KYBERNETIKA — VOLUME *51* (2015), NUMBER 6, PAGES 1023–1034

NECESSARY CONDITIONS FOR VECTOR OPTIMIZATION IN INFINITE DIMENSION

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In the paper we present second-order necessary conditions for constrained vector optimization problems in infinite–dimensional spaces. In this way we generalize some corresponding results obtained earlier.

Keywords: $C^{1,1}$–function, ℓ–stable function, generalized second-order directional derivative, Dini derivative, vector optimization

Classification: 49K10, 49J52, 49J50, 90C29, 90C30

1. INTRODUCTION

The research of second-order optimality conditions is very important from both theoretical and practical point of view. Let us recall the following monographs containing a lot of information on generalized second-order derivatives and their applications in optimization: [25, 31, 35].

In this paper, we will study a certain vector constrained optimization problem. Let X, Y, Z be normed linear spaces, $f\colon X \to Y$, $g\colon X \to Z$ be functions, and let $C \subset Y$ and $K \subset Z$ be closed convex pointed cones with $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} K \neq \emptyset$. For the definitions and properties of such cones, see e. g. [23, 34, 35].

We will consider the problem

$$\min f(x), \qquad\qquad \text{subject to } g(x) \in -K. \tag{1}$$

A feasible point x_0 (i. e. $g(x_0) \in -K$) is said to be *a local weakly efficient point* of problem (1) if there exists a neighbourhood U of x_0 such that

$$(f(U \cap g^{-1}(-K)) - f(x_0)) \cap (-\operatorname{int} C) = \emptyset.$$

The problem (1) was studied e. g. in [16, 17, 18, 19, 20, 26, 27, 32]. The obtained results were surpassed in 2011, when I. Ginchev [14] and D. Bednařík with K. Pastor [8] published indepedently the following equivalent result (Theorem 1.1). We recall that the equivalence was shown in [11].

We will need some next notions around problem (1) to remind Theorem 1.1.

DOI: 10.14736/kyb-2015-6-1023

First, for a cone $C \subset X$, we define

$$C^* = \{c^* \in X^*; \langle c^*, c\rangle \geq 0, \quad \forall c \in C\}$$

and by S_{X^*} we denote the unit sphere in X^*, i.e. the set $\{x^* \in X^*; \|x^*\| = 1\}$.

Further, we recall that a function $f\colon X \to Y$, where X and Y are normed linear spaces, is *strictly differentiable at* $x \in X$ if it has Fréchet derivative $f'(x) \in \mathcal{L}(X, Y)$ at x such that it holds

$$\lim_{y\to x, t\downarrow 0} \sup_{h\in S_X} \|\frac{1}{t}(f(y+th) - f(y)) - f'(x)h\| = 0.$$

Supposing that a function $f\colon X \to Y$ is Fréchet differentiable at $x \in X$, we define *the second-order Hadamard directional derivative* $D_2 f(x; u)$ *of* f *at* x *in the direction* $u \in X$ in the following way:

$$\begin{aligned} D_2 f(x;u) &= \operatorname{Limsup}_{t\downarrow 0, v\to u} \frac{f(x+tv) - f(x) - tf'(x)u}{t^2/2} \\ &= \Big\{ y \in Y;\ \exists (t_n, u_n) \to (0^+, u), \\ &\qquad y = \lim_{n\to\infty} \frac{f(x + t_n u_n) - f(x) - t_n f'(x)u}{t_n^2/2} \Big\}. \end{aligned}$$

Finally, for problem (1) we denote

$$K(g(x_0)) = \{\gamma(z + g(x_0)) : \gamma \geq 0, z \in K\}.$$

Theorem 1.1. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$ be strictly differentiable at $x_0 \in \mathbb{R}^n$. If x_0 is a local weakly efficient point of problem (1), then

