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STABILITY ANALYSIS AND ABSOLUTE SYNCHRONIZATION OF A THREE–UNIT DELAYED NEURAL NETWORK

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In this paper, we consider a three-unit delayed neural network system, investigate the linear stability, and obtain some sufficient conditions ensuring the absolute synchronization of the system by the Lyapunov function. Numerical simulations show that the theoretically predicted results are in excellent agreement with the numerically observed behavior.

Keywords: absolute synchronization, delay, linear stability, neural network

Classification: 34D06, 34D20

1. INTRODUCTION

Artificial neural networks widely exist in lots of subjects, such as content-addressable memories, neocortex, cerebellum, hippocampus, and even in chemistry and electrical design. Information is stored as stable equilibrium points of the system, which has been investigated and also become the subject of much recent activity.

It is not difficult to check that neural networks are complex and filled with lots of rich nonlinear dynamics, especially the dynamics of the delayed neural networks, which are even richer and more complicated [29]. In general, it has two ways to deal with these kinds of problems: One is to linearize the system near the equilibrium; drive the conditions in this way concerning the local stability around an equilibrium. The other method is to obtain some sufficient conditions ensuring the stability of the system by constructing a suitable Lyapunov function. In order to obtain a clear and deep understanding of the dynamics of neural networks, more and more experts investigate and study the delayed neural networks models with isolated neuron and two neurons [1, 2–7, 9, 11–13, 15–18, 28]. It is hoped to promote our understanding about the large networks by discussing the dynamics of such small networks. In fact, these theoretical results can help us to improve the understanding of the dynamics of the system and are also important complements to the experimental and numerical investigations exploiting analog circuits and digital computers. Hopfield studied a simplified neural network model in which each neuron is represented by a linear circuit consisting of a resistor and a capacitor and is connected to the other neurons via nonlinear sigmoidal activation functions [14, 20, 21,

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30]. Dhamala studied that synchronization of individual neurons plays the crucial role in the emergence of pathological rhythmic brain activity in Parkinsons disease, essential tremor, and epilepsies [10]. Delayed feedback suppressed or facilitated the collective synchrony in an ensemble of global coupled oscillators [22, 23, 26, 27, 31]. Artificial neural network reveals the complex dynamic properties of the biological neural network system by simulating the structure and function of human brain cells through the circuit [8, 19, 24, 25, 32]. Herein we consider the dynamic characteristic of bidirectional ring network model with the connections between three neurons, which is shown in Figure 1. This kind of ring network is a kind of common loop network, which has been found in a lot of neural structures, such as neocortex, cerebellum, hippocampus, and even in chemistry and electrical design. We can know the basic mechanisms of recurrent network by studying the ring network. Due to the complexity of the analysis, many researchers study and discuss the sufficient condition of stability by traditional Lyapunov approach. However, very few papers about stability analysis of bidirectional ring network model by analyzing the characteristic of the eigenvalues can be found and very limited, especially in high dimensional bidirectional ring network model with delay.

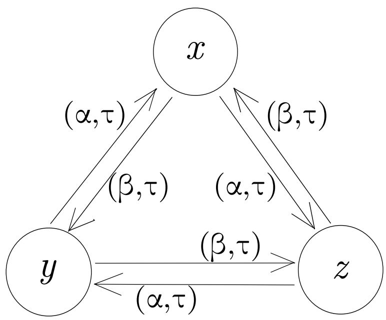

Fig. 1. Architecture of the model described by (1.1).

In this paper, we consider the dynamic characteristic of bidirectional ring network model (1.1) with the connections between three constant neurons, which is given as the following model (see Figure 1):

$$\begin{cases} \dot{x} &= -x + \alpha f(y(t-\tau)) + \beta f(z(t-\tau)), \\ \dot{y} &= -y + \alpha f(z(t-\tau)) + \beta f(x(t-\tau)), \\ \dot{z} &= -z + \alpha f(x(t-\tau)) + \beta f(y(t-\tau)), \end{cases} \tag{1.1}$$

where τ is nonnegative and denotes the synaptic transmission delay, and the strength of the self and nearest-neighbour coupling is denoted by α and β, respectively. They are the nonzero connection weights. $f : R \to R$ is the activation function. Throughout this paper, we always assume that $\alpha\beta \neq 0$, and $f : R \to R$ is adequately smooth, and satisfies the following conditions:

(C1) $f(0) = 0, f'(0) = 1$;

(C2) $f(x) \neq 0$ for $x \neq 0$.

