# FURTHER STUDY ON COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF ARRAYS OF ROWWISE ASYMPTOTICALLY ALMOST NEGATIVELY ASSOCIATED RANDOM VARIABLES

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In this paper, the authors further studied the complete convergence for weighted sums of arrays of rowwise asymptotically almost negatively associated (AANA) random variables with non-identical distribution under some mild moment conditions. As an application, the Marcinkiewicz–Zygmund type strong law of large numbers for weighted sums of AANA random variables is obtained. The results not only generalize the corresponding ones of Wang et al. [19], but also partially improve the corresponding ones of Huang et al. [8].

*Keywords:* arrays of rowwise AANA random variables, complete convergence, Marcinkiewicz–Zygmund type strong law of large numbers, weighted sums

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## 1. INTRODUCTION

Firstly, let us recall the definitions of dependence structures.

**Definition 1.1.** A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets $A_1$ and $A_2$ of $\{1, 2, \cdots, n\}$,

$$\mathrm{Cov}(f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)) \le 0, \tag{1.1}$$

whenever $f_1$ and $f_2$ are coordinatewise non-decreasing for every variable (or coordinatewise non-increasing for every variable) such that this covariance exists. An infinite family of random variables $\{X_n, n \ge 1\}$ is said to be NA if every finite subfamily is NA.

An array of random variables $\{X_{ni}, i \ge 1, n \ge 1\}$ is called rowwise NA if for every $n \ge 1$, $\{X_{ni}, i \ge 1\}$ is a sequence of NA random variables.

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**Definition 1.2.** A sequence of random variables $\{X_n, n \geq 1\}$ is said to be AANA if there exists a nonnegative sequence $u(n) \to 0$ as $n \to \infty$ such that

$$\operatorname{Cov}\left(f_1\left(X_n\right), f_2\left(X_{n+1}, \cdots, X_{n+k}\right)\right) \leq u(n)\left(\operatorname{Var}\left(f_1(X_n)\right) \operatorname{Var}\left(f_2\left(X_{n+1}, \cdots, X_{n+k}\right)\right)\right)^{1/2}, \tag{1.2}$$

for all $n, k \geq 1$ and for all coordinatewise non-decreasing continuous functions $f_1$ and $f_2$ whenever the variances exist. $\{u(n), n \geq 1\}$ is the so-called mixing coefficient sequence.

An array of random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is called rowwise AANA if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of AANA random variables with the mixing coefficients $\{u(i), i \geq 1\}$ in each row.

The concept of NA was introduced by Joag–Dev and Proschan [11], and the concept of AANA was introduced by Chandra and Ghosal [5]. These concepts of dependent random variables are very useful in reliability theory and applications. The family of AANA random variables contains NA sequences and independent sequences (with $u(n) = 0, n \geq 1$) and some more sequences of random variables which are not much deviated from being NA random variables.

Since the concept of AANA random variables was firstly introduced by Chandra and Ghosal [5], in the past decades, many authors have studied this concept and provided some interesting results and applications. For example, we refer to [5, 6, 7, 8, 12, 14, 15, 18, 19, 20, 22, 23, 24, 25, 26]. Hence, extending the limit properties of AANA random variables has very important significance in the theory and application.

As Bai and Cheng [3] remarked, many useful linear statistics based on a random sample, for example, least-squares estimators, nonparametric regression function estimators and jackknife estimates are weighted sums of i.i.d. random variables. In this respect, studies of limit properties for these weighted sums are meaningful. But in many practical applications, the assumption of independent is not plausible. So, it is of interest of many authors to extend the independent case to the dependent cases.

We will introduce the definition of stochastic domination in this paper.

**Definition 1.3.** A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable $X$ if there exists a positive constant $C$ such that

$$P(|X_n| \geq x) \leq CP(|X| \geq x), \tag{1.3}$$

for all $x \geq 0$ and $n \geq 1$.

An array of rowwise random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ is said to be stochastically dominated by a random variable $X$ if there exists a positive constant $C$ such that

$$P\left(|X_{ni}| \geq x\right) \leq CP\left(|X| \geq x\right), \tag{1.4}$$

for all $x \geq 0$, $i \geq 1$ and $n \geq 1$.

