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# GENERALIZED VERSIONS OF MV-ALGEBRAIC CENTRAL LIMIT THEOREMS

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MV-algebras can be treated as non-commutative generalizations of boolean algebras. The probability theory of MV-algebras was developed as a generalization of the boolean algebraic probability theory. For both theories the notions of state and observable were introduced by abstracting the properties of the Kolmogorov's probability measure and the classical random variable. Similarly, as in the case of the classical Kolmogorov's probability, the notion of independence is considered. In the framework of the MV-algebraic probability theory many important theorems (as the individual ergodic theorem and the laws of large numbers for observables) were proved. In particular, the central limit theorem (CLT) for sequences of independent and identically distributed observables was considered. In this paper, for triangular arrays of independent, not necessarily identically distributed observables of MV-algebras, we have proved the Lindeberg and the Lyapunov central limit theorems, and the Feller theorem. To show that the generalization proposed by us is essential, we discuss examples of applications of the proved MV-algebraic versions of theorems.

*Keywords:* MV-algebra, MV-algebraic probability, central limit theorem

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## 1. INTRODUCTION

Introduction of the notion of MV-algebras by Chang in [5] was an important contribution to the theory of algebraic systems, especially to the $\aleph_0$-valued propositional calculus developed in [10] and [30]. MV-algebras can be considered as non-commutative generalizations of boolean algebras. The literature concerning the general theory of MV-algebras is very rich and it contains many interesting results (see e. g., [7] and references therein). Special cases of the general MV-algebras are the MV-algebras of fuzzy sets.

Fundamentals of the boolean algebraic probability theory were created by Carathéodory and von Neumann. In [4] Carathéodory presented basic notions of point-free probability. The author replaced probability measures on $\sigma$-algebras, considered in the classical Kolmogorov theory, by strictly positive probability measures defined on $\sigma$-complete boolean algebras. In contradistinction to classical random variables, which are measurable real-valued functions defined on the event space $\Omega$, their algebraic counterparts are functions from the borelian $\sigma$-algebra $B(\mathbb{R})$ into the $\sigma$-boolean algebra of events. In turn, Birkhoff

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and von Neumann in [3] identified properties of a quantum system $S$ with projections in the algebra $\mathcal{B}(\mathcal{H})$ of continuous linear operators on the Hilbert space $\mathcal{H}$ of the system $S$ (or equivalently with elements of the space $\mathcal{L}_{\mathcal{H}}$ of closed linear subspaces of $\mathcal{H}$). In the Birkhoff–von Neumann approach, observables of $S$ were built of the projections and a state (corresponding to the "physical state" of $S$) was defined as a map $m : \mathcal{L}_{\mathcal{H}} \to [0, 1]$ satisfying the following conditions: (i) $m(\mathcal{H}) = 1$; (ii) if $(A_n) \subset \mathcal{L}_{\mathcal{H}}$ and $A_n$ are pairwise orthogonal, then

$$m\left(\bigvee_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n),$$

where $\bigvee_{n=1}^{\infty} A_n$ is the closed linear subspace of $\mathcal{H}$ generated by the union $\bigcup_{n=1}^{\infty} A_n$.

However, for systems with infinitely many degrees of freedom (e. g., occurring in quantum statistical mechanics and quantum field theory) the Hilbert space representation is not unique (see [28]). On the other hand, when quantum mechanical events are described only vaguely, a fuzzy approach should be incorporated to the model (see [8]). Therefore besides quantum logics considered in [9, 18, 29, 31] as orthomodular posets, fuzzy models of quantum mechanics were studied in many papers. Piasecki in [17] introduced the notion of P-measure defined on a family of fuzzy subsets of a given set. Riečan in [22] used this fuzzy measure for building the theory of F-quantum spaces. The mentioned theory was further developed by many authors (e. g., Chovanec, Dvurečenskij, and Mesiar in [8, 11]). In the fuzzy quantum logic model the Zadeh connectives, used in F-quantum spaces, were replaced by their Łukasiewicz counterparts (see [20, 21]). Riečan in [23] introduced the notion of observables to the model. For details concerning the theories of F-quantum spaces and fuzzy quantum logics, we refer the reader to [29] and references therein.