(i) there exists $(c^*, k^*) \in ((C^* \times K(g(x_0))^*) \setminus \{(0,0)\})$ such that

$$c^* \circ f'(x_0) + k^* \circ g'(x_0) = 0 \tag{2}$$

(ii) for $u \in \mathbb{R}^n$ if $(f,g)'(x_0)u \in -(C \times K(g(x_0)) \setminus \operatorname{int}(C \times K(g(x_0))))$, then for every $(y_0, z_0) \in D_2(f,g)(x_0; u)$ there exists $(c^*, k^*) \in ((C^* \times K(g(x_0))^*) \setminus \{(0,0)\})$ such that (2) is true and

$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle \geq 0. \tag{3}$$

2. ℓ–STABILITY

In some previous papers, the second-order optimality conditions were stated for $C^{1,1}$ functions, see e.g. [2, 9, 10, 16, 17, 19, 20, 21, 22] and references therein. We recall that *a* $C^{1,1}$ *function* is a function which is differentiable with a locally Lipschitz derivative.

In 2007, the concept of $\ell-$stability was introduced to diminish the $C^{1,1}$ property in solving some second-order scalar optimization problems [3]. A function $f\colon X \to \mathbb{R}$, where X is a normed linear space, is *ℓ–stable at $x \in X$* if there exist a neighborhood $\mathcal{U}$ of x and a $K > 0$ such that

$$|f^{\ell}(y;h) - f^{\ell}(x;h)| \le K\|y - x\|, \quad \forall y \in \mathcal{U},\ \forall h \in S_X,$$

where

$$f^{\ell}(y;h) = \liminf_{t\downarrow 0} \frac{f(y+th) - f(y)}{t}.$$

The properties of $\ell-$stable at some point functions were studied e. g. in [1, 4, 5, 6, 7, 8, 11, 12, 14, 15, 28, 29, 30] for both scalar and vector functions. Among the others, the sufficient second-order optimality condition for problem (1) was stated indepedently in [14] and [8] in terms of $\ell-$stable at some point functions.

Now, we recall the definition of $\ell-$stability for vector functions possibly for infinite dimension. We say that a function $f\colon X \to Y$, where X and Y are normed linear spaces, is *ℓ–stable at $x \in X$* provided that there are a neighborhood $\mathcal{U}$ of x and a constant $K > 0$ such that

$$|f^{\ell}(y;h)(\gamma) - f^{\ell}(x;h)(\gamma)| \le K\|y - x\|,$$

for every $y \in \mathcal{U}$, for every $h \in S_X$ and for every $\gamma \in S_{Y^*}$.

The symbol $f^{\ell}(x;h)(\gamma)$ denotes *the lower Dini directional derivative of f at x in the direction $h \in X$ with respect to the linear functional $\gamma \in Y^*$*. It is defined by the formula:

$$f^{\ell}(x;h)(\gamma) := \liminf_{t\downarrow 0} \frac{\langle \gamma, f(x+th) - f(x)\rangle}{t}.$$

Of course, $f^{\ell}(x;h) = f^{\ell}(x;h)(1)$ for scalar functions.

3. INFINITE DIMENSION

The following differentiable property of $\ell-$stable at a point functions was obtained in [33, Theorem 3.1], consult also [12].

Theorem 3.1. Let X be a normed linear space, Y a Banach space, and $f\colon X \to Y$ be a continuous function near $x \in X$. If f is an ℓ–stable function at x, then f is strictly differentiable at x.

In the sequel, we will need a certain mean value theorem.

Lemma 3.2. (Pastor [33, Lemma 3.2]) Let X and Y be normed linear spaces, $f\colon X \to Y$ be a continuous function, $\gamma \in Y^*$ and let $a, b \in X$. Then there are points $\xi_1, \xi_2 \in (a, b)$ such that

$$f^{\ell}(\xi_1; b-a)(\gamma) \le \langle \gamma, f(b) - f(a)\rangle \le f^{\ell}(\xi_2; b-a)(\gamma).$$

The following lemma generalizes the analogous result from [7, Lemma 6], where we supposed that X was a finite-dimensional space and that Y was a Banach space having the Radon–Nikodým property.