The rest of the paper is organized as follows: In Section 2, we discuss the linear stability of the trivial solution of system (1.1). We study the absolute synchronization of the system exploiting the Lyapunov function in Section 3. Section 4 is devoted to numerical simulations and we conclude this paper in Section 5.

2. LINEAR STABILITY OF THE TRIVIAL FIXED POINT

In this section, we focus on the linear stability of the trivial fixed point $(x, y, z) = (0, 0, 0)$ of the nonlinear DDE(1.1). Linearizing (1.1) about it produces

$$\left\{\begin{array}{lcl}\dot{x} & = & -x + \alpha y(t-\tau) + \beta z(t-\tau),\\ \dot{y} & = & -y + \alpha z(t-\tau) + \beta x(t-\tau),\\ \dot{z} & = & -z + \alpha x(t-\tau) + \beta y(t-\tau).\end{array}\right. \tag{2.1}$$

The characteristic matrix of (2.1) is

$$\triangle(\mu, \lambda) = \begin{pmatrix} \lambda+1 & -\alpha e^{-\lambda\tau} & -\beta e^{-\lambda\tau} \\ -\beta e^{-\lambda\tau} & \lambda+1 & -\alpha e^{-\lambda\tau} \\ -\alpha e^{-\lambda\tau} & -\beta e^{-\lambda\tau} & \lambda+1 \end{pmatrix},$$

and hence the characteristic equation is

$$0 = \det\triangle(\mu, \lambda) = (\lambda+1)^3 - 3\alpha\beta e^{-2\lambda\tau}(\lambda+1) - (\alpha^3+\beta^3)e^{-3\lambda\tau} = \chi_1(\lambda)\chi_2(\lambda), \tag{2.2}$$

where

$$\chi_1(\lambda) = \lambda + 1 - (\alpha+\beta)e^{-\lambda\tau}$$

and

$$\chi_2(\lambda) = \Big(\lambda + 1 + \frac{\alpha+\beta}{2}e^{-\lambda\tau}\Big)^2 + \frac{3}{4}(\alpha-\beta)^2 e^{-2\lambda\tau}.$$

So either

$$\lambda + 1 - (\alpha+\beta)e^{-\lambda\tau} = 0 \tag{2.3}$$

or

$$\Big(\lambda + 1 + \frac{\alpha+\beta}{2}e^{-\lambda\tau}\Big) = \pm i\frac{\sqrt{3}}{2}(\alpha-\beta)e^{-\lambda\tau}. \tag{2.4}$$

It is well known that the trivial fixed point of the nonlinear DDE (1.1) is locally asymptotically stable if all the roots λ of the characteristic equation (2.2) satisfy $Re\lambda < 0$. Our goal in this section is to give the largest subset of the parameters α, β, and τ, respectively. All the roots of the characteristic equation (2.2) have negative real parts in corresponding largest subset. We shall refer this subset as to the stability region of the trivial fixed point.

Substituting $\lambda = \mu + i\omega$ into the left sides of both (2.3) and (2.4) and separating the real and imaginary parts, we can obtain

$$R_{(2.3)}(\mu, \omega) = \mu + 1 - (\alpha+\beta)e^{-\mu\tau}\cos(\omega\tau),$$

$$I_{(2.3)}(\mu, \omega) = \omega + (\alpha+\beta)e^{-\mu\tau}\sin(\omega\tau),$$

$$R_{(2.4)}(\mu,\omega) = \mu + 1 + \alpha e^{-\mu\tau}\cos\left(\omega\tau \pm \frac{\pi}{3}\right) + \beta e^{-\mu\tau}\cos\left(\omega\tau \mp \frac{\pi}{3}\right),$$

$$I_{(2.4)}(\mu,\omega) = \omega - \alpha e^{-\mu\tau}\sin\left(\omega\tau \pm \frac{\pi}{3}\right) - \beta e^{-\mu\tau}\sin\left(\omega\tau \mp \frac{\pi}{3}\right).$$

It follows that if $\lambda = \mu + i\omega$ is a solution to (2.3) then

$$\mu = -1 + (\alpha+\beta)e^{-\mu\tau}\cos(\omega\tau), \tag{2.5}$$

$$\omega = -(\alpha+\beta)e^{-\mu\tau}\sin(\omega\tau), \tag{2.6}$$

and if $\lambda = \mu + i\omega$ is a solution to (2.4) then

$$\mu = -1 - \alpha e^{-\mu\tau}\cos\left(\omega\tau \pm \frac{\pi}{3}\right) - \beta e^{-\mu\tau}\cos\left(\omega\tau \mp \frac{\pi}{3}\right), \tag{2.7}$$

$$\omega = \alpha e^{-\mu\tau}\sin\left(\omega\tau \pm \frac{\pi}{3}\right) + \beta e^{-\mu\tau}\sin\left(\omega\tau \mp \frac{\pi}{3}\right). \tag{2.8}$$

Theorem 2.1. If $|\alpha+\beta| < 1$ and $\alpha\beta > 0$, then all the roots of the characteristic equation (2.2) have negative real parts.