The fuzzy quantum logic of all measurable functions with values in the interval $[0, 1]$ can be considered as a prototype of a general MV-algebra. The possibilities of application of MV-algebras for the description of quantum mechanical systems with infinitely many degrees of freedom were discussed in [28]. In the probability theory of MV-algebras, being a generalization of the boolean algebraic probability theory, the notions of state and observable were introduced, similarly to the algebraic probability case, by abstracting the properties of probability measure and classical random variable. In the literature two notions of a state of an MV-algebra $M$ are considered. The first one, introduced in [14], is a normalized additive functional $s : M \to [0, 1]$, for which $\sigma$-additivity is not assumed. However, the $\sigma$-additivity of $s$ is recovered via the Kroupa-Panti theorem (see e. g., [15]) and Riesz representation. The second notion of state is used in the present paper, similarly as in [28], and it corresponds to $\sigma$-states in von Neumann algebras. In this case a state $s : M \to [0, 1]$ is assumed to be a normalized $\sigma$-additive functional. A probability MV-algebra is a pair $(M, m)$, where $M$ is a $\sigma$-complete MV-algebra and $m : M \to [0, 1]$ is a faithful state on $M$. Then for each observable $x : \mathcal{B}(\mathbb{R}) \to M$, the composite map $m_x : \mathcal{B}(\mathbb{R}) \to [0, 1]$, defined by the equality $m_x(A) = m(x(A))$ for each $A \in \mathcal{B}(\mathbb{R})$, is a probability measure.

The most essential results in the MV-algebraic probability theory were obtained by Riečan. One of the most important theorems of the probability theory is the central limit theorem. The authors in [29] considered a probability MV-algebra $(M, m)$ and a sequence of independent observables of $M$ with the same distribution, such that their expected value and variance were equal to $a \in \mathbb{R}$ and $\sigma^2 > 0$, respectively. They proved that, for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} m\left(\frac{x_1 + x_2 + \cdots + x_n - na}{\sigma\sqrt{n}}(-\infty, t)\right) = \Phi(t),$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution. Summarization of main results of the MV-algebraic probability theory, including the mentioned above version of the central limit theorem, the laws of large numbers, and the individual ergodic theorem one can find in [27, 28, 29]. In [24] the central limit theorem (CLT) for independent, identically distributed observables was proved in a more general setting. A variant of the martingale convergence theorem for MV-algebras of fuzzy sets was proved in [25]. The existing MV-algebraic probability theory was also applied in the Atanassov's IFS setting (see e. g., [26]), which shows the possibility of further development of this theory. There are also many interesting results of other authors concerning ($\sigma$-additive) states and observables of MV-algebras (see e. g., Chovanec [6], Mesiar [12], and Pulmannová [19]).

In this paper we consider limit behavior of the row sums of triangular arrays of independent MV-observables. The aim of the paper is to prove general versions of the central limit theorem for MV-observables, i. e., the Lindeberg CLT and the Lyapounov CLT. Moreover, we prove an MV-algebraic version of the Feller theorem, which is a converse of the Lindeberg CLT for null arrays of observables. Considering different distributions in the central limit theorem is important from the application's point of view (see e. g., [16]). To illustrate the fact that our generalization is essential, we present two examples of the application of the Lindeberg and the Lyapounov CLT for sequences of observables with convergent scaled sums. The first example concerns a sequence of observables with discrete distributions, satisfying the Lyapounov condition. In the second one, the distributions of observables are continuous, the Lindeberg condition holds and simultaneously the Lyapounov condition fails. To our best knowledge such an example has been never proposed, even in the classical probability theory. In both cases observables have different distributions.

The paper is organized as follows. Section 2 contains preliminaries from the classical probability theory and the theory of MV-algebras. Main results are included in Section 3, where the MV-algebraic versions of the Lindeberg CLT, Lyapounov's CLT, and the Feller theorem are proved. Examples of applications of the limit theorems are presented in Section 4. Section 5 contains conclusions.

