, $y \in (0,1)$ be arbitrary such that $x \neq y$ and $y \in [\frac{x}{2}, 2x]$ for $x \in (0, \frac{1}{2})$. Let us show that $y \in \mathcal{I}_T{}^{(x)}$. Suppose that $y \notin \mathcal{I}_T{}^{(x)}$. That is, y is comparable to x according to $\preceq_T$. Then, $y < x$ and $y \preceq_T x$ or $x < y$ and $x \preceq_T y$.

• Firstly, let $y < x$ and $y \preceq_T x$. Then, there exists an elements $\ell \in [0,1]$ such that $T(x,\ell) = y$. Since $x \neq y$, it must be $\ell \neq 1$. Since $x \neq 1$ and $\ell \neq 1$, by the definition of T, it is obtained that $T(x,\ell) = y = \frac{x\ell}{2}$. Since $\frac{x}{2} \leq y$, we have that $\ell = \frac{2y}{x} \geq 1$, a contradiction. So it is obtained that $y \npreceq_T x$, that is $y \in \mathcal{I}_T{}^{(x)}$.

• Similarly, let $x < y$ and $x \preceq_T y$. Then, for some $\ell^* \in [0,1]$, $T(y,\ell^*) = x$. Since $x \neq y$, it must be $\ell^* \neq 1$. Since $y \neq 1$ and $\ell^* \neq 1$, by the definition of T, it is obtained that $T(y,\ell^*) = x = \frac{y\ell^*}{2}$. Since $y \leq 2x$, we have that $\ell^* = \frac{2x}{y} \geq 1$, a contradiction. So it is obtained that $x \npreceq_T y$, that is $y \in \mathcal{I}_T{}^{(x)}$. So, it is obtained that $\{y \in (0,1) \mid y \in [\frac{x}{2}, 2x] \text{ and } x \neq y\} \subseteq \mathcal{I}_T{}^{(x)}$ for $x \in (0, \frac{1}{2})$. Consequently, we have that $\mathcal{I}_T{}^{(x)} = \{y \in (0,1) \mid y \in [\frac{x}{2}, 2x] \text{ and } x \neq y\}$ for $x \in (0, \frac{1}{2})$.

a_2) Similarly, it can be show that $\mathcal{I}_T{}^{(x)} = \{y \in (0,1) \mid y \in [\frac{x}{2}, 1) \text{ and } x \neq y\}$ for $x \in [\frac{1}{2}, 1)$.

(ii) Let $y \in (0,1)$ be arbitrary such that $x \neq y$ for $x \in (0,1)$. Let us show that $y \in \mathcal{I}_{T_D}{}^{(x)}$. Suppose that $y \notin \mathcal{I}_{T_D}{}^{(x)}$. That is, y is comparable to x according to $\preceq_{T_D}$. Then, $y < x$ and $y \preceq_{T_D} x$ or $x < y$ and $x \preceq_{T_D} y$.

• Firstly, let $y < x$ and $y \preceq_{T_D} x$. Then, for some $m \in [0,1]$, $T_D(x,m) = y$. Since $x \neq y$, it must be $m \neq 1$. Since $x, m \neq 1$, by the definition of T_D, it is obtained that $y = 0$, a contradiction. So, we have that $y < x$ and $y \npreceq_{T_D} x$, that is $y \in \mathcal{I}_{T_D}{}^{(x)}$.

• Secondly, let $x < y$ and $x \preceq_{T_D} y$. Then, for some $k \in [0, 1]$, $T_D(y, k) = x$. Since $x \neq y$, it must be $k \neq 1$. Since $y, k \neq 1$, by the definition of T_D, it is obtained that $x = 0$, a contradiction. So, we have that $x < y$ and $x \npreceq_{T_D} y$, that is $y \in \mathcal{I}_{T_D}{}^{(x)}$. Thus, we have that $\{y \in (0, 1) \mid x \neq y\} \subseteq \mathcal{I}_{T_D}{}^{(x)}$. Conversely, for any t-norm T, it is clear that $\mathcal{I}_{T_D}{}^{(x)} \subseteq (0, 1)$ for $x \in [0, 1]$. Thus it is obtained that $\mathcal{I}_{T_D}{}^{(x)} = \{y \in (0, 1) \mid x \neq y\}$ for $x \in (0, 1)$.

For example, since $\frac{1}{3} \preceq_T \frac{3}{4}$, it is obtained that $\frac{3}{4} \notin \mathcal{I}_T{}^{(\frac{1}{3})}$. But on the other hand $\frac{3}{4} \in \mathcal{I}_{T_D}{}^{(\frac{1}{3})}$. So, it is obtained that $\mathcal{I}_T{}^{(\frac{1}{3})} \neq \mathcal{I}_{T_D}{}^{(\frac{1}{3})}$.