Lemma 3.3. Let X be a normed linear space, Y a Banach space, and $f\colon X \to Y$ be a continuous function near $x \in X$. If f is an ℓ–stable function at x, then there exists an $\alpha > 0$ such that

$$\begin{aligned}
&\forall R > 0\ \exists \delta > 0\ \forall u, w \in X : \|u\| \le R, \|w\| \le R, \forall t \in (0,\delta) : \\
&\left\| \frac{2}{t^2}(f(x+tu) - f(x) - tf'(x)u) - \frac{2}{t^2}(f(x+tw) - f(x) - tf'(x)w) \right\| \\
&\le \alpha(\|u\| + \|w\|)\|u - w\|. \qquad (4)
\end{aligned}$$

P r o o f. Note that by Theorem 3.1 f is strictly differentiable at x. Suppose that $\mathcal{U}$ denotes a neighborhood of x on which f is continuous and a constant $K > 0$ is such that

$$|f^\ell(y;h)(\xi) - f^\ell(x;h)(\xi)| \le K\|y - x\|, \quad \forall y \in \mathcal{U}, \forall h \in S_X, \forall \xi \in S_{Y^*}.$$

Let us consider an auxiliary function $g\colon X \to Y$ defined by $g(z) := f(z) - f'(x)z, z \in X$.

There is an $\eta > 0$ such that $B(x,\eta) \subset \mathcal{U}$. Further, we fix $R > 0$ and consider $\delta > 0$ such that $\delta R < \eta$. Then for arbitrary $u \in X$ and $w \in X$ satisfying $\|u\| \le R$, $\|w\| \le R$, and for every $t \in (0,\delta)$ we have $x + tu \in B(x,\eta)$, $x + tw \in B(x,\eta)$. We fix u, w with the previous properties. Then for certain $y_t \in (x + tu, x + tw)$, $\xi_t \in S_{Y^*}$, it holds due to Lemma 3.2, the Hahn–Banach theorem and ℓ–stability:

$$\begin{aligned}
& \left\|\frac{2}{t^2}(f(x+tu) - f(x) - tf'(x)u) - \frac{2}{t^2}(f(x+tw) - f(x) - tf'(x)w)\right\| \\
= {} & \frac{2}{t^2}\|g(x+tu) - g(x+tw)\| = \frac{2}{t^2}|\langle \xi_t, g(x+tu) - g(x+tw)\rangle| \\
\le {} & \frac{2}{t}|g^\ell(y_t; u - w)(\xi_t)| = \frac{2}{t}|f^\ell(y_t; u-w)(\xi_t) - \langle \xi_t, f'(x)(u - w)\rangle| \\
\le {} & \frac{2}{t}K\|y_t - x\|\|u - w\|.
\end{aligned}$$

Since for some $\mu \in (0,1)$ we have $y_t = \mu(x+tu) + (1-\mu)(x+tw)$, then we can derive:

$$\begin{aligned}
\|y_t - x\| &= \|\mu(x+tu) + (1-\mu)(x+tw) - x\| \\
&= t\|\mu u + (1-\mu)w\| \\
&\le t(\mu\|u\| + (1-\mu)\|w\|) \\
&\le t(\|u\| + \|w\|).
\end{aligned}$$

Now, letting $\alpha := 2K > 0$ we get our inequality (4). □

Theorem 3.4. Let X be a normed linear space, Y, Z be Banach spaces, $f\colon X \to Y$, $g\colon X \to Z$ be continuous functions near $x \in X$ which are ℓ–stable at x. Let x be a local weakly efficient point for problem (1). Then the following two conditions are satisfied for each $u \in S_X$:

(i) $(f,g)'(x)u \notin -\operatorname{int}(C \times K)$,

(ii) if $(f,g)'(x)u \in -((C \times K) \setminus \operatorname{int}(C \times K))$, then for all $(y,z) \in D_2(f,g)(x;u)$ it holds

$$\operatorname{conv}\{(y,z), \operatorname{Im}(f,g)'(x)\} \cap (-\operatorname{int}(C \times K)) = \emptyset.$$