Proof. Let $R(\mu) = \mu + 1 - |\alpha+\beta|e^{-\mu\tau}$. Obviously,

$$R_{2.3}(\mu,\omega) \geq R(\mu) \quad \text{and} \quad R_{2.4}(\mu,\omega) \geq R(\mu). \tag{2.9}$$

Since $|\alpha+\beta| < 1$ and $\alpha\beta > 0$, we have

$$R(0) = 1 - |\alpha+\beta| > 0.$$

Since $\frac{\mathrm{d}R(\mu)}{\mathrm{d}\mu} = 1 + |\alpha+\beta|\tau e^{-\mu\tau} > 0$, it implies that $R(\mu) > 0$ for $\mu > 0$. Therefore, it follows from (2.9) that

$$R_{(2.3)}(\mu,\omega) > 0 \quad \text{and} \quad R_{(2.4)}(\mu,\omega) > 0 \quad for \quad \mu > 0 \quad \text{and} \quad \omega \in R. \tag{2.10}$$

Assume that $\lambda = \mu + i\omega$ is a solution to (2.2). By (2.10), we only need to show that $\lambda \neq i\omega$. If $|\alpha+\beta| < 1$ and $\alpha\beta > 0$, this is true from $R(0) = 1 - |\alpha+\beta| > 0$ and (2.9). This completes the proof. □

Theorem 2.1 presents a delay-independent sufficient condition for the linear stability of the trivial solution. In other words, under the condition that If $|\alpha+\beta| < 1$ and $\alpha\beta > 0$, the delay τ is harmless to (1.1). In the following, we will give some delay-dependent conditions about the linear stability of the trivial solution of (1.1).

Theorem 2.2. Assume $\sqrt{2}\beta > -1$, $0 \leq \tau \leq -\frac{1}{2\alpha}$, $-2 < \alpha < \beta < 0$. Then all the roots of the characteristic equation (2.2) have negative real parts .

Proof. Let $\lambda = \mu + i\omega$ be a root of (2.2). Since the roots of (2.2) appear in complex conjugate pairs, without loss of generality, we can assume that $\omega \geq 0$. By way of contradiction, we assume that $\mu \geq 0$. We will finish the proof in the following two cases.

First, we assume that μ and ω satisfy (2.5) and (2.6) simultaneously. It follows from (2.6) that $\omega \leq -\alpha - \beta$ and hence $0 \leq \omega\tau \leq 1 < \frac{\pi}{3}$. It also follows from (2.5) and (2.6) that

$$M_1(\mu, \omega) = 0, \tag{2.11}$$

where

$$M_1(\mu,\omega) = (\mu+1)^2 + \omega^2 - 2\alpha(\mu+1)e^{-\mu\tau}\cos(\omega\tau) + 2\alpha\omega e^{-\mu\tau}\sin(\omega\tau) + \alpha^2 e^{-2\mu\tau} - \beta^2 e^{-2\mu\tau}.$$

For the fixed ω, we have

$$\begin{aligned} M_1(0,\omega) &= 1 + \omega^2 - 2\alpha\cos(\omega\tau) + 2\alpha\omega\sin(\omega\tau) + \alpha^2 - \beta^2 \\ &\geq 1 - 2\alpha\cos(\omega\tau) + \omega^2 + 2\alpha\omega(\omega\tau) + \alpha^2 - \beta^2 \quad (\text{because}\sin(\omega\tau) \leq \omega\tau) \\ &= 1 - 2\alpha\cos(\omega\tau) + \omega^2(1 + 2\alpha\tau) + \alpha^2 - \beta^2 \\ &> 0 \quad (\text{because} -1 \leq 2\alpha\tau \leq 0). \end{aligned}$$