In the following, some hypothesis conditions are introduced:

(H1) There exist some $\delta$ with $0 < \delta < 1$ and some $\alpha$ with $0 < \alpha < 2$ such that $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O\left(n^{\delta}\right)$ and assume further that $EX_{ni} = 0$ if $1 < \alpha < 2$.

(H2) The mixing coefficient sequence $\{u(n), n \geq 1\}$ satisfies $\sum_{n=1}^{\infty} u^{1/(M-1)}(n) < \infty$ for some $M \in \left(3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}\right]$, where $k$ is a positive integer.

(H3) For some $h > 0$ and $\gamma > 0$,

$$E \exp\left(h|X|^{\gamma}\right) < \infty.$$

(H4) There exists some $\alpha$ with $0 < \alpha < 2$ such that $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n)$ and assume further that $EX_{ni} = 0$ if $1 < \alpha < 2$.

Wang et al. [19] proved the following theorems for weighted sums of arrays of rowwise AANA random variables.

**Theorem 1.4.** Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables which is stochastically dominated by a random variable $X$ and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers. Suppose that the above conditions (H1) – (H3) are satisfied, then for $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha s-2} P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty, \tag{1.5}$$

where $s \geq 1/\alpha$ and $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$.

**Theorem 1.5.** Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables which is stochastically dominated by a random variable $X$ and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers. Suppose that the above conditions (H2) – (H4) are satisfied, then for $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty, \tag{1.6}$$

where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$.

Recently, Huang et al.[8] consider the cases of $\alpha \neq \gamma$ and $\alpha = \gamma$ for $0 < \alpha \leq 2$ respectively under much weaker moment conditinons, and obtain the following results.

**Theorem 1.6.** Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable $X$, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying.

$$A_{\beta} = \lim_{n\to\infty} \sup A_{\beta,n} < \infty, \quad A_{\beta,n} = \frac{1}{n}\sum_{i=1}^{n} |a_{ni}|^{\beta}, \tag{1.7}$$

where $\beta = \max(\alpha, \gamma)$ for some $0 < \alpha \leq 2$ and $\gamma > 0$. Assume that $EX_n = 0$ for $1 < \alpha \leq 2$ and $E|X|^\beta < \infty$. Then,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} X_i\right| > \varepsilon b_n\right) < \infty \quad \text{for } \forall \varepsilon > 0, \tag{1.8}$$

where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$.

**Theorem 1.7.** Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable $X$, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying (1.7) for $\beta = \alpha$ and some $0 < \alpha \leq 2$. Assume that $EX_n = 0$ for $1 < \alpha \leq 2$ and $E|X|^\alpha \log(1 + |X|) < \infty$. Then (1.8) holds, where $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$.

For more details about the strong convergence theorems for weighted sums of dependent sequences, one can refer to Cai [4], Jing and Liang [10], Zhou et al. [27], Wang et al. [21], Huang and Wang [9], Shen and Wu [13], Sung [16, 17], and so on.

Inspired by the above theorems obtained by Wang et al. [19] and Huang et al. [8], in this work, we will further study the complete convergence for weighted sums of arrays of rowwise AANA random variables under some mild moment conditions, which are weaker than the above hypothesis condition (H3). Some complete convergence for the maximum weighed sums of arrays of rowwise AANA random variables are obtained without the assumption of identical distribution. As an application, the Marcinkiewicz–Zygmund type strong law of large numbers for weighted sums of AANA random variables is obtained. The results not only generalize the corresponding ones of Wang et al. [19], but also partially improve the corresponding ones of Huang et al. [8].

## 2. MAIN RESULTS AND PROOFS

The proofs of our main results in this paper are based on the following lemmas.

**Lemma 2.1.** (Yuan and An [25]) Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with the mixing coefficients $\{u(n), n \geq 1\}$, let $\{f_n, n \geq 1\}$ be a sequence of all non-decreasing (or all non-increasing) continuous functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of AANA random variables with the mixing coefficients $\{u(n), n \geq 1\}$.