## 2. PRELIMINARIES

### 2.1. The classical central limit theorems

We introduce a necessary theoretical base of our further considerations.

We denote by $\mathbb{N}$ the set of all positive integers.

Let $X$ be a non-empty set. A class $\mathcal{X}$ of subsets of $X$ is called a *$\sigma$-algebra* if it contains $X$ itself and is closed under the formation of complements and countable unions, i. e.,

**(i)** $X \in \mathcal{X}$;

**(ii)** $A \in \mathcal{X}$ implies $A^c \in \mathcal{X}$;

**(iii)** $A_1, A_2, \ldots \in \mathcal{X}$ implies $A_1 \cup A_2 \cup \ldots \in \mathcal{X}$.

A *measurable space* is a pair $(X, \mathcal{X})$ consisting of a non-empty set $X$ and a $\sigma$-algebra $\mathcal{X}$ of subsets of $X$.

Let $(X, \mathcal{X})$ and $(X', \mathcal{X}')$ be two measurable spaces. A mapping $T : X \to X'$ is measurable $\mathcal{X}/\mathcal{X}'$ if $T^{-1}(A') \in \mathcal{X}$ for each $A' \in \mathcal{X}'$.

We call a set function $\mu : \mathcal{X} \to [0, \infty]$ a *measure* and a triple $(X, \mathcal{X}, \mu)$ a *measure space* if $(X, \mathcal{X})$ is a measurable space and

**(1)** $\mu(\varnothing) = 0$;

**(2)** if $A_1, A_2, \ldots \in \mathcal{X}$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

In particular, $\mu$ is a *probability measure* if $\mu(X) = 1$. Then $(X, \mathcal{X}, \mu)$ is a *probability space.*

Let $\{k_n\}_{n\in\mathbb{N}}$ be a fixed sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty$. Let $\left\{\left(\Omega_{(n)}, \mathcal{S}_{(n)}, P_{(n)}\right)\right\}_{n\in\mathbb{N}}$ be a sequence of probability spaces.

For each $n \in \mathbb{N}$ and a real-valued random variable $X$ on $\left(\Omega_{(n)}, \mathcal{S}_{(n)}, P_{(n)}\right)$, we denote by $\mathcal{E}_{(n)}X$ the expected value of $X$ and by $\mathcal{D}^2_{(n)}X$ the variance of $X$ with respect to $P_{(n)}$.

**Definition 2.1.** Let, for each positive integer $n \in \mathbb{N}$, $\{X_{n1}, X_{n2}, \ldots, X_{nk_n}\}$ be a sequence of independent real-valued random variables on $\left(\Omega_{(n)}, \mathcal{S}_{(n)}, P_{(n)}\right)$. Then

$$\{X_{nj} : 1 \leq j \leq k_n\}_{n\in\mathbb{N}}$$

is called a *triangular array of independent random variables.*

We present a slightly modified versions of the Lindeberg CLT, the Lyapunov CLT, and the Feller theorem formulated in [1]. We assume more generally that expected values of $X_{nj}$, $1 \leq j \leq k_n$, $n \in \mathbb{N}$, are not necessarily equal to zero.

**Definition 2.2.** Let $\{X_{nj} : 1 \leq j \leq k_n\}_{n\in\mathbb{N}}$ be a triangular array of independent random variables such that

$$\mathcal{E}_{(n)}X^2_{nj} < \infty, \quad 1 \leq j \leq k_n, \ n \in \mathbb{N};$$
$$S^2_n = \sum_{j=1}^{k_n} \mathcal{D}^2_{(n)}X_{nj} \in (0, \infty), \ n \in \mathbb{N}. \tag{1}$$

Then $\{X_{nj} : 1 \leq j \leq k_n\}_{n\in\mathbb{N}}$ is said to satisfy the *Lindeberg condition* if for each $\varepsilon > 0$

$$L_n(\varepsilon) = \frac{1}{S^2_n} \sum_{j=1}^{k_n} \mathcal{E}_{(n)}\left(\left(X_{nj} - \mathcal{E}_{(n)}X_{nj}\right)^2 I_{|X_{nj} - \mathcal{E}_{(n)}X_{nj}| > \varepsilon S_n}\right) \xrightarrow[n\to\infty]{} 0. \tag{2}$$