Taking the partial derivative of $M_1(\mu,\omega)$ with respect to μ, we have

$$\begin{aligned} &\frac{\partial M_1(\mu,\omega)}{\partial\mu} \\ &= 2(\mu+1) - 2\alpha e^{-\mu\tau}\cos(\omega\tau) + 2\alpha\tau(\mu+1)e^{-\mu\tau}\cos(\omega\tau) - 2\alpha\omega\tau e^{-\mu\tau}\sin(\omega\tau) \\ &\quad -2\alpha^2\tau e^{-2\mu\tau} + 2\beta^2\tau e^{-2\mu\tau} \\ &= 2(\mu+1)[1 + \alpha\tau e^{-\mu\tau}\cos(\omega\tau)] - 2\alpha e^{-\mu\tau}[\cos(\omega\tau) + \alpha\tau e^{-\mu\tau}] - 2\alpha\omega\tau e^{-\mu\tau}\sin(\omega\tau) \\ &\quad +2\beta^2\tau e^{-2\mu\tau}. \end{aligned}$$

Noticing that $\alpha < 0$ for $0 \leq \tau \leq -\frac{1}{2\alpha}$. Then the last two terms $-2\alpha\omega\tau e^{-\mu\tau}\sin(\omega\tau)$ and $2\beta^2\tau e^{-2\mu\tau}$ are nonnegative.

On the other hand, due to

$$0 \leq \tau \leq -\frac{1}{2\alpha} \quad \text{and} \quad \frac{1}{2} = \cos\frac{\pi}{3} < \cos 1 \leq \cos(\omega\tau) \leq 1,$$

we have

$$(\mu+1)[1 + \alpha\tau e^{-\mu\tau}\cos(\omega\tau)] \geq (\mu+1)\Big(1 - \frac{1}{2}\Big) \geq 0$$

and

$$-\alpha e^{-\mu\tau}[\cos(\omega\tau) + \alpha\tau e^{-\mu\tau}] \geq -\alpha e^{-\mu\tau}\Big(\cos 1 - \frac{1}{2}\Big) > 0.$$

So $\frac{\partial M_1(\mu,\omega)}{\partial\mu} > 0$. This, combined with $M_1(0,\omega) > 0$, implies that $M_1(\mu,\omega) > 0$ for $\mu \geq 0$, which contradicts (2.11).

Now, we assume that μ and ω simultaneously satisfy (2.7) and (2.8). We first assume that

$$\mu = -1 - \alpha e^{-\mu\tau}\cos\Big(\omega\tau + \frac{\pi}{3}\Big) - \beta e^{-\mu\tau}\cos\Big(\omega\tau - \frac{\pi}{3}\Big) \tag{2.12}$$

$$\omega = \alpha e^{-\mu\tau} \sin\Big(\omega\tau + \frac{\pi}{3}\Big) + \beta e^{-\mu\tau} \sin\Big(\omega\tau - \frac{\pi}{3}\Big). \tag{2.13}$$

It follows from (2.13) easily that $\omega \leq -\alpha - \beta$ and hence $0 \leq \omega\tau \leq 1 < \frac{\pi}{3}$. Then

$$\frac{\pi}{3} \leq \omega\tau + \frac{\pi}{3} < \frac{2\pi}{3} \quad \text{and} \quad -\frac{\pi}{3} \leq \omega\tau - \frac{\pi}{3} < 0.$$

Thus

$$\begin{aligned} \omega &= \alpha e^{-\mu\tau} \sin\Big(\omega\tau + \tfrac{\pi}{3}\Big) + \beta e^{-\mu\tau} \sin\Big(\omega\tau - \tfrac{\pi}{3}\Big) \\ &< \beta e^{-\mu\tau} \sin\Big(\omega\tau - \tfrac{\pi}{3}\Big) \\ &< -\beta. \end{aligned}$$

This, combined with the assumptions $0 \leq \tau \leq -\frac{1}{2\alpha}$, gives us

$$0 \leq \omega\tau < \frac{1}{2} < \frac{\pi}{6}.$$

Then we can obtain

$$\frac{\pi}{3} \leq \omega\tau + \frac{\pi}{3} < \frac{\pi}{2} \quad , \quad -\frac{\pi}{3} \leq \omega\tau - \frac{\pi}{3} < -\frac{\pi}{6}.$$

Then

$$\begin{aligned} \alpha e^{-\mu\tau} \sin\Big(\omega\tau + \tfrac{\pi}{3}\Big) + \beta e^{-\mu\tau} \sin\Big(\omega\tau - \tfrac{\pi}{3}\Big) &< \alpha e^{-\mu\tau} \sin \tfrac{\pi}{3} + \beta e^{-\mu\tau} \sin\Big(-\tfrac{\pi}{3}\Big) \\ &= \tfrac{\sqrt{3}}{2} e^{-\mu\tau}(\alpha - \beta) \\ &< 0. \end{aligned}$$