**Lemma 2.2.** (Yuan and An [25]) Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with the mixing coefficients $\{\mu(n), n \geq 1\}$, $EX_n = 0$. If the mixing coefficient sequence satisfies $\sum_{i=1}^{\infty} u^{1/(M-1)}(i) < \infty$ for some $M \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1}]$, where $k$ is a positive integer, then there exists a positive constant $C = C(M)$ depending only on $M$ such that for all $n \geq 1$,

$$E\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} X_i\right|^M\right) \leq C\left(\sum_{i=1}^{n} E|X_i|^M + \left(\sum_{i=1}^{n} EX_i^2\right)^{M/2}\right). \tag{2.1}$$

**Lemma 2.3.** (Adler and Rosalsky [1]; Adler, Rosalsky and Taylor [2]) Suppose that $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise random variables which is stochastically dominated by a random variable $X$. For $\forall \alpha > 0$ and $t > 0$, the following statements hold:

$$E|X_{ni}|^{\alpha} I\left(|X_{ni}| \leq t\right) \leq C_1 \left(E|X|^{\alpha} I\left(|X| \leq t\right) + t^{\alpha} P\left(|X| > t\right)\right), \tag{2.2}$$

$$E|X_{ni}|^{\alpha} I\left(|X_{ni}| > t\right) \leq C_2 E|X|^{\alpha} I\left(|X| > t\right), \tag{2.3}$$

where $C_1$ and $C_2$ are positive constants.

Throughout this paper, the symbol $C$ represents a positive constant whose value may be different in various places, and $a_n = O\left(b_n\right)$ stands for $a_n \leq C b_n$. Let $I\left(A\right)$ be the indicator function of the set $A$. Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables with the mixing coefficients $\{\mu\left(i\right), i \geq 1\}$ in each row, and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of AANA random variables with the mixing coefficients $\{\mu\left(n\right), n \geq 1\}$.

**Theorem 2.4.** Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of rowwise AANA random variables which is stochastically dominated by a random variable $X$ and $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real constants. If there exists

$$\lambda > \max\left\{\alpha^2 p,\ \alpha + 2,\ \alpha + \frac{\alpha\left(\alpha p - 1\right)}{1 - \delta},\ \alpha\left(\alpha p - 1\right) + 2\delta\right\}$$

and $\alpha p \geq 1$ such that

$$E|X|^{\lambda} < \infty. \tag{2.4}$$

When the hypothesis conditions (H1) (with $0 < \alpha \leq 2$) and (H2) are satisfied, then,

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1 \leq m \leq n} \left|\sum_{i=1}^{m} a_{ni} X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \text{for } \forall \varepsilon > 0, \tag{2.5}$$

where $b_n = n^{1/\alpha} (\log n)^{1/\gamma}$ and $\gamma > 0$.

P r o o f. For fixed $n \geq 1$, define

$$X_i^{(n)} = -b_n I(X_{ni} < -b_n) + X_{ni} I(|X_{ni}| \leq b_n) + b_n I(X_{ni} > b_n),\ i \geq 1,$$

$$T_m^{(n)} = \sum_{i=1}^{m} \left(a_{ni} X_i^{(n)} - E a_{ni} X_i^{(n)}\right),\ m = 1, 2, \cdots, n,$$

$$A = \bigcap_{i=1}^{n} \left(X_{ni} = X_i^{(n)}\right), \quad B = \bar{A} = \bigcup_{i=1}^{n} \left(X_{ni} \neq X_i^{(n)}\right) = \bigcup_{i=1}^{n} \left(|X_{ni}| > b_n\right),$$

$$E_n = \left(\max_{1 \leq m \leq n} \left|\sum_{i=1}^{m} a_{ni} X_{ni}\right| > \varepsilon b_n\right).$$

