SYNCHRONIZATION OF TWO COUPLED HINDMARSH–ROSE NEURONS

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This paper is concerned with synchronization of two coupled Hind-marsh-Rose (HR) neurons. Two synchronization criteria are derived by using nonlinear feedback control and linear feedback control, respectively. A synchronization criterion for FitzHugh–Nagumo (FHN) neurons is derived as the application of control method of this paper. Compared with some existing synchronization results for chaotic systems, the contribution of this paper is that feedback gains are only dependent on system parameters, rather than dependent on the norm bounds of state variables of uncontrolled and controlled HR neurons. The effectiveness of our results are demonstrated by two simulation examples.

Keywords: coupled neurons, Hindmarsh–Rose neurons, synchronization, feedback control

Classification: 34D06

1. INTRODUCTION

Since Pecora and Carroll (1990) [19] studied synchronization of chaotic systems, chaotic synchronization has been widely used in the secure communication, oscillator networks and neural networks during the last 20 years, see for example, [2, 3, 4, 11, 12, 13, 14, 15, 16, 17, 20, 22, 23, 24, 25, 27, 28, 29, 30].

Some models for bursting behaviors, repetitive and patterned activities of molluscan neurones can be mathematically represented as the Hindmarsh–Rose (HR) model [1, 5, 6, 7, 27]. Synchronization of coupled HR neurons has played a key role in the neuronal information processing and communication within the brain area [2, 7, 21, 26, 27]. Some papers have studied how HR neurons can achieve synchronization. Hrg (2013) [8] investigated synchronization of two HR neurons, but the couplings between two neurons were unidirectional. Kuntanapreeda [10] studied synchronization of two unified chaotic systems in which the control method can be applied to studying the master-slave synchronization of HR neurons. Nguyen and Hong [18] investigated synchronization of chaotic FitzHugh–Nagumo neurons which were the special cases of HR neurons. In [10, 18], the norm bounds of state variables of both controlled and uncontrolled chaotic systems were used to derive synchronization criteria. However, it is difficult or impossible to estimate the norm bounds of state variables of two coupled HR neurons (controlled

DOI: 10.14736/kyb-2015-5-0784

and uncontrolled HR neurons). Thus, how to avoid to use the norm bounds of state variables of controlled and uncontrolled neurons to achieve global synchronization for two coupled HR neurons is the motivation of this paper.

In this paper, we will use the nonlinear feedback control and linear feedback control to achieve synchronization for two coupled HR neurons, respectively. The main contribution of this paper is that the feedback gains are dependent on the system parameters, rather than the norm bounds of state variables of uncontrolled and controlled HR neurons. As the application of the control method, a synchronization criterion is derived for FHN neurons. Two examples will be used to reveal the effectiveness of our synchronization results.

2. PROBLEM STATEMENT

Consider the following HR neuron model described by

$$\begin{cases} \dot{y}_1(t) = ay_1^2(t) - y_1^3(t) + y_2(t) - y_3(t) + I, \\ \dot{y}_2(t) = 1 + by_1^2(t) - y_2(t), \\ \dot{y}_3(t) = c(y_1(t) + 1.56) - 0.006y_3(t), \end{cases} \tag{1}$$

where $y_1(t)$, $y_2(t)$, $y_3(t)$ are the membrane potential, the recovery variable for the current of Na^+ or K^+ ions, and the adaptation current for the current of Ca^+ ions, respectively; I represents the external applied current for mimicking the membrane current of neurons; a, b, and c are system constants.

We consider the following two coupled HR neurons

$$\begin{cases} \dot{y}_1(t) = ay_1^2(t) - y_1^3(t) + y_2(t) - y_3(t) - p(y_1(t) - z_1(t)) + I, \\ \dot{y}_2(t) = 1 + by_1^2(t) - y_2(t), \\ \dot{y}_3(t) = c(y_1(t) + 1.56) - 0.006y_3(t), \end{cases} \tag{2}$$

and

$$\begin{cases} \dot{z}_1(t) = az_1^2(t) - z_1^3(t) + z_2(t) - z_3(t) - p(z_1(t) - y_1(t)) + I + u(t), \\ \dot{z}_2(t) = 1 + bz_1^2(t) - z_2(t), \\ \dot{z}_3(t) = c(z_1(t) + 1.56) - 0.006z_3(t), \end{cases} \tag{3}$$

where system (2) is the uncontrolled HR neuron, system (3) is the controlled HR neuron; p is the coupling strength; $u(t)$ is the control; the initial condition is $y_1(0) = y_{1_0}$, $y_2(0) = y_{2_0}$, $y_3(0) = y_{3_0}$, $z_1(0) = z_{1_0}$, $z_2(0) = z_{2_0}$, and $z_3(0) = z_{3_0}$.

Let

$$u(t) = k(y_1(t) - z_1(t)) - k_0(y_1^2(t) + z_1^2(t))(y_1(t) - z_1(t)), \tag{4}$$

where k and k_0 are gains which need to be determined.