P r o o f. In order to prove (i) fix $u \in X$ arbitrarily. Suppose that $x \in X$ is a local weakly efficient point for problem (1) and $g'(x)u \in -\operatorname{int} K$. Then there exists a sequence $\{x + t_k u\}_{k=1}^{+\infty} \subset X$, $t_k \downarrow 0$, such that

$$(g(x+t_k u) - g(x))/t_k \in -\operatorname{int} K$$

$$g(x+t_k u) \in g(x) - \operatorname{int} K \subset -K - K = -K.$$

Hence, every point $x + t_k u$, $k \in \mathbb{N}$, is feasible and we obtain

$$f(x+t_k u) - f(x) \notin -\operatorname{int} C$$

$$(f(x+t_k u) - f(x))/t_k \notin -\operatorname{int} C$$

for all k large enough. Now letting $k \to +\infty$ we get that $f'(x)u \notin -\operatorname{int} C$. Note that Theorem 3.1 guarantees the existence of $f'(x)$ and $g'(x)$.

In order to prove the second condition we will assume on the contrary that there is a $u \in S_X$ such that $(f,g)'(x)u \in -((C \times K) \setminus \operatorname{int}(C \times K))$, and for some $(y,z) \in D_2(f,g)(x;u)$ it holds:

$$\operatorname{conv}\{(y,z), \operatorname{Im}(f,g)'(x)\} \cap (-\operatorname{int}(C \times K)) \neq \emptyset.$$

In other words, there exist a $\overline{\lambda} \in [0,1]$ and a $w \in X$ so that

$$(1-\overline{\lambda})(y,z)(u) + \overline{\lambda}(f,g)'(x)w \in -\operatorname{int}(C \times K). \tag{5}$$

Since $(-\operatorname{int}(C \times K))$ is open, the above formula gives the existence of an $\varepsilon > 0$ such that

$$(1-\lambda)(y,z)(u) + \lambda(f,g)'(x)w \in -\operatorname{int}(C \times K), \quad \forall \lambda \in (\overline{\lambda} - \varepsilon, \overline{\lambda} + \varepsilon).$$

Thus, we can suppose, without loss of generality, that $\overline{\lambda} \in (0,1)$ in formula (5).

Let sequences $\{t_k\}_{k=1}^{\infty}$, $t_k \downarrow 0$, and $\{u_k\}_{k=1}^{\infty}$, $u_k \to u$ satisfy

$$\{(2/t_k^2)(f(x+t_k u_k) - f(x) - t_k f'(x)u)\} \longrightarrow y$$

$$\{(2/t_k^2)(g(x+t_k u_k) - g(x) - t_k g'(x)u)\} \longrightarrow z$$

as $k \to +\infty$. We put

$$v_k := u_k + \{\overline{\lambda} t_k w / 2(1-\overline{\lambda})\}.$$

Observe that $v_k \to u$ as $k \to +\infty$, and $w = (2(1-\overline{\lambda})(v_k - u_k))/(\overline{\lambda} t_k)$. We claim that $(2/t_k^2)(g(x+t_k v_k) - g(x+t_k u_k)) \to \overline{\lambda} g'(x)w/(1-\overline{\lambda})$ as $k \to +\infty$. Indeed, by the Hahn–Banach Theorem, Lemma 3.2 and the definition of ℓ–stability, there are $\xi_k \in S_{Z^*}$, $y_k \in (x+t_k u_k, x+t_k v_k)$ and $L > 0$ such that for almost all $k \in \mathbb{N}$ it holds

$$\begin{aligned} & \|(2/t_k^2)(g(x+t_k v_k) - g(x+t_k u_k)) - \overline{\lambda} g'(x)w/(1-\overline{\lambda})\| \\ = \ & \langle \xi_k, (2/t_k^2)(g(x+t_k v_k) - g(x+t_k u_k)) - \overline{\lambda} g'(x)w/(1-\overline{\lambda})\rangle \\ \leq \ & \overline{\lambda} g^{\ell}(y_k; w)(\xi_k)/(1-\overline{\lambda}) - \overline{\lambda} g^{\ell}(x; w)(\xi_k)/(1-\overline{\lambda}) \\ \leq \ & L\overline{\lambda}\|y_k - x\|\|w\|/(1-\overline{\lambda}) \to 0 \text{ as } k \to +\infty. \end{aligned}$$