This contradicts the fact that $\omega \geq 0$. Finally, we assume that

$$\mu = -1 - \alpha e^{-\mu\tau} \cos\Big(\omega\tau - \frac{\pi}{3}\Big) - \beta e^{-\mu\tau} \cos\Big(\omega\tau + \frac{\pi}{3}\Big), \tag{2.14}$$

$$\omega = \alpha e^{-\mu\tau} \sin\Big(\omega\tau - \frac{\pi}{3}\Big) + \beta e^{-\mu\tau} \sin\Big(\omega\tau + \frac{\pi}{3}\Big). \tag{2.15}$$

Then it follows from (2.14) and (2.15) that

$$M_2(\mu, \omega) = 0, \tag{2.16}$$

where

$$\begin{aligned} M_2(\mu, \omega) &= (\mu+1)^2 + \omega^2 - \alpha^2 e^{-2\mu\tau} - \beta^2 e^{-2\mu\tau} - 2\alpha\beta e^{-2\mu\tau} \cos\Big(-\tfrac{2\pi}{3}\Big) \\ &= (\mu+1)^2 + \omega^2 - \alpha^2 e^{-2\mu\tau} - \beta^2 e^{-2\mu\tau} + \alpha\beta e^{-2\mu\tau} \end{aligned}$$

and

$$M_2(0, \omega) = 1 + \omega^2 - \alpha^2 - \beta^2 + \alpha\beta > 0.$$

Herein we have used the condition $\alpha > -2$. Again, taking the partial derivative of $M_2(\mu, \omega)$ with respect to μ, we have

$$\begin{aligned} \frac{\partial M_2(\mu, \omega)}{\partial \mu} &= 2(\mu+1) + 2\alpha^2 \tau e^{-2\mu\tau} + 2\beta^2 \tau e^{-2\mu\tau} - 2\alpha\beta\tau e^{-2\mu\tau} \\ &= 2[\mu + 1 + \alpha^2 \tau e^{-2\mu\tau} + \beta^2 \tau e^{-2\mu\tau} - \alpha\beta\tau e^{-2\mu\tau}]. \end{aligned}$$

Noticing

$$-1+\alpha\beta\tau e^{-2\mu\tau} \le -1+\alpha\beta\Big(-\frac{1}{2\alpha}\Big) < -1+\alpha\beta\frac{1}{-\alpha-\beta} < -1-\frac{\alpha}{2}$$

and $\alpha > -2$, we have

$$\frac{\partial M_2(\mu,\omega)}{\partial\mu} > 0.$$

This, combined with $M_2(0,\omega) > 0$, implies that $M_2(\mu,\omega) > 0$ for $\mu \ge 0$, which is contradictory to (2.16). This completes the proof. □

In the remaining of this section, we will give two results for the unstability of the trivial solution. One is delay-independent, and the other is delay-dependent.

Theorem 2.3. Assume that $\alpha+\beta > 1$. Then the characteristic equation (2.2) has a root with positive real part for any τ.

Proof. Under the assumption, we have that

$$\chi_1(0) = 1-\alpha-\beta < 0.$$

It is easy to check

$$\lim_{\lambda\to+\infty}\chi_1(\lambda) = \lim_{\lambda\to+\infty}[\lambda+1-(\alpha+\beta)e^{-\lambda\tau}] = +\infty.$$

It follows from the mean value theorem that there exists a $\lambda^* > 0$ such that $\chi_1(\lambda^*) = 0$. So (2.2) has a positive real root. This completes the proof. □

Theorem 2.4. Assume that $\tau > 1$, $\ln\tau + \tau - (\alpha+\beta) < 0$. Then the characteristic equation (2.2) has a root with positive real part.

Proof. Under the assumption, we can obtain that

$$\chi_1\Big(\frac{\ln\tau}{\tau}\Big) = \frac{\ln\tau}{\tau} + 1 - (\alpha+\beta)e^{-\tau\frac{\ln\tau}{\tau}} = \frac{\ln\tau}{\tau} + 1 - (\alpha+\beta)\frac{1}{\tau} < 0 \quad \Big(\frac{\ln\tau}{\tau} > 0\Big),$$

$$\lim_{\lambda\to+\infty}\chi_1(\lambda) = \lim_{\lambda\to+\infty}[\lambda+1-(\alpha+\beta)e^{-\lambda\tau}] = +\infty.$$

Then it is not difficult to check that there exists $\lambda^* > 0$ such that $\chi_1(\lambda^*) = 0$. So (2.2) has a positive real root. This completes the proof. □

3. THE ABSOLUTE SYNCHRONIZATION OF THE SYSTEM

In this section, we will study the absolute synchronization of the delayed network system (1.1). Throughout this section, we need to assume: f satisfies the following conditions:

(1) $f'(x) > 0$, $\forall x \in \mathbb{R}$;

(2) $f''(x)x < 0$, $\forall x \ne 0$;

(3) $f'''(0) < 0$;

(4) $-\infty < \lim_{x\to\pm\infty} f(x) < +\infty$.