We define three error variables $e_i(t) = y_i(t) - z_i(t)$, $i = 1, 2, 3$. From (2) and (4), we have the following error system

$$\begin{cases} \dot{e}_1(t) = ae_1(t)(y_1(t) + z_1(t)) - e_1(t)(y_1^2(t) + z_1^2(t) + y_1(t)z_1(t)) \\ \qquad\qquad + e_2(t) - e_3(t) - 2pe_1(t) - ke_1(t) + k_0(y_1^2(t) + z_1^2(t))e_1(t), \\ \dot{e}_2(t) = be_1(t)(y_1(t) + z_1(t)) - e_2(t), \\ \dot{e}_3(t) = ce_1(t) - 0.006e_3(t), \end{cases} \tag{5}$$

where the initial condition is $e_1(0) = y_{1_0} - z_{1_0}$, $e_2(0) = y_{2_0} - z_{2_0}$, and $e_3(0) = y_{3_0} - z_{3_0}$.

This paper intends to obtain the synchronization criteria for two coupled HR neuron (2), i. e., to find k_0 and k such that

$$\lim_{t\to\infty} \|y_i(t) - z_i(t)\| = 0, \ i = 1, 2, 3,$$

which also means that the error system described by (5) is globally asymptotically stable.

3. SYNCHRONIZATION CRITERIA

In this section, we provide two synchronization criteria for the system described by (2), (3) and (4), which also guarantees the stability of the error system (5).

Now we can give the following result.

Proposition 1. Two coupled HR neurons described by (2), (3) and (4) can achieve global synchronization, i. e., the error system described by (5) is globally asymptotically stable, if

$$\begin{cases} k_0 < \frac{-b^2+4-|b^2-2|}{4}, \\ k > \frac{(\frac{b}{2}+a)^2(8k_0^2-12k_0+2b^2k_0+4-b^2)}{(3-b^2+2b^2k_0+4k_0^2-8k_0)(4-b^2-4k_0)} + \frac{1}{4} + \frac{(c-1)^2}{0.024} - 2p. \end{cases} \tag{6}$$

P r o o f . We use the following Lyapunov function

$$V(t) = \frac{e_1^2(t) + e_2^2(t) + e_3^2(t)}{2}. \tag{7}$$

Taking the derivative of $V(t)$ with respect to t along the trajectory of (5) yields

$$\begin{aligned} \frac{dV(t)}{dt} &= -\left(\frac{(b(y_1(t) + z_1(t)) + 1)}{2}e_1(t) - e_2(t)\right)^2 \\ &\qquad - 0.006\left(\frac{c-1}{0.012}e_1(t) - e_3(t)\right)^2 + \theta(t)e_1^2(t) \end{aligned} \tag{8}$$

where

$$\begin{aligned} \theta(t) &= a(y_1(t) + z_1(t)) - (y_1^2(t) + z_1^2(t) + y_1(t)z_1(t)) \\ &\quad + \frac{(b(y_1(t) + z_1(t)) + 1)^2}{4} + \frac{(c-1)^2}{0.024} - 2p - k + k_0(y_1^2(t) + z_1^2(t)). \end{aligned} \tag{9}$$

By virtue of equations described by (8) and (9), we have $\theta(t) < 0$ which can ensure $\frac{dV(t)}{dt} < 0$. Thus, we need to find k_0 and k such that $\theta(t) < 0$.

Rewriting the right-side of the equation (9) as a function of $z_1(t)$, we have

$$\begin{aligned}\theta(t) &= \left(\frac{b^2}{4} - 1 + k_0\right) z_1^2(t) + \left(a + \frac{b}{2} + \left(\frac{b^2}{2} - 1\right) y_1(t)\right) z_1(t) \\ &\quad + \left(\frac{b^2}{4} - 1 + k_0\right) y_1^2(t) + \left(a + \frac{b}{2}\right) y_1(t) - \tilde{k}_1, \end{aligned} \tag{10}$$

where $\tilde{k}_1 = 2p + k - \frac{1}{4} - \frac{(c-1)^2}{0.024}$.

If

$$\frac{b^2}{4} - 1 + k_0 < 0 \quad \text{and} \quad \tilde{\theta}(t) < 0 \tag{11}$$

where

$$\begin{aligned}\tilde{\theta}(t) &= \left(a + \frac{b}{2} + (\frac{b^2}{2} - 1\right) y_1(t))^2 - 4\left(\frac{b^2}{4} - 1 + k_0\right)\left(\left(\frac{b^2}{4} - 1 + k_0\right) y_1^2(t) \right. \\ &\quad \left. + \left(a + \frac{b}{2}\right) y_1(t) - \tilde{k}_1\right),\end{aligned}$$

then $\theta(t) < 0$.