It is easy to check that for $\forall\, \varepsilon > 0$,

$$E_n = E_n A \cup E_n B \subset \left( \max_{1\le m\le n} \left| \sum_{i=1}^{m} a_{ni} X_i^{(n)} \right| > \varepsilon b_n \right) \bigcup \left( \max_{1\le m\le n} |X_{ni}| > b_n \right),$$

which implies that

$$\begin{aligned} P(E_n) &\le P\left( \max_{1\le m\le n} \left| \sum_{i=1}^{m} a_{ni} X_i^{(n)} \right| > \varepsilon b_n \right) + P\left( \max_{1\le m\le n} |X_{ni}| > b_n \right) \\ &\le P\left( \max_{1\le m\le n} \left| T_m^{(n)} \right| > \varepsilon b_n - \max_{1\le m\le n} \left| \sum_{i=1}^{m} E a_{ni} X_i^{(n)} \right| \right) + \sum_{i=1}^{n} P\left( |X_{ni}| > b_n \right). \end{aligned} \tag{2.6}$$

Firstly we will show that

$$b_n^{-1} \max_{1\le m\le n} \left| \sum_{i=1}^{m} E a_{ni} X_i^{(n)} \right| \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

By $\max_{1\le m\le n} |a_{ni}|^\alpha \le \sum_{i=1}^n |a_{ni}|^\alpha = O(n^\delta)$, we know that $\max_{1\le m\le n} |a_{ni}| \le C n^{\delta/\alpha}$ for $0 < \alpha \le 2$. Hence,

$$\sum_{i=1}^{n} |a_{ni}|^k = \sum_{i=1}^{n} |a_{ni}|^\alpha |a_{ni}|^{k-\alpha} \le C n^\delta n^{\delta(k-\alpha)/\alpha} = C n^{\delta k/\alpha} \quad \text{for } \forall\ k \ge \alpha. \tag{2.8}$$

For $0 < \alpha \le 1$, it follows from (2.2) of Lemma 2.3, (2.8) (for $k = 1$), the $c_r$ inequality, the Markov inequality and (2.4) that

$$\begin{aligned} & b_n^{-1} \max_{1\le m\le n} \left| \sum_{i=1}^{m} E a_{ni} X_i^{(n)} \right| \le C b_n^{-1} \sum_{i=1}^{n} \left| E a_{ni} X_i^{(n)} \right| \\ \le\ & C b_n^{-1} \sum_{i=1}^{n} |a_{ni}| E|X_{ni}| I(|X_{ni}| \le b_n) + C \sum_{i=1}^{n} |a_{ni}| P(|X_{ni}| > b_n) \\ \le\ & C b_n^{-1} n^{\delta/\alpha} E|X| I(|X| \le b_n) + C n^{\delta/\alpha} P(|X| > b_n) \\ \le\ & C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} E|X| I(b_{k-1} < |X| \le b_k) + C n^{\delta/\alpha} b_n^{-\lambda} E|X|^\lambda \\ \le\ & C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} b_k E|X|^\lambda b_{k-1}^{-\lambda} + C n^{\delta/\alpha} n^{-\lambda/\alpha} (\log n)^{-\lambda/\gamma} \\ \le\ & C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} k^{1/\alpha} (\log k)^{1/\gamma} (k-1)^{-\lambda/\alpha} (\log(k-1))^{-\lambda/\gamma} + C n^{\delta/\alpha} n^{-\lambda/\alpha} (\log n)^{-\lambda/\gamma} \\ \le\ & C b_n^{-1} n^{\delta/\alpha} \sum_{k=2}^{n} k^{1/\alpha - \lambda/\alpha} + C n^{\delta/\alpha} n^{-\lambda/\alpha} (\log n)^{-\lambda/\gamma} \\ \le\ & C n^{-1/\alpha} (\log n)^{-1/\gamma} n^{\delta/\alpha} n^{1+1/\alpha-1/\lambda} + C n^{\delta/\alpha} n^{-\lambda/\alpha} (\log n)^{-\lambda/\gamma} \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.9}$$

It follows from $\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n^{\delta})$ and the Hölder inequality that

$$\sum_{i=1}^{n} |a_{ni}|^{k} \leq \left(\sum_{i=1}^{n} \left(|a_{ni}|^{k}\right)^{\frac{\alpha}{k}}\right)^{\frac{k}{\alpha}} \left(\sum_{i=1}^{n} 1\right)^{\frac{\alpha-k}{\alpha}} \leq Cn \quad \text{for } 1 \leq k < \alpha. \tag{2.10}$$