Definition 3.1. The solution x^φ of the system (1.1) is asymptotic synchronous, if its one $\omega-$ finite set is in the following synchronous image set:

$$\{\psi = (\psi_1, \cdots, \psi_3)^T \in C([-\tau, 0], \mathbb{R}^3) : \psi_1 = \psi_2, \psi_2 = \psi_3\}.$$

If all the solutions of system (1.1) are asymptotic synchronous, then the system (1.1) is absolute synchronous, i. e. the following conditions hold for all nonnegative τ:

$$\lim_{t\to\infty} |x(t) - y(t)| = 0, \lim_{t\to\infty} |y(t) - z(t)| = 0, \lim_{t\to\infty} |z(t) - x(t)| = 0.$$

Theorem 3.1. If $\alpha + \beta < 1$, then all the solutions of system (1.1) are asymptotic synchronous for all the nonnegative τ.

P r o o f. We consider the given solution $x : [-\tau, \infty) \to \mathbb{R}^3$ of system (1.1). Set

$$m(t) = x(t) - y(t), n(t) = y(t) - z(t), l(t) = z(t) - x(t).$$

Exploiting the system (1.1), we can obtain for $t \geq 0$,

$$\begin{aligned}
m'(t) &= -m(t) + \alpha[f(y(t-\tau)) - f(z(t-\tau))] + \beta[f(z(t-\tau)) - f(x(t-\tau))] \\
&= -m(t) + \alpha p_1(t)n(t-\tau) + \beta p_2(t)l(t-\tau), \\
n'(t) &= -n(t) + \alpha[f(z(t-\tau)) - f(x(t-\tau))] + \beta[f(x(t-\tau)) - f(y(t-\tau))] \\
&= -n(t) + \alpha p_2(t)l(t-\tau) + \beta p_3(t)m(t-\tau), \\
l'(t) &= -l(t) + \alpha[f(x(t-\tau)) - f(y(t-\tau))] + \beta[f(y(t-\tau)) - f(z(t-\tau))] \\
&= -l(t) + \alpha p_3(t)m(t-\tau) + \beta p_1(t)n(t-\tau),
\end{aligned}$$

where

$$p_1(t) = \int_0^1 f'(sy(t-\tau) + (1-s)z(t-\tau))\,\mathrm{d}s,$$

$$p_2(t) = \int_0^1 f'(sz(t-\tau) + (1-s)x(t-\tau))\,\mathrm{d}s,$$

$$p_3(t) = \int_0^1 f'(sx(t-\tau) + (1-s)y(t-\tau))\,\mathrm{d}s.$$

Using the property of f, there exists $p^* \in (0, 1]$ such that $p_{1,2,3}(t) \leq p^*$ for all $t \geq 0$.

We consider the Lyapunov function candidate:

$$V(t) = m^2(t) + n^2(t) + l^2(t) + (\alpha+\beta)p^* \left(\int_{t-\tau}^t m^2(s)\,\mathrm{d}s + \int_{t-\tau}^t n^2(s)\,\mathrm{d}s + \int_{t-\tau}^t l^2(s)\,\mathrm{d}s \right).$$

Thus the differential coefficient of V is described by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t) &= 2m(t)(-m(t) + \alpha p_1(t)n(t-\tau) + \beta p_2(t)l(t-\tau)) \\
&\quad +2n(t)(-n(t) + \alpha p_2(t)l(t-\tau) + \beta p_3(t)m(t-\tau)) \\
&\quad +2l(t)(-l(t) + \alpha p_3(t)m(t-\tau) + \beta p_1(t)n(t-\tau)) \\
&\quad +(\alpha+\beta)p^*(n^2(t) - n^2(t-\tau) + m^2(t) - m^2(t-\tau) + l^2(t) - l^2(t-\tau)) \\
&\leq -2(m^2(t) + n^2(t) + l^2(t)) + 2(\alpha+\beta)p^*(m^2(t) + n^2(t) + l^2(t)) \\
&\leq -2(1-\alpha-\beta)(m^2(t) + n^2(t) + l^2(t)) < 0.
\end{aligned}$$

Therefore, we have

$$\lim_{t\to+\infty} m(t) = \lim_{t\to+\infty} n(t) = \lim_{t\to+\infty} l(t) = 0.$$

$\square$

4. NUMERICAL SIMULATION EXAMPLE

In this section, some numerical results of simulating system (1.1) are presented at different data of α, β, τ. In the simulations, we will find that the theoretically predicted results are in excellent agreement with the numerically observed behavior.