Rearranging the right-side of $\tilde{\theta}(t)$ as a function of $y_1(t)$, we have

$$\begin{aligned}\tilde{\theta}(t) &= (-3 + b^2 - 4k_0^2 - 2b^2 k_0 + 8k_0) y_1^2(t) + 2\left(\frac{b}{2} + a\right)(1 - 2k_0) y_1(t) \\ &\quad + \left(\frac{b}{2} + a\right)^2 - 4\left(1 - \frac{b^2}{4} - k_0\right)\tilde{k}_1.\end{aligned} \tag{12}$$

If

$$-3 + b^2 - 4k_0^2 - 2b^2 k_0 + 8k_0 < 0 \quad \text{and} \quad \hat{\theta}(t) < 0 \tag{13}$$

where

$$\begin{aligned}\hat{\theta}(t) &= 4\left(\frac{b}{2} + a\right)^2 (1 - 2k_0)^2 - 4(-3 + b^2 - 4k_0^2 - 2b^2 k_0 + 8k_0)\left(\frac{b}{2} + a\right)^2 \\ &\quad -16(-3 + b^2 - 4k_0^2 - 2b^2 k_0 + 8k_0)\left(-1 + \frac{b^2}{4} + k_0\right)\tilde{k}_1,\end{aligned} \tag{14}$$

then $\tilde{\theta}(t) < 0$. From (13), (14), we obtain that

$$\tilde{k}_1 = 2p + k - \frac{1}{4} - \frac{(c-1)^2}{0.024} > \frac{(\frac{b}{2} + a)^2 (8k_0^2 - 12k_0 + 2b^2 k_0 + 4 - b^2)}{(3 - b^2 + 2b^2 k_0 + 4k_0^2 - 8k_0)(4 - b^2 - 4k_0)}$$

will ensure $\hat{\theta}(t) < 0$. Therefore,

$$k > \frac{(\frac{b}{2} + a)^2 (8k_0^2 - 12k_0 + 2b^2 k_0 + 4 - b^2)}{(3 - b^2 + 2b^2 k_0 + 4k_0^2 - 8k_0)(4 - b^2 - 4k_0)} + \frac{1}{4} + \frac{(c-1)^2}{0.024} - 2p \tag{15}$$

can guarantee $\theta(t) < 0$ and $k_0 < \frac{-b^2+4-|b^2-2|}{4}$ can guarantee $-3+b^2-4k_0^2-2b^2k_0+8k_0 < 0$ and $\frac{b^2}{4} - 1 + k_0 < 0$. It follows from (6) that

$$\frac{dV(t)}{dt} < 0 \tag{16}$$

for all $e_i(t) \neq 0,\ i = 1, 2, 3$. Moreover, it is clear that

$$\frac{dV(t)}{dt} = 0,\ for\ e_1(t) = e_2(t) = e_3(t) = 0. \tag{17}$$

By virtue of LaSalle Invariant principle, the solution of the error system described by (5), which starts from arbitrary initial value, will be convergent to the largest invariant set which is constrained in $\frac{dV(t)}{dt} = 0$ as $t \to \infty$ [9]. Thus, (16) and (17) can ensure that the error system (5) is globally asymptotically stable, which also means that two coupled HR neurons described by (2), (3) and (4) can achieve global synchronization. This completes the proof. □

Remark 1. In [10] and [18], trajectories of chaotic systems were assumed to be bounded which were used to derive the synchronization criteria. However, it is difficult or impossible to estimate the bound of trajectories of two coupled systems (2) and (3). Compared with synchronization criteria in [10] and [18], the norm bounds of trajectories of systems (2) and (3) are not used to derive synchronization criteria in Proposition 1, and gains k_0 and k are dependent on system constants a, b, c and p which is the main contribution of this paper.

If $k_0 = 0$, the control (4) reduces to the following linear feedback control

$$u(t) = k(y_1(t) - z_1(t)). \tag{18}$$

We consider the following two coupled HR neurons with linear feedback control (18)

$$\begin{cases} \dot{y}_1(t) = ay_1^2(t) - y_1^3(t) + y_2(t) - y_3(t) - p(y_1(t) - z_1(t)) + I, \\ \dot{y}_2(t) = 1 + by_1^2(t) - y_2(t), \\ \dot{y}_3(t) = c(y_1(t) + 1.56) - 0.006y_3(t), \end{cases} \tag{19}$$

and

$$\begin{cases} \dot{z}_1(t) = az_1^2(t) - z_1^3(t) + z_2(t) - z_3(t) - p(z_1(t) - y_1(t)) + I + k(y_1(t) - z_1(t)), \\ \dot{z}_2(t) = 1 + bz_1^2(t) - z_2(t), \\ \dot{z}_3(t) = c(z_1(t) + 1.56) - 0.006z_3(t), \end{cases} \tag{20}$$

where $y_i(t)$, $z_i(t)$, $i = 1, 2, 3$, a, b, c, p, and I are the same as those defined in (2); the initial condition is $y_1(0) = y_{1_0}$, $y_2(0) = y_{2_0}$, $y_3(0) = y_{3_0}$, $z_1(0) = z_{1_0}$, $z_2(0) = z_{2_0}$, and $z_3(0) = z_{3_0}$.