Since

$$\begin{aligned} & \lim_{k\to+\infty} (2/t_k^2)(g(x+t_k v_k) - g(x) - t_k g'(x)u) \\ = \ & \lim_{k\to+\infty} (2/t_k^2)(g(x+t_k u_k) - g(x) - t_k g'(x)u) \\ + \ & \lim_{k\to+\infty} (2/t_k^2)(g(x+t_k v_k) - g(x+t_k u_k)) = z + \overline{\lambda} g'(x)w/(1-\overline{\lambda}) \in -\operatorname{int} K \end{aligned}$$

we derive

$$g(x+t_k v_k) \in g(x) + t_k g'(x)u - \operatorname{int} K \subset -K - K - \operatorname{int} K \subset -\operatorname{int} K$$

for almost all $k \in \mathbb{N}$.

Hence, every point $x + t_k v_k$ is feasible if k is large enough. We can proceed analogously for f – we get

$$f(x+t_k v_k) - f(x) \in t_k f'(x)u - \operatorname{int} C \subset -C - \operatorname{int} C \subset -\operatorname{int} C$$

for almost all $k \in \mathbb{N}$, a contradiction. □

Theorem 3.5. Let X be a normed linear space, Y, Z be Banach spaces, $f \colon X \to Y$ and $g \colon X \to Z$ be continuous functions near $x \in X$ which are ℓ–stable at x. If x is a local weakly efficient point of problem (1), then

(i) there exists a $(c^*, k^*) \in ((C^* \times K^*) \setminus \{(0,0)\})$ such that

$$c^* \circ f'(x_0) + k^* \circ g'(x_0) = 0 \tag{6}$$

(ii) for any $u \in X$, if $(f,g)'(x)u \in -((C \times K) \setminus \operatorname{int}(C \times K))$, then for every $(y_0, z_0) \in D_2(f,g)(x;u)$ there exists a $(c^*, k^*) \in ((C^* \times K^*) \setminus \{(0,0)\})$ such that (6) is true and

$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle \geq 0. \tag{7}$$

Proof.

(i) By Theorem 3.4 (i) and the separation theorem (see e.g. [13, Corollary 2.13]) there are $(c^*, k^*) \in ((Y^* \times Z^*) \setminus \{(0,0)\})$ and $\alpha \in \mathbb{R}$ such that for every $u \in X$ and for every $(c, k) \in -(C \times K)$ we have

$$\langle c^*, f'(x)u\rangle + \langle k^*, g'(x)u\rangle \geq \alpha, \tag{8}$$

$$\langle c^*, c\rangle + \langle k^*, k\rangle \leq \alpha. \tag{9}$$

Since $(f, g)'(x)X$ and $C \times K$ are cones, it holds $\alpha = 0$. Then, the inequality (8) becomes the equality (6). Setting $k = 0$ in (9), we obtain $c^* \in C^*$, and setting $c = 0$ in (9), we obtain $k^* \in K^*$.

(ii) Using Theorem 3.4 (ii) and the separation theorem, one has (8), (9), and in addition

$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle \geq \alpha.$$

Similarly as in (i), $\alpha = 0$, $c^* \in C^*$, $k^* \in K^*$, and thus formulas (6) and (7) hold.

□

4. COMPARISON OF THEOREMS

Remark 4.1. Comparing Theorem 1.1 and Theorem 3.5, we can say that in finite–dimensional setting the optimality condition from Theorem 1.1 is tighter in general. Indeed, for an arbitrary $z_0 \in K$ we can write

$$z_0 = 1(z_0 - g(x_0) + g(x_0)),$$

and because $g(x_0) \in -K$ and K is a cone, we have $z_0 - g(x_0) \in K$. Therefore $z_0 \in K(g(x_0))$, and thus $K \subset K(g(x_0))$. Then $K(g(x_0))^* \subset K^*$.