Example 4.1. We consider the system as follows:

$$\begin{cases} \dot{x} &= -x + \alpha\tanh(y(t-\tau)) + \beta\tanh(z(t-\tau)), \\ \dot{y} &= -y + \alpha\tanh(z(t-\tau)) + \beta\tanh(x(t-\tau)), \\ \dot{z} &= -z + \alpha\tanh(x(t-\tau)) + \beta\tanh(y(t-\tau)). \end{cases} \tag{4.1}$$

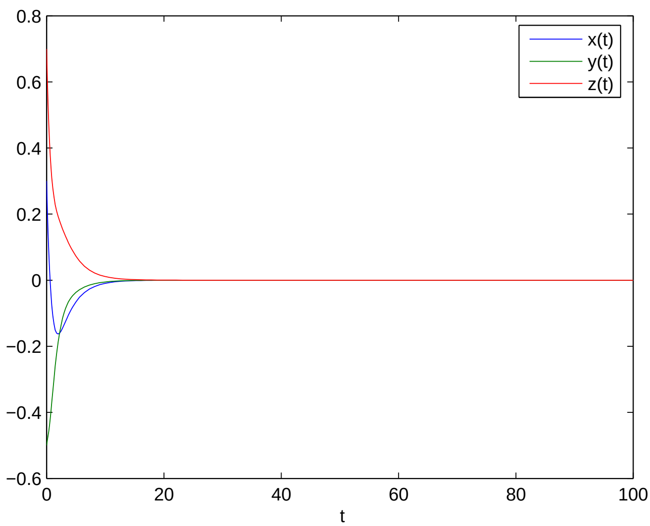

Fig. 2. $\alpha = -0.5$, $\beta = -0.25$, $\tau = 0.25$.

Choosing $\alpha = -0.5$, $\beta = -0.25$, $\tau = 0.25$, by Theorem 2.1, we know that the equilibrium $(0, 0, 0)$ is asymptotically stable, which is shown in Figure 2. Choosing $\alpha = -0.75$, $\beta = -0.5$, $\tau = 0.5$, by Theorem 2.2, we can find that the equilibrium $(0, 0, 0)$ is asymptotically stable, which is shown in Figure 3. Choosing $\alpha = 1$, $\beta = 2$, $\tau = 0.5$, by Theorem 2.3, we know that the equilibrium $(0, 0, 0)$ is not stable, which is shown in Figure 4. Choosing $\alpha = 3$, $\beta = 2$, $\tau = 2$, by Theorem 2.4, we can find that the

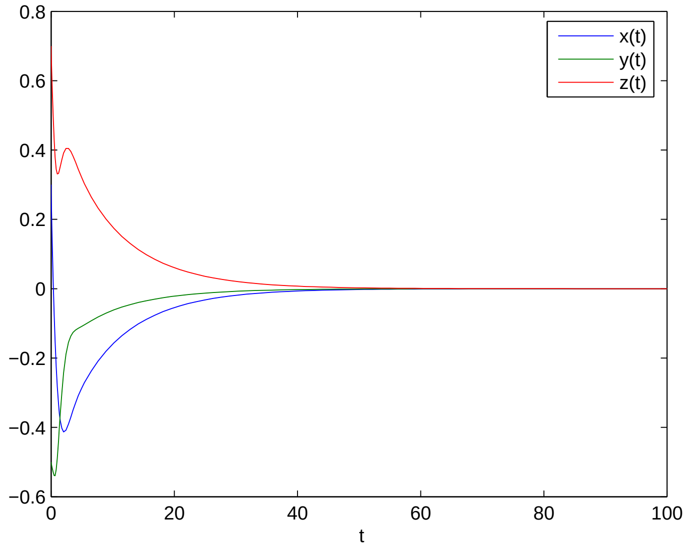

Fig. 3. $\alpha = -0.75$, $\beta = -0.5$, $\tau = 0.5$.