We define three error variables $e_i(t) = y_i(t) - z_i(t),\ i = 1, 2, 3$. From (19), we have the following error system

$$\begin{cases} \dot{e}_1(t) = ae_1(t)(y_1(t) + z_1(t)) - e_1(t)(y_1^2(t) + z_1^2(t) + y_1(t)z_1(t)) \\ \qquad\qquad + e_2(t) - e_3(t) - 2pe_1(t) - ke_1(t), \\ \dot{e}_2(t) = be_1(t)(y_1(t) + z_1(t)) - e_2(t), \\ \dot{e}_3(t) = ce_1(t) - 0.006e_3(t), \end{cases} \tag{21}$$

where the initial condition is $e_1(0) = y_{1_0} - z_{1_0}$, $e_2(0) = y_{2_0} - z_{2_0}$, and $e_3(0) = y_{3_0} - z_{3_0}$.

Using the Lyapunov function (7) and the similar proof method to Proposition 1, we have the following result.

Proposition 2. Two coupled HR neurons described by (19) and (20) can achieve global synchronization, i. e., the error system described by (21) is globally asymptotically stable, if

$$\begin{cases} 3 - b^2 > 0, \\ k > \frac{(\frac{b}{2} + a)^2}{(3 - b^2)} + \frac{1}{4} + \frac{(c-1)^2}{0.024} - 2p. \end{cases} \tag{22}$$

Remark 2. In Proposition 1, a synchronization criterion is derived by using the nonlinear feedback control (4). In Proposition 2, a synchronization criterion is obtained by using the linear feedback control (18). It should be pointed out that an additional constraint for $3 > b^2$ is required in Proposition 2. If $b^2 > 3$, Proposition 2 can not be used to derive the synchronization criterion. Proposition 1 can be used for any b such that $b^2 \geq 3$ or $b^2 < 3$.

Remark 3. In [24], some synchronization criteria for chaotic systems were derived by using the backstepping method, in which the final control $u(t)$ was obtained after several virtual controls were designed. The backstepping method can also be applied to achieving synchronization of HR neurons, in which the control $u(t)$ could be *nonlinear*. Using our control method, a *linear* control $u(t)$ can be derived by Proposition 2.

If $k_0 = 0$ and $k = 0$, we consider the following two coupled HR neurons

$$\begin{cases} \dot{y}_1(t) = ay_1^2(t) - y_1^3(t) + y_2(t) - y_3(t) - p(y_1(t) - z_1(t)) + I, \\ \dot{y}_2(t) = 1 + by_1^2(t) - y_2(t), \\ \dot{y}_3(t) = c(y_1(t) + 1.56) - 0.006y_3(t), \end{cases} \tag{23}$$

and

$$\begin{cases} \dot{z}_1(t) = az_1^2(t) - z_1^3(t) + z_2(t) - z_3(t) - p(z_1(t) - y_1(t)) + I, \\ \dot{z}_2(t) = 1 + bz_1^2(t) - z_2(t), \\ \dot{z}_3(t) = c(z_1(t) + 1.56) - 0.006z_3(t), \end{cases} \tag{24}$$

without control $u(t)$, where $y_i(t)$, $z_i(t)$, $i = 1, 2, 3$, a, b, c, p, and I are the same as those defined in (2); the initial condition is $y_1(0) = y_{1_0}$, $y_2(0) = y_{2_0}$, $y_3(0) = y_{3_0}$, $z_1(0) = z_{1_0}$, $z_2(0) = z_{2_0}$, and $z_3(0) = z_{3_0}$.

We define three error variables $e_i(t) = y_i(t) - z_i(t),\ i = 1, 2, 3$, which can lead to the following error system

$$\begin{cases} \dot{e}_1(t) = ae_1(t)(y_1(t) + z_1(t)) - e_1(t)(y_1^2(t) + z_1^2(t) + y_1(t)z_1(t)) \\ \qquad\qquad + e_2(t) - e_3(t) - 2pe_1(t), \\ \dot{e}_2(t) = be_1(t)(y_1(t) + z_1(t)) - e_2(t), \\ \dot{e}_3(t) = ce_1(t) - 0.006e_3(t), \end{cases} \tag{25}$$

where the initial condition is $e_1(0) = y_{1_0} - z_{1_0}$, $e_2(0) = y_{2_0} - z_{2_0}$, and $e_3(0) = y_{3_0} - z_{3_0}$.

From Proposition 2, we have the following result.