Let $\{Y_n\}_{n=0}^{\infty}$ be a collection of random variables and let $F_n$ denote the cumulative distribution functions of $Y_n$, $n \geq 0$. Then $\{Y_n\}_{n=1}^{\infty}$ is said to *converge in distribution* to $Y_0$ if for every $t \in C(F_0)$

$$\lim_{n\to\infty} F_n(t) = F_0(t),$$

where $C(F_0) = \{t \in \mathbb{R} : F_0 \text{ is continuous at } t\}$.

We will write $Y_n \underset{n\to\infty}{\longrightarrow} N(0,1)$ in distribution if $\{Y_n\}_{n=1}^{\infty}$ converge in distribution to a random variable $Y_0$ and $Y_0$ has the standard normal distribution $N(0,1)$.

For details concerning the convergence in distribution, we refer the reader to [1].

**Theorem 2.3.** (Lindeberg CLT) Let $\{X_{nj} : 1 \leq j \leq k_n\}_{n\in\mathbb{N}}$ be a triangular array of independent random variables satisfying (1) and the Lindeberg condition (2). Then

$$\frac{\sum_{j=1}^{k_n} X_{nj} - \sum_{j=1}^{k_n} \mathcal{E}_{(n)} X_{nj}}{S_n} \underset{n\to\infty}{\longrightarrow} N(0,1)$$

in distribution.

In the next definition we introduce the Lyapunov condition.

**Definition 2.4.** A triangular array $\{X_{nj} : 1 \leq j \leq k_n\}_{n\in\mathbb{N}}$ of independent random variables satisfying (1) is said to satisfy the *Lyapunov condition* if there exists $\delta > 0$ such that

$$\frac{1}{S_n^{2+\delta}} \sum_{j=1}^{k_n} \mathcal{E}_{(n)} |X_{nj} - \mathcal{E}_{(n)} X_{nj}|^{2+\delta} \underset{n\to\infty}{\longrightarrow} 0. \tag{3}$$

**Theorem 2.5.** (Lyapounov CLT) Let $\{X_{nj} : 1 \leq j \leq k_n\}_{n\in\mathbb{N}}$ be a triangular array of independent random variables satisfying (1) and Lyapounov's condition (3). Then

$$\frac{\sum_{j=1}^{k_n} X_{nj} - \sum_{j=1}^{k_n} \mathcal{E}_{(n)} X_{nj}}{S_n} \underset{n\to\infty}{\longrightarrow} N(0,1)$$

in distribution.

The following Feller theorem can be treated as converse of the Lindeberg CLT.

**Theorem 2.6.** (Feller) Let $\{X_{nj} : 1 \leq j \leq k_n\}_{n\in\mathbb{N}}$ be a triangular array of independent random variables satisfying (1) and such that, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \max_{1\leq j\leq k_n} P(|X_{nj}| > \varepsilon S_n) = 0.$$

If

$$\frac{\sum_{j=1}^{k_n} X_{nj} - \sum_{j=1}^{k_n} \mathcal{E}_{(n)} X_{nj}}{S_n} \underset{n\to\infty}{\longrightarrow} N(0,1)$$

in distribution, then the Lindeberg condition (2) is satisfied.

### 2.2. MV-algebras

Basic MV-algebraic notions and preliminary results concerning MV-algebras one can find in [7] (see also [13]). However, for benefit of the reader, we recall some of them in this subsection. We also present basic elements of the MV-algebraic probability theory from [28] with minor modifications.