For $1 < \alpha \leq 2$, it follows from $EX_n = 0$, (2.3) of Lemma 2.3, (2.10), the $c_r$ inequality, the Markov inequality and (2.4) again that

$$\begin{aligned} b_n^{-1} \max_{1\leq m\leq n} \left|\sum_{i=1}^{m} Ea_{ni}X_i^{(n)}\right| &\leq Cb_n^{-1} \max_{1\leq m\leq n} \left|\sum_{i=1}^{m} Ea_{ni}\left(X_{ni}I\left(|X_{ni}|\leq b_n\right)+b_nI\left(|X_{ni}|>b_n\right)\right)\right| \\ &\leq Cb_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X_{ni}|I(|X_{ni}|>b_n) + C\sum_{i=1}^{n}|a_{ni}|P(|X_{ni}|>b_n) \\ &\leq Cb_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X|I(|X|>b_n) + CnP(|X|>b_n) \\ &\leq Cb_n^{-1}nE|X|I(|X|>b_n) + Cn\frac{E|X|^{\lambda}}{b_n^{\lambda}} \\ &= Cb_n^{-1}n\sum_{k=n}^{\infty} E\,|X|\,I\left(b_k < |X| \leq b_{k+1}\right) + Cnn^{-\lambda/\alpha}(\log n)^{-\lambda/\gamma} \\ &\leq Cb_n^{-1}n\sum_{k=n}^{\infty} b_{k+1}P\left(|X|>b_k\right) + Cnn^{-\lambda/\alpha}(\log n)^{-\lambda/\gamma} \\ &\leq Cb_n^{-1}n\sum_{k=n}^{\infty} b_{k+1}\frac{E|X|^{\lambda}}{b_k^{\lambda}} + Cnn^{-\lambda/\alpha}(\log n)^{-\lambda/\gamma} \\ &\leq Cb_n^{-1}n\sum_{k=n}^{\infty} \frac{(k+1)^{1/\alpha}(\log(k+1))^{1/\gamma}}{k^{\lambda/\alpha}(\log k)^{\lambda/\gamma}} + Cnn^{-\lambda/\alpha}(\log n)^{-\lambda/\gamma} \\ &\leq Cb_n^{-1}\sum_{k=n}^{\infty} k^{1/\alpha-\lambda/\alpha+1} + Cnn^{-\lambda/\alpha}(\log n)^{-\lambda/\gamma} \\ &\leq Cn^{2-\lambda/\alpha}(\log n)^{-1/\gamma} + Cn^{1-\lambda/\alpha}(\log n)^{-\lambda/\gamma} \to 0 \quad \text{as } n\to\infty. \end{aligned} \tag{2.11}$$

By (2.9) and (2.11), we can obtain (2.7) immediately. Hence, for $n$ large enough,

$$P(E_n) \leq P\left(\max_{1\leq m\leq n}\left|T_m^{(n)}\right| > \frac{\varepsilon b_n}{2}\right) + \sum_{i=1}^{n} P(|X_{ni}|>b_n). \tag{2.12}$$

To prove (2.5), it suffices to prove that

$$I \triangleq \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n} P(|X_{ni}|>b_n) < \infty, \tag{2.13}$$

$$J \triangleq \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le m\le n}\left|T_m^{(n)}\right| > \frac{\varepsilon b_n}{2}\right) < \infty. \tag{2.14}$$

It follows from (1.4), the Markov inequality and (2.4) that

$$\begin{aligned} I &\triangleq \sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n} P(|X_{ni}|>b_n) \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n} P(|X|>b_n) \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-1}\frac{E|X|^{\lambda}}{b_n^{\lambda}} \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-1} n^{-\lambda/\alpha}(\log n)^{-\lambda/\gamma} < \infty \quad (\text{ since } \lambda > \alpha^2 p). \end{aligned} \tag{2.15}$$