Corollary 1. Two coupled HR neurons described by (23) and (24) can achieve global synchronization, i. e., the error system described by (25) is globally asymptotically stable, if

$$\begin{cases} 3 - b^2 > 0, \\ p > \frac{(\frac{b}{2}+a)^2}{2(3-b^2)} + \frac{1}{8} + \frac{(c-1)^2}{0.048}. \end{cases} \tag{26}$$

Remark 4. The FitzHugh–Nagumo (FHN) model has been widely used to study the dynamical evolution of brain neurons. The mathematical model of FHN can be described as

$$\begin{cases} \dot{y}_1(t) = y_1(t)(y_1(t) - 1)(1 - ry_1(t)) - y_2(t) + \frac{l}{w}\cos(wt) + I, \\ \dot{y}_2(t) = dy_1(t) - vy_2(t), \end{cases} \tag{27}$$

where $y_1(t), y_2(t)$ are the state variables; r, l, w, d and v are system constants. Two coupled FHN neurons can be described as

$$\begin{cases} \dot{y}_1(t) = y_1(t)(y_1(t) - 1)(1 - ry_1(t)) - y_2(t) - p(y_1(t) - z_1(t)) + \frac{l}{w}\cos(wt) + I, \\ \dot{y}_2(t) = dy_1(t) - vy_2(t), \end{cases} \tag{28}$$

and

$$\begin{cases} \dot{z}_1(t) = z_1(t)(z_1(t)-1)(1-rz_1(t))-z_2(t)-p(z_1(t)-y_1(t))+\frac{l}{w}\cos(wt)+I+u(t), \\ \dot{z}_2(t) = dz_1(t) - vz_2(t), \end{cases} \tag{29}$$

where $z_1(t), z_2(t)$ are the state variables; p is the coupling strength; $u(t)$ is the control; the initial condition is $y_1(0) = y_{1_0}$, $y_2(0) = y_{2_0}$, $z_1(0) = z_{1_0}$, and $z_2(0) = z_{2_0}$. It should be pointed out that our control method for HR neurons is still valid for the FHN neurons. Choosing the control (4) and defining two error variables $e_i(t) = y_i(t) - z_i(t)$, $i = 1, 2$, we have the error system

$$\begin{cases} \dot{e}_1(t) = -re_1(t)(y_1^2(t) + z_1^2(t) + y_1(t)z_1(t)) + (r+1)(y_1(t) + z_1(t))e_1(t) \\ \qquad\qquad - e_1(t) - e_2(t) - 2pe_1(t) - ke_1(t) + k_0(y_1^2(t) + z_1^2(t))e_1(t), \\ \\ \dot{e}_2(t) = de_1(t) - ve_2(t), \end{cases} \tag{30}$$

where the initial condition is $e_1(0) = y_{1_0} - z_{1_0}$ and $e_2(0) = y_{2_0} - z_{2_0}$.

Using the Lyapunov function (7), we can obtain the following Corollary for FHN neurons.

Corollary 2. Two coupled FHN neurons described by (28) and (29) can achieve global synchronization, i. e., the error system described by (30) is globally asymptotically stable, if

$$\begin{cases} r - k_0 > 0, \\ \\ k > \frac{(d-1)^2}{4v} - 1 - 2p - \frac{(r+1)^2}{4(k_0-r)} - \frac{(r+1)^2(2k_0-r)}{4(2k_0-3r)(k_0-r)}. \end{cases} \tag{31}$$

4. EXAMPLES

Example 1. Consider the HR model

$$\begin{cases} \dot{y}_1(t) = 3y_1^2(t) - y_1^3(t) + y_2(t) - y_3(t) + 3.2, \\ \\ \dot{y}_2(t) = 1 - 5y_1^2(t) - y_2(t), \\ \\ \dot{y}_3(t) = 0.024(y_1(t) + 1.56) - 0.006y_3(t). \end{cases} \tag{32}$$

If we choose the initial condition of (32) as $y_{1_0} = 0.3$, $y_{2_0} = 0.3$, and $y_{3_0} = 3$, there is a chaotic attractor which can be demonstrated by Figure 1.

Let $p = 0.1$. Consider the following two coupled HR neurons

$$\begin{cases} \dot{y}_1(t) = 3y_1^2(t) - y_1^3(t) + y_2(t) - y_3(t) - 0.1(y_1(t) - z_1(t)) + 3.2, \\ \\ \dot{y}_2(t) = 1 - 5y_1^2(t) - y_2(t), \\ \\ \dot{y}_3(t) = 0.024(y_1(t) + 1.56) - 0.006y_3(t), \end{cases} \tag{33}$$

and

$$\begin{cases} \dot{z}_1(t) = 3z_1^2(t) - z_1^3(t) + z_2(t) - z_3(t) - 0.1(z_1(t) - y_1(t)) + 3.2 + u(t), \\ \\ \dot{z}_2(t) = 1 - 5z_1^2(t) - z_2(t), \\ \\ \dot{z}_3(t) = 0.024(z_1(t) + 1.56) - 0.006z_3(t), \end{cases} \tag{34}$$