**Definition 2.7.** An *MV-algebra* $(M, 0, 1, \neg, \oplus, \odot)$ is a system where $M$ is a non-empty set, the operation $\oplus$ is associative-commutative with a neutral element 0, $\neg 0 = 1$, $\neg 1 = 0$, and additionally, for all $x, y \in M$ $x \oplus 1 = 1$,

$$y \oplus \neg (y \oplus \neg x) = x \oplus \neg (x \oplus \neg y)$$

and for the operation $\odot$

$$x \odot y = \neg (\neg x \oplus \neg y) \, .$$

In every MV-algebra $(M, 0, 1, \neg, \oplus, \odot)$ the relation $\leq$ given by

$$x \leq y \Leftrightarrow x \odot \neg y = 0$$

is a partial order. Furthermore, the operations $\vee$ and $\wedge$ given by

$$x \vee y = \neg (\neg x \oplus y) \oplus y$$

and

$$x \wedge y = \neg (\neg x \vee \neg y)$$

make $M$ into a distributive lattice (called the underlying lattice of $M$) with least element 0 and greatest element 1.

**Definition 2.8.** An MV-algebra $M$ is *$\sigma$-complete* if its underlying lattice is $\sigma$-complete, i. e., every non-empty countable subset of $M$ has the supremum in $M$. An MV-algebra $M$ is *complete* if every non-empty subset of $M$ has the supremum in $M$.

We will use the following notations.
Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of a set $X$. Then $A_n \nearrow A$ iff $\{A_n\}_{n=1}^{\infty}$ is monotone (i. e., $A_1 \subseteq A_2 \subseteq \cdots$) and $\bigcup_n^{\infty} A_n = A$.
For a sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers, $x_n \nearrow x$ iff $x_1 \leq x_2 \leq \cdots$ and $x = \sup_i x_i$.
Finally, for $\{b_n\}_{n=1}^{\infty}$ included in an MV-algebra $M$, $b_n \nearrow b$ iff $b_1 \leq b_2 \leq \cdots$ and $b = \sup_i b_i$ with respect to the underlying order of $M$.
As noted above, in the present paper we use the $\sigma$-additive states.

**Definition 2.9.** Let $M$ be a $\sigma$-complete MV-algebra. A *state* on $M$ is a map $m : M \to [0, 1]$ satisfying the following conditions for all $a, b, c \in M$ and $\{a_n\}_{n=1}^{\infty}$:

**(i)** $m(1) = 1$;

**(ii)** if $b \odot c = 0$, then $m(b \oplus c) = m(b) + m(c)$;

**(iii)** if $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

We say that $m$ is *faithful* if $m(x) \neq 0$ whenever $x \neq 0$ and $x \in M$.

**Definition 2.10.** A *probability MV-algebra* is a pair $(M, m)$, where $M$ is a $\sigma$-complete MV-algebra and $m$ is a faithful state on $M$.

Each probability MV-algebra is complete (see [15], Theorem 13.8).

Let $n \in \mathbb{N}$ and $\mathcal{P}$ be the family of bounded rectangles of the form

$$(a_1, b_1] \times \cdots \times (a_n, b_n] : a_i, b_i \in \mathbb{R},\ a_i < b_i,\ i = 1, 2, \ldots, n.$$

The $\sigma$-algebra $\mathcal{B}(\mathbb{R}^n)$ of Borel subsets of $\mathbb{R}^n$ is the $\sigma$-algebra generated by $\mathcal{P}$ (i. e., the smallest $\sigma$-algebra of subsets of $\mathbb{R}^n$ containing $\mathcal{P}$).

**Definition 2.11.** Let $M$ be a $\sigma$-complete MV-algebra. An *$n$-dimensional observable* of $M$ is a map $x : \mathcal{B}(\mathbb{R}^n) \to M$ satisfying the following conditions:

**(i)** $x(\mathbb{R}^n) = 1$;

**(ii)** whenever $A, B \in \mathcal{B}(\mathbb{R}^n)$ and $A \cap B = \varnothing$, then

$$x(A) \odot x(B) = 0 \quad \text{and} \quad x(A \cup B) = x(A) \oplus x(B);$$

**(iii)** for all $A, A_1, A_2, \ldots \in \mathcal{B}(\mathbb{R}^n)$, if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

**Theorem 2.12.** Let $M$ be a $\sigma$-complete MV-algebra with an $n$-dimensional observable $x : \mathcal{B}(\mathbb{R}^n) \to M$ and a state $m$. Then the map $m_x : \mathcal{B}(\mathbb{R}^n) \to [0,1]$ given by

$$m_x(A) = (m \circ x)(A) = m(x(A)),\ A \in \mathcal{B}(\mathbb{R}^n),$$

is a probability measure on $\mathcal{B}(\mathbb{R}^n)$.