Now, it is an open question whether or not we can replace K^* by $K(g(x_0))^*$ in Theorem 3.5.

Remark 4.2. Further, in finite–dimensional setting, Theorem 1.1 requires only strict differentiability at the considered point. Having in mind Theorem 3.1, it is another open question whether or not we can replace ℓ–stability by strict differentiability in Theorem 3.5.

On the other hand, Theorem 3.5 can help to find a local weakly efficient point of problem (1) in infinite dimension in contrast to Theorem 1.1. We will demonstrate this fact by the following example which was inspired by Example 1 in [7].

Example 4.3. Consider the sequence $a_n = 1/n$, $n = 1, 2, \dots$ Then

$$\lim_{n\to\infty} \frac{a_{n+1} + a_n^2}{a_{n+1} + a_n} = \frac{1}{2} > 0.$$

Let us define a function $\varphi\colon [0, +\infty) \to \mathbb{R}$ as follows.

$$\varphi(u) = \begin{cases} a_1, & \text{if } u > a_1, \\ \frac{a_n^2 - a_{n+1}}{a_n - a_{n+1}}(u - a_{n+1}) + a_{n+1}, & \text{if } u \in (a_{n+1}, a_n], \\ 0, & \text{if } u = 0. \end{cases}$$

Next, we will define a function $f\colon \mathbb{R} \to \mathbb{R}$ via the Riemann integral:

$$r(x) := \int_0^{|x|} \varphi(u)\,\mathrm{d}u, \quad x \in \mathbb{R}.$$

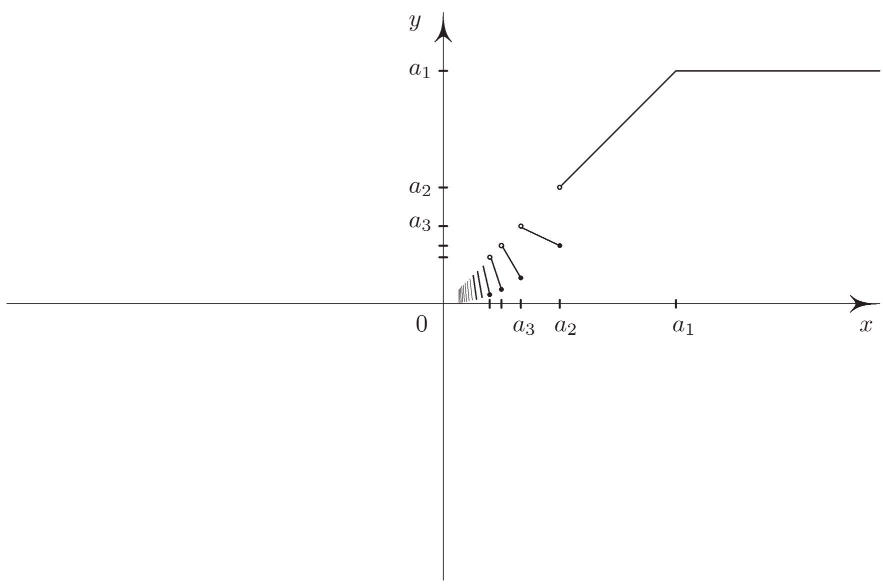

Fig. 1. Function φ.

It is easy to see r is not of class $\mathrm{C}^{1,1}$ on any neighborhood of $x = 0$. Furthermore $r'(0) = 0$, r is ℓ–stable at $x = 0$, and $\liminf_{t\downarrow 0} r(t)/(2/t^2) > \varepsilon$ for some $\varepsilon > 0$ (for details see [BP2, Example 2]). By definition of φ, we can show that for any $x > 0$, we have $r(x) \le x^2/2$. Now we consider a function $f\colon \mathbb{R} \to \ell_2$ defined as follows

$$f(t) := \left\{\frac{r(t)}{2^n}\right\}_{n=1}^{+\infty} \in \ell_2,$$

where $\ell_2 = \{\{a_n\}_{n=1}^{+\infty} : \sum_{n=1}^{+\infty} |a_n|^2 < +\infty\}$ with the norm