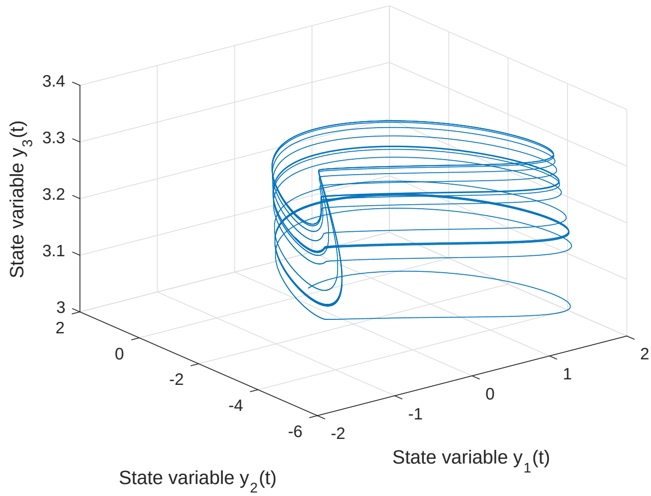

Fig. 1. The phase figure of the HR neuron with $a = 3$, $b = -5$, and $c = 0.024$.

where the control $u(t)$ is defined by (4). The error system is

$$\begin{cases} \dot{e}_1(t) = 3e_1(t)(y_1(t) + z_1(t)) - e_1(t)(y_1^2(t) + z_1^2(t) + y_1(t)z_1(t)) \\ \qquad\qquad + e_2(t) - e_3(t) - 0.2e_1(t) - ke_1(t) + k_0(y_1^2(t) + z_1^2(t))e_1(t), \\ \dot{e}_2(t) = -5e_1(t)(y_1(t) + z_1(t)) - e_2(t), \\ \dot{e}_3(t) = 0.024e_1(t) - 0.006e_3(t), \end{cases} \tag{35}$$

Since $3 < b^2$, Proposition 2 fails to derive the synchronization criterion. By using Proposition 1 and the control (4), we have

$$\begin{cases} k_0 < \frac{-b^2+4-|b^2-2|}{4} = -11, \\ k > \frac{(\frac{b}{2}+a)^2(8k_0^2-12k_0+2b^2k_0+4-b^2)}{(3-b^2+2b^2k_0+4k_0^2-8k_0)(4-b^2-4k_0)} + \frac{1}{4} + \frac{(c-1)^2}{0.024} - 2p = 39.75. \end{cases}$$

Choosing $k_0 = -11.1$ and $k = 40$, the control (4) is

$$u(t) = 40e_1(t) + 11.1(z_1^2(t) + y_1^2(t))e_1(t).$$

We give Figures 2, 3, and Figure 4 for variables $y_i(t)$, $z_i(t)$, and $e_i(t)$, $i = 1, 2, 3$, respectively, where the initial condition is $y_{1_0} = 0.3$, $y_{2_0} = 0.3$, $y_{3_0} = 3$, $z_{1_0} = 1.3$, $z_{2_0} = 1.3$, and $z_{3_0} = 2$. Figure 4 indicates that error system (5) is globally asymptotically stable, i. e., two coupled HR neurons described by (33) and (34) can achieve global synchronization.

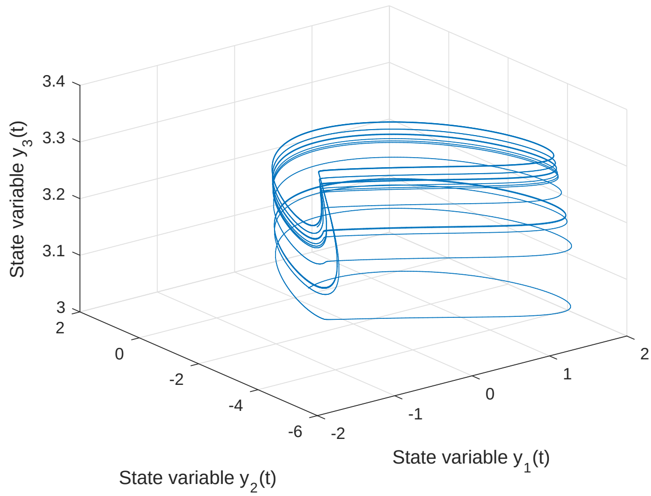

Fig. 2. The simulation result for state variables $y_1(t),\ y_2(t),\ y_3(t)$.

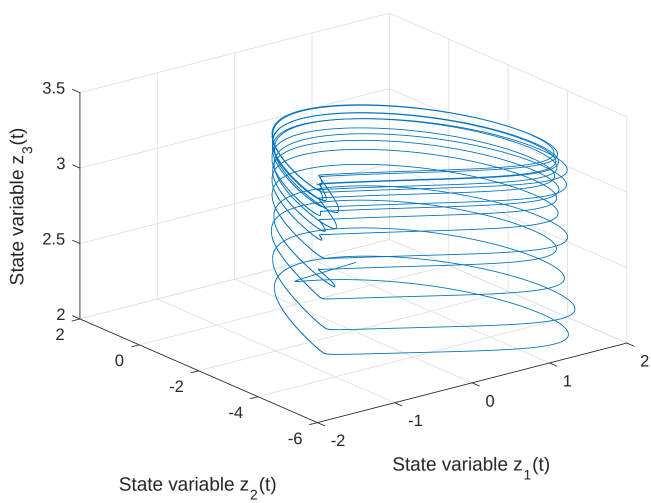

Fig. 3. The simulation result for state variables $z_1(t),\ z_2(t),\ z_3(t)$ with $k_0 = -11.1,\ k = 40$.