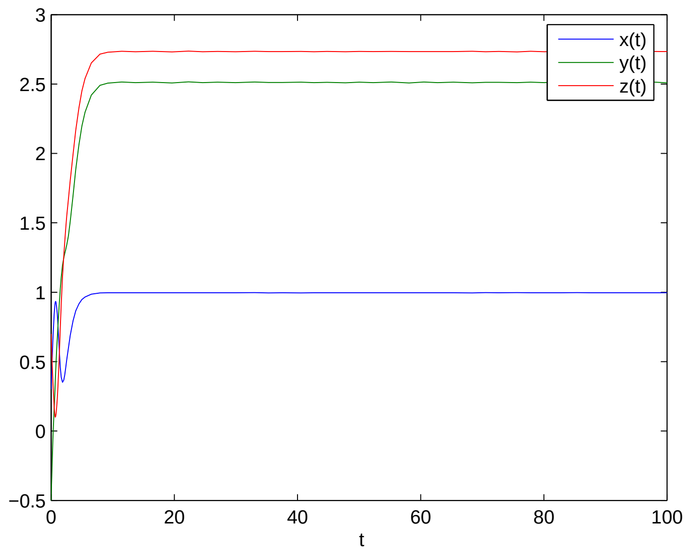

Fig. 4. $\alpha = 1$, $\beta = 2$, $\tau = 0.5$.

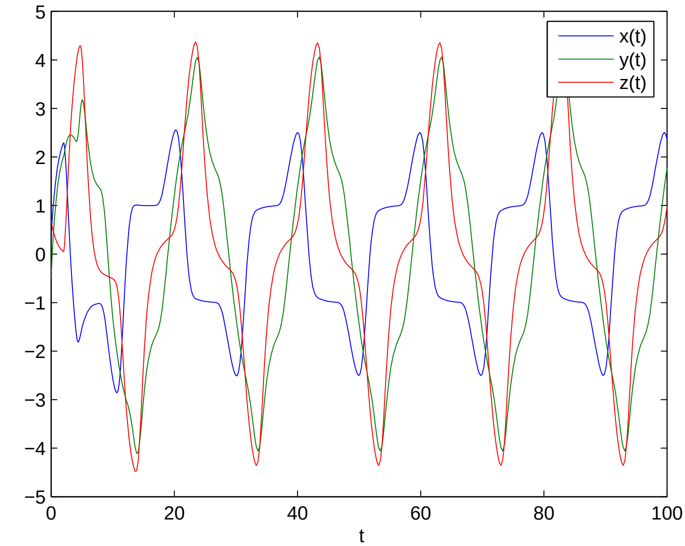

Fig. 5. $\alpha = 3$, $\beta = 2$, $\tau = 2$.

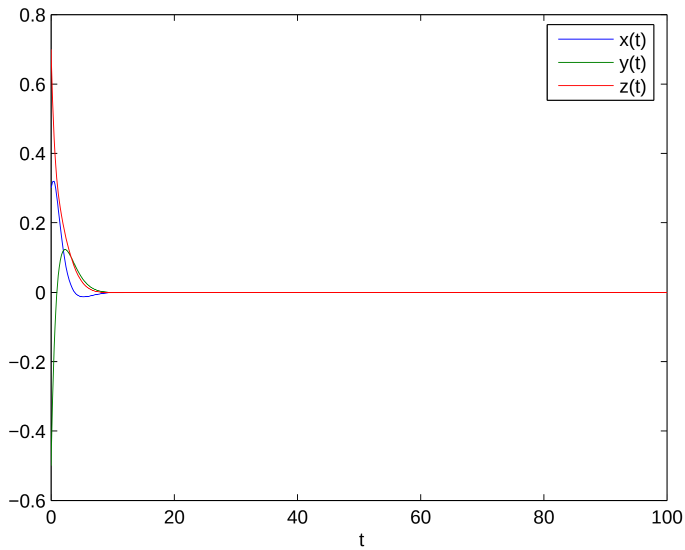

Fig. 6. $\alpha = 0.5$,$\beta = 0.25$,$\tau = 0.25$.

equilibrium $(0,0,0)$ is not stable, which is shown in Figure 5. Choosing $\alpha = 0.5$, $\beta = 0.25$, $\tau = 0.25$, by Theorem 3.1, we have that the system (1.1) is absolute synchronous, i. e. the following conditions hold for all nonnegative τ, which is shown in Figure 6. These numerical simulations are in accordance with the theoretical results of this paper.

5. CONCLUSIONS

We investigate the linear stability of trivial solution of three-unit delayed neural network and obtain the sufficient condition of absolute synchronization of the system in this paper. Meanwhile, the performances of numerical simulations have demonstrated the correctness of the theoretical results.

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