5. CONCLUSION

We have defined the set of incomparable elements with respect to the triangular order for any t-norm on a bounded lattice $(L, \leq, 0, 1)$. Also we have introduced and studied an equivalence relation β_L in (3) defined on the class of all t-norms on L. We have shown that any two continuous t-norms on $[0, 1]$ are equivalent by the introduced equivalence relation. As shown by examples, all left-continuous t-norms on $[0, 1]$ do not form an equivalence class in our approach. Further we have shown when an equivalence class of the infimum t-norm $T_\wedge$ on L, is the set of all divisible t-norms on L.

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REFERENCES

[1] E. Aşıcı and F. Karaçal: On the T-partial order and properties. Inf. Sci. *267* (2014), 323–333. DOI:10.1016/j.ins.2014.01.032

[2] G. Birkhoff: Lattice Theory. Third edition. Providence 1967.

[3] B. De Baets and R. Mesiar: Triangular norms on product lattices. Fuzzy Sets Syst. *104* (1999), 61–75. DOI:10.1016/s0165-0114(98)00259-0

[4] B. De Baets and R. Mesiar: Triangular norms on the real unit square. In: Proc. 1999 EUSFLAT-ESTYLF Joint Conference, Palma de Mallorca 1999, pp. 351–354.

[5] J. Casasnovas and G. Mayor: Discrete t-norms and operations on extended multisets. Fuzzy Sets Syst. *159* (2008), 1165–1177. DOI:10.1016/j.fss.2007.12.005

[6] S. Gottwald: A Treatise on Many-valued Logic. Studies in Logic and Computation, Research Studies Press, Baldock 2001.

[7] F. Karaçal and E. Aşıcı: Some notes on T-partial order. J. Inequal. Appl. *2013* (2013), 219. DOI:10.1186/1029-242x-2013-219

[8] F. Karaçal and M. N. Kesicioğlu: A T-partial order obtained from t-norms. Kybernetika *47* (2011), 300–314.

[9] F. Karaçal and Y. Sağıroğlu: Infinetely $\bigvee$- distributive t-norm on complete lattices and pseudo-complements. Fuzzy Sets Syst. *160* (2009), 32–43.

[10] F. Karaçal and Dj. Khadjiev: $\bigvee$-distributive and infinitely $\bigvee$-distributive t-norms on complete lattice. Fuzzy Sets Syst. *151* (2005), 341–352. DOI:10.1016/j.fss.2008.03.022

[11] F. Karaçal: On the direct decomposability of strong negations and S-implication operators on product lattices. Inf. Sci. *176* (2006), 3011–3025. DOI:10.1016/j.ins.2005.12.010

[12] M. N. Kesicioğlu, F. Karaçal and R. Mesiar: Order-equivalent triangular norms. Fuzzy Sets Syst. *268* (2015), 59-71.

[13] D. Khadjiev and F. Karaçal: Pairwise comaximal elements and the classification of all $\vee$-distributive triangular norms of length 3. Inf. Sci. *204* (2012), 36–43. DOI:10.1016/j.ins.2012.03.014

[14] E. P. Klement, R. Mesiar, and E. Pap,: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000. DOI:10.1007/978-94-015-9540-7

[15] X. Liang and W. Pedrycz: Logic-based fuzzy networks: A study in system modeling with triangular norms and uninorms. Fuzzy Sets Syst. *160* (2009), 3475–3502. DOI:10.1016/j.fss.2009.04.014

[16] K. C. Maes and A. Mesiarova-Zemankova: Cancellativity properties for t-norms and t-subnorms. Inf. Sci. *179* (2009), 135–150. DOI:10.1016/j.ins.2008.11.035

[17] J. Martin, G. Mayor, and J. Torrens: On locally internal monotonic operations. Fuzzy Sets Syst. *137* (2003), 27–42. DOI:10.1016/s0165-0114(02)00430-x

[18] H. Mitsch: A natural partial order for semigroups. Proc. Am. Math. Soc. *97* (1986), 384–388. DOI:10.1090/s0002-9939-1986-0840614-0

[19] S. Saminger: On ordinal sums of triangular norms on bounded lattices. Fuzzy Sets Syst. *157* (2006), 1403–1413. DOI:10.1016/j.fss.2005.12.021

[20] B. Schweizer and A. Sklar: Probabilistic Metric Spaces. Elsevier, Amsterdam 1983.

[21] B. Schweizer and A. Sklar: Espaces métriques aléatoires. C. R. Acad. Sci. Paris Sér. A *247* (1958), 2092–2094.

[22] B. Schweizer and A. Sklar: Statistical metric spaces. Pacific J. Math. *10* (1960), 313–334. DOI:10.2140/pjm.1960.10.313

[23] B. Schweizer and A. Sklar: Associative functions and abstract semigroups. Publ. Math. Debrecen *10* (1963), 69–81.

Emel Aşıcı, Department of Software Engineering, Faculty of Technology, Karadeniz Technical University, 61830 Trabzon. Turkey.
e-mail: emelkalin@hotmail.com

Funda Karaçal, Department of Mathematics, Karadeniz Technical University, 61080 Trabzon. Turkey.
e-mail: fkaracal@yahoo.com