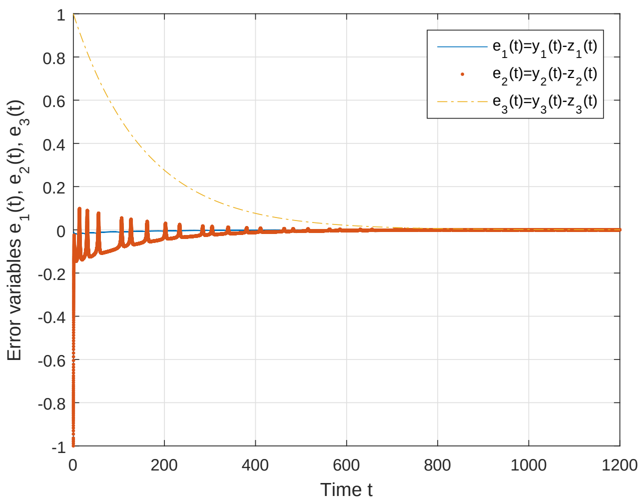

Fig. 4. The simulation result for error variables $e_1(t)$, $e_2(t)$, $e_3(t)$ with $k_0 = -11.1$, $k = 40$.

Example 2. Consider the HR model (1)

$$\begin{cases} \dot{y}_1(t) = -y_1^2(t) - y_1^3(t) + y_2(t) - y_3(t) + 3.2, \\ \dot{y}_2(t) = 1 - 1.5y_1^2(t) - y_2(t), \\ \dot{y}_3(t) = 0.76(y_1(t) + 1.56) - 0.006y_3(t). \end{cases} \tag{36}$$

If we choose the initial condition of (1) as $y_{1_0} = 1.3$ and $y_{2_0} = 1.3$, $y_{3_0} = 4$, a chaotic attractor can be demonstrated by Figure 5.

Let $p = 0.1$. Now we consider the coupled HR neurons

$$\begin{cases} \dot{y}_1(t) = -y_1^2(t) - y_1^3(t) + y_2(t) - y_3(t) - 0.1(y_1(t) - z_1(t)) + 3.2, \\ \dot{y}_2(t) = 1 - 1.5y_1^2(t) - y_2(t), \\ \dot{y}_3(t) = 0.76(y_1(t) + 1.56) - 0.006y_3(t), \end{cases} \tag{37}$$

and

$$\begin{cases} \dot{z}_1(t) = -z_1^2(t) - z_1^3(t) + z_2(t) - z_3(t) - 0.1(z_1(t) - y_1(t)) + 3.2 + u(t), \\ \dot{z}_2(t) = 1 - 1.5z_1^2(t) - z_2(t), \\ \dot{z}_3(t) = 0.76(z_1(t) + 1.56) - 0.006z_3(t). \end{cases} \tag{38}$$

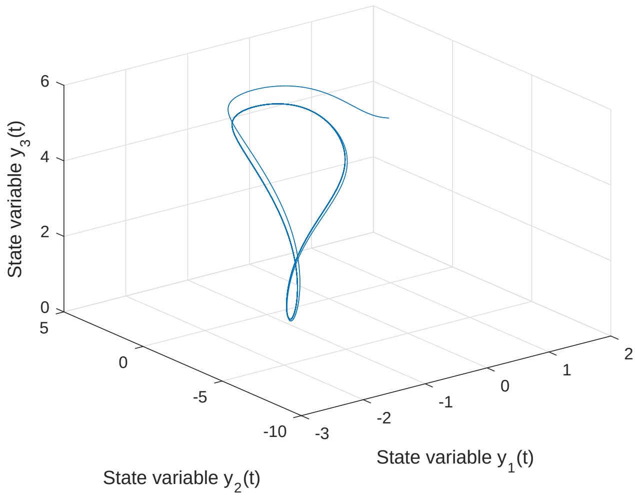

Fig. 5. The phase figure of the HR neuron with $a = -1$, $b = -1.5$, $c = 0.76$.

where the initial condition is $y_{1_0} = 1.3$, $y_{2_0} = 1.3$, $y_{3_0} = 4$, $z_{1_0} = 2.3$, $z_{2_0} = 2.3$, and $z_{3_0} = 2$. The control $u(t)$ is defined by (18), in which gain k can be determined later. The error system can be obtained as following

$$\begin{cases} \dot{e}_1(t) = -e_1(t)(y_1(t) + z_1(t)) - e_1(t)(y_1^2(t) + z_1^2(t) + y_1(t)z_1(t)) \\ \qquad\qquad + e_2(t) - e_3(t) - 2pe_1(t) - ke_1(t), \\ \dot{e}_2(t) = -1.5e_1(t)(y_1(t) + z_1(t)) - e_2(t), \\ \dot{e}_3(t) = 0.76e_1(t) - 0.006e_3(t). \end{cases} \tag{39}$$

Since $3 > b^2$, we can use Proposition 2 to obtain $k > \frac{(\frac{b}{2}+a)^2}{(3-b^2)} + \frac{1}{4} + \frac{(c-1)^2}{0.024} - 2p = 6.5333$. We choose $k = 6.6$. The control is $u(t) = 6.6e_1(t)$.