For fixed $n \ge 1$, it is easily seen that $\{X_i^{(n)} - EX_i^{(n)}, i \ge 1, n \ge 1\}$ is still a sequence of AANA random variables by Lemma 2.1. Hence, it follows from Lemma 2.2 and the Markov inequality (for $M \in \left(3\times 2^{k-1}, 4\times 2^{k-1}\right]$, where the integer number $k \ge 1$) that

$$\begin{aligned} J &\triangleq \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le m\le n}\left|T_m^{(n)}\right| > \frac{\varepsilon b_n}{2}\right) \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} E\left(\max_{1\le m\le n}\left|T_m^{(n)}\right|^M\right) \\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}\left(\sum_{i=1}^{n}|a_{ni}|^M E\left|X_i^{(n)}\right|^M + \left(\sum_{i=1}^{n} a_{ni}^2 E(X_i^{(n)})^2\right)^{M/2}\right) \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}\sum_{i=1}^{n}|a_{ni}|^M E\left|X_i^{(n)}\right|^M + C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}\left(\sum_{i=1}^{n} a_{ni}^2 E(X_i^{(n)})^2\right)^{M/2} \\ &\triangleq J_1 + J_2. \end{aligned} \tag{2.16}$$

Take a suitable constant $M$ such that

$$\max\left\{3\times 2^{k-1}, \frac{\alpha(\alpha p-1)}{1-\delta}\right\} < M < \min\left\{4\times 2^{k-1}, \lambda-\alpha, \frac{\lambda-\alpha^2 p+\alpha}{\delta}\right\},$$

which implies that

$$\lambda > \alpha + M,\quad \frac{\lambda}{\alpha} - \frac{M}{\alpha} > 1,\quad \lambda > \alpha^2 p - \alpha + M\delta,\quad \frac{\lambda}{\alpha} - \alpha p + 2 - \frac{M\delta}{\alpha} > 1,$$

$$\alpha p - 2 + \frac{M\delta}{\alpha} - \frac{M}{\alpha} < -1,\quad M > \alpha.$$

It follows from (2.8), (2.2) of Lemma 2.3 and the Markov inequality that

$$
\begin{aligned}
J_1 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}\sum_{i=1}^{n}|a_{ni}|^M \left(E|X_{ni}|^M I\left(|X_{ni}|\le b_n\right)+b_n^M P\left(|X_{ni}|>b_n\right)\right)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}\sum_{i=1}^{n}|a_{ni}|^M \left(E|X|^M I\left(|X|\le b_n\right)+b_n^M P\left(|X|>b_n\right)\right)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}\sum_{i=1}^{n}|a_{ni}|^M E|X|^M I\left(|X|\le b_n\right)\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n}|a_{ni}|^M P\left(|X|>b_n\right)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} n^{M\delta/\alpha} E|X|^M I\left(|X|\le b_n\right)+C\sum_{n=1}^{\infty} n^{\alpha p-2} n^{M\delta/\alpha} P\left(|X|>b_n\right)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2+M\delta/\alpha} b_n^{-M}\sum_{k=2}^{n} E|X|^M I\left(b_{k-1}<|X|\le b_k\right)\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2+M\delta/\alpha}\frac{E|X|^\lambda}{b_n^\lambda}\\
&\le C\sum_{k=2}^{\infty} b_k^M P\left(|X|>b_{k-1}\right)\sum_{n=k}^{\infty} n^{\alpha p-2+M\delta/\alpha-M/\alpha}(\log n)^{-M/\gamma}\\
&\quad + C\sum_{n=1}^{\infty}\frac{n^{\alpha p-2+M\delta/\alpha}}{n^{\lambda/\alpha}(\log n)^{\lambda/\gamma}}\\
&\le C\sum_{k=3}^{\infty}\frac{k^{M/\alpha}(\log k)^{M/\gamma}}{(k-1)^{\lambda/\alpha}(\log(k-1))^{\lambda/\gamma}}+C\sum_{n=1}^{\infty}\frac{n^{\alpha p-2+M\delta/\alpha}}{n^{\lambda/\alpha}(\log n)^{\lambda/\gamma}}<\infty.
\end{aligned}
\tag{2.17}
$$

The final inequality is based on the fact that $\lambda>\alpha+M$ and $\frac{\lambda}{\alpha}-\alpha p+2-\frac{M\delta}{\alpha}>1$.