$$\|\{a_n\}\| := \sqrt{\sum_{n=1}^{+\infty} |a_n|^2}.$$

It is well known that $(\ell_2, \|\cdot\|)$ is a Banach space and that $\ell_2^* = \ell_2$. We will define

$$C = \left\{ x = \{x_n\}_{n=1}^{+\infty} \in \ell_2 : \sum_{n=1}^{+\infty} \frac{x_n}{(\sqrt{2})^n} \geq \frac{1}{2}\|\{x_n\}\| \right\}.$$

Then

$$C^* = \left\{ a = \{a_n\}_{n=1}^{+\infty} \in \ell_2 : \sum_{n=1}^{+\infty} a_n x_n \geq 0,\ \forall x = \{x_n\}_{n=1}^{+\infty} \in C \right\}.$$

We note that the considered cone C is a special case of a more general type of cones satisfying $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} C^* \neq \emptyset$, for details see [24].

For any $t \in \mathbb{R}$ and $\xi = \{a_n\}_{n=1}^{+\infty} \in S_{\ell_2^*}$ we have:

$$\begin{aligned} f^\ell(t; \pm 1)(\xi) &= \liminf_{s \downarrow 0} \frac{\langle \xi, f(t \pm s) - f(t) \rangle}{s} \\ &= \liminf_{s \downarrow 0} \frac{\left\langle \xi, \left\{ \frac{r(t \pm s)}{2^n} \right\}_{n=1}^{+\infty} - \left\{ \frac{r(t)}{2^n} \right\}_{n=1}^{+\infty} \right\rangle}{s} \\ &= \liminf_{s \downarrow 0} \frac{1}{s} \sum_{n=1}^{+\infty} a_n \left\{ \frac{r(t \pm s)}{2^n} - \frac{r(t)}{2^n} \right\} \\ &= \liminf_{s \downarrow 0} \frac{r(t \pm s) - r(t)}{s} \sum_{n=1}^{+\infty} \frac{a_n}{2^n} = r^\ell(t; \pm 1) \sum_{n=1}^{+\infty} \frac{a_n}{2^n}. \end{aligned}$$

From the properties of r we deduce that $f'(0) = 0$ and that f is ℓ–stable at $t = 0$. It can be easily shown that it holds

$$D_2 f(0; 1) = D_2 f(0, -1) \subset \left\{ \{y_n\}_{n=1}^{+\infty} \in \ell_2 : y_n > \frac{\varepsilon}{2^n},\ \forall n \in \mathbb{N} \right\}.$$

Further, we define $g \colon \mathbb{R} \to \mathbb{R} : g(t) = t$, and

$$K = \{s; s \geq 0\} = K^*.$$

We have $g'(0) = 1$, $D_2 g(0; 1) = D_2 g(0, -1) = \{0\}$.

Now, we can see that Theorem 3.5 admits for 0 to be a local weakly efficient point. Indeed, condition (i) of Theorem 3.5 is satisfied if we take

$$c^* = \left\{ \frac{1}{(\sqrt{2})^n} \right\}_{n=1}^{+\infty}, \quad k^* = 0.$$

Condition (ii) from Theorem 3.5 is also satisfied for the previous choice of c^* and k^*, because

$$\langle c^*, y_0\rangle + \langle k^*, z_0\rangle = \langle c^*, y_0\rangle = \sum_{n=1}^{+\infty} \frac{y_n}{(\sqrt{2})^n} > \sum_{n=1}^{+\infty} \frac{\varepsilon}{(2^{\frac{3}{2}})^n} = \frac{\varepsilon}{2^{\frac{3}{2}} - 1} > 0$$

for every $y_0 \in D_2 f(0;1) = D_2 f(0,-1)$ and $z_0 = 0$.

ACKNOWLEDGEMENT

This work was partially supported by the student projects IGA-PrF-2014-028 and IGA-PrF-2015-013 of the Palacky University.

(Received September 9, 2014)

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