Figures 6, 7, and Figure 8 demonstrate variables of $y_i(t)$, $z_i(t)$, $e_i(t)$, $i = 1, 2, 3$, with $k = 6.6$, respectively. From Figure 8, we know that error system (39) is globally asymptotically stable which means that two coupled HR neurons described by (37) and (38) can achieve global synchronization.

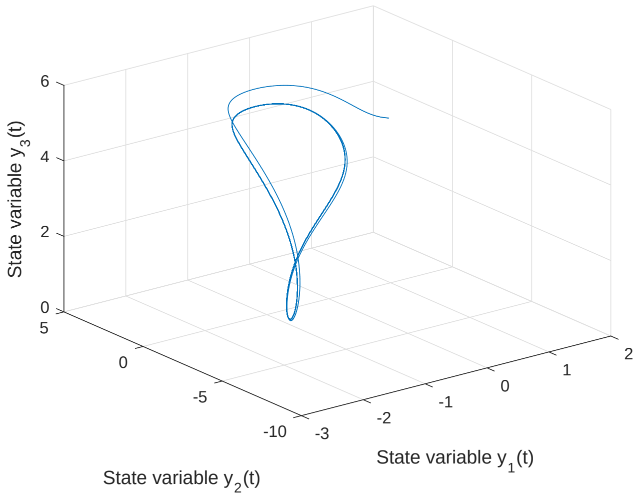

Fig. 6. The simulation result for state variables $y_1(t)$, $y_2(t)$, $y_3(t)$.

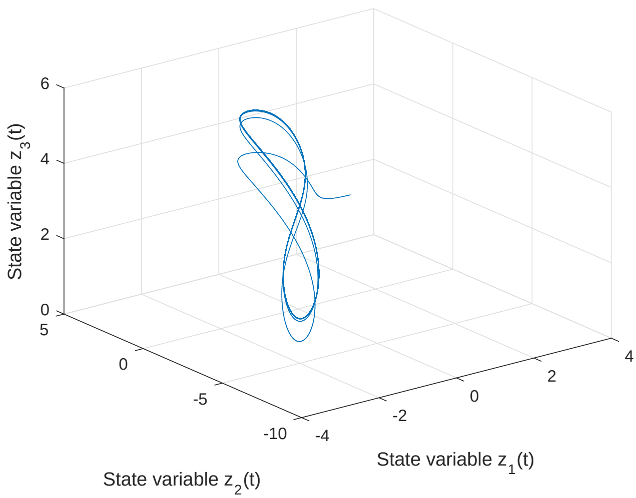

Fig. 7. The simulation result for state variables $z_1(t)$, $z_2(t)$, $z_3(t)$ with $k = 6.6$.

5. CONCLUSION

We have derived two global synchronization criteria for two coupled HR neurons by using the nonlinear feedback control and linear feedback control, respectively. The

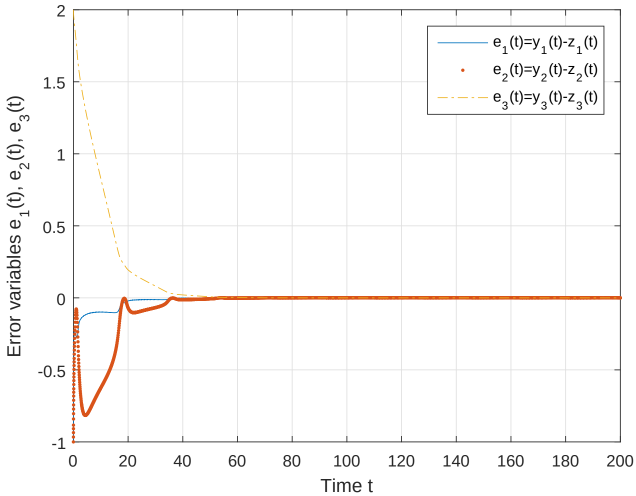

Fig. 8. The simulation result for error variables $e_1(t)$, $e_2(t)$, $e_3(t)$ with $k = 6.6$.

feedback control gains are dependent on the system parameters. The norm bounds of state variables of controlled and uncontrolled HR neurons are not used to derive synchronization criteria. We have applied the control method to derive a synchronization criterion for FHN neurons. We have used two simulation examples to illustrate the effectiveness of our results. In this paper, the synchronization of two coupled HR neurons are studied. How to derive synchronization criteria for networks of HR neurons by using the linear control is our future research focus.

ACKNOWLEDGEMENT

This work was partially supported by the key project of Youth Science Fund of Jiangxi China under Grant 20131542040017, the project of Science Fund of Jiangxi Education Department China under Grant GJJ14349, and the project of of Jiangxi E-commerce High Level Engineering Technology Research Center.

(Received September 25, 2014)

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