It follows from (2.2) of Lemma 2.3, (2.8) and the $c_r$ inequality again that

$$
\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}\left(\sum_{i=1}^{n} a_{ni}^2 EX_{ni}^2 I(|X_{ni}|\le b_n)+b_n^2 P(|X_{ni}|>b_n)\right)^{M/2}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M}\left(\sum_{i=1}^{n} a_{ni}^2 EX^2 I(|X|\le b_n)+b_n^2 P(|X|>b_n)\right)^{M/2}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} n^{M\delta/\alpha}\left(EX^2 I(|X|\le b_n)\right)^{M/2}\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2} n^{M\delta/\alpha}\left(P(|X|>b_n)\right)^{M/2}\\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2} b_n^{-M} n^{M\delta/\alpha} E|X|^M I(|X|\le b_n)+C\sum_{n=1}^{\infty} n^{\alpha p-2} n^{M\delta/\alpha} P(|X|>b_n)\\
&< \infty \quad \text{(see the proof of (2.17))}.
\end{aligned}
\tag{2.18}
$$

Hence, the desired result of (2.6) follows immediately from the above statements. The proof of Theorem 2.4 is completed. □

In Theorem 2.4, if the hypothesis condition (H1) (with $0 < \alpha \le 2$) is replaced by the hypothesis condition (H4) (with $0 < \alpha \le 2$), assume that there exists some $\lambda > \alpha + 2$ satisfying (2.4). Then, we can get the following result for the special case of $\alpha p = 1$. The proof is similar to that of Theorem 2.4, so we omit the detailed proof.

**Theorem 2.5.** Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of rowwise AANA random variables which is stochastically dominated by a random variable $X$ and $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real constants. When the hypothesis conditions (H2), (H4) (with $0 < \alpha \le 2$) and (2.4) for some $\lambda > \alpha + 2$ are satisfied, then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_{ni}\right| > \varepsilon b_n\right) < \infty \text{ for } \forall \varepsilon > 0, \tag{2.19}$$

where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ and $\gamma > 0$.

In Theorems 2.4 − 2.5, the array of rowwise AANA random variables is replaced with the sequence of AANA random variables, i. e. let $\{X_n, n \ge 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable $X$. We can provide the following Marcinkiewicz–Zygmund type strong law of large numbers for weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ of AANA random variables. Their proofs are similar to that of Theorem 2.4, so the details are omitted.

**Theorem 2.6.** Let $\{X_n, n \ge 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable $X$ and $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real constants. When the hypothesis conditions (H1) (with $0 < \alpha \le 2$), (H2) and (2.4) are satisfied, then,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_{i}\right| > \varepsilon b_n\right) < \infty \quad \text{for } \forall \varepsilon > 0. \tag{2.20}$$

$$\lim_{n\to\infty} \frac{1}{b_n}\left|\sum_{i=1}^{n} a_{ni}X_i\right| = 0 \quad \text{a.s. ,} \tag{2.21}$$

where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ and $\gamma > 0$.

**Theorem 2.7.** Let $\{X_n, n \ge 1\}$ be a sequence of AANA random variables which is stochastically dominated by a random variable $X$ and $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers. When the hypothesis conditions (H2), (H4) (with $0 < \alpha \le 2$) and (2.4) for some $\lambda > \alpha + 2$ are satisfied, then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le m\le n}\left|\sum_{i=1}^{m} a_{ni}X_{i}\right| > \varepsilon b_n\right) < \infty \quad \text{for } \forall \varepsilon > 0, \tag{2.22}$$

and (2.21) hold, where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$.

**Remark 2.8.** Wang et al. [19] investigated the complete convergence for weighted sums of arrays of rowwise AANA random variables and AANA random variables under an exponential moment condition, respectively. Compared with Wang et al. [19], it is worth pointing out that our conclusions are obtained under much weaker moment conditions. In addition, in Theorem 2.5 and Theorem 2.7, we consider both of the cases of $\alpha \neq \gamma$ and $\alpha = \gamma$. The only defect is that our moment condition $E|X|^{\lambda}$ for some $\lambda > \alpha + 2$ is stronger than the corresponding ones of the above Theorem 1.6 and Theorem 1.7. So, our main results partially improve the results obtained by Huang et al. [8].

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