Proof. It suffices to prove the following two conditions for the set function $m_x : \mathcal{B}(\mathbb{R}^n) \to [0,1]$ :

**i)** $m_x(\mathbb{R}^n) = 1$;

**ii)** if $A_1, A_2, \ldots \in \mathcal{B}(\mathbb{R}^n)$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$m_x\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m_x(A_i).$$

Condition i) follows straightforwardly from the definition of state and observable. Indeed, $m_x(\mathbb{R}^n) = m(x(\mathbb{R}^n)) = m(1) = 1$.

Let $A_1, A_2, \ldots \in \mathcal{B}(\mathbb{R}^n)$ and $A = \bigcup_{i=1}^{\infty} A_i$. To prove condition ii), we define

$$E_n = \bigcup_{i=1}^{n} A_i,\ n = 1, 2, \ldots$$

Clearly, $E_n \nearrow A$ and therefore, from the definition of state and observable,

$$x(E_n) \nearrow x(A) \text{ and } m_x(E_n) \nearrow m_x(A). \tag{4}$$

Moreover, for each $k \geq 2$,

$$x(E_k) = x(E_{k-1} \cup A_k) = x(E_{k-1}) \oplus x(A_k) \text{ and } x(E_{k-1}) \odot x(A_k) = 0.$$

Therefore

$$\begin{aligned} m_x(E_k) &= m(x(E_{k-1}) \oplus x(A_k)) = m(x(E_{k-1})) + m(x(A_k)) \\ &= m_x(E_{k-1}) + m_x(A_k). \end{aligned} \tag{5}$$

Applying (5) for $k = n, n-1, \ldots, 2$, we obtain the equality

$$\begin{aligned} m_x(E_n) &= m_x(E_{n-1}) + m_x(A_n) = m_x(E_{n-2}) + m_x(A_{n-1}) + m_x(A_n) \\ &= \cdots = \sum_{i=1}^{n} m_x(A_i). \end{aligned} \tag{6}$$

Finally, from formulas (6) and (4),

$$\sum_{i=1}^{\infty} m_x(A_i) = \lim_{n\to\infty} m_x(E_n) = m_x(A),$$

which ends the proof of condition ii). □

**Definition 2.13.** Let $(M, m)$ be a probability MV-algebra. Let $x : \mathcal{B}(\mathbb{R}) \to M$ be an observable of $M$. Then $x$ is said to be *integrable* in $(M, m)$, and we write $x \in L^1_m$, if the *expectation*

$$\mathbb{E}(x) = \int_{\mathbb{R}} t m_x(\mathrm{d}t)$$

exists. We say that $x$ is *square-integrable* in $(M, m)$, and we write $x \in L^2_m$, if $\int_{\mathbb{R}} t^2 m_x(\mathrm{d}t)$ exists. Then the *variance* of $x$ also exists and is described by the equality

$$\mathbb{D}^2(x) = \int_{\mathbb{R}} t^2 m_x(\mathrm{d}t) - (\mathbb{E}(x))^2 = \int_{\mathbb{R}} (t - \mathbb{E}(x))^2 m_x(\mathrm{d}t).$$

More generally, we write $x \in L^p_m$ for $p \geq 1$ if $\int_{\mathbb{R}} |t|^p m_x(\mathrm{d}t) < \infty$.

**Definition 2.14.** Let $(M, m)$ be a probability MV-algebra. We say that observables $x_1, x_2, \ldots, x_n$ are *independent* (with respect to $m$) if there exists an $n$-dimensional observable $h : \mathcal{B}(\mathbb{R}^n) \to M$ such that

$$\begin{aligned} m(h(C_1 \times C_2 \times \cdots \times C_n)) &= m(x_1(C_1)) \cdot m(x_2(C_2)) \cdot \cdots \cdot m(x_n(C_n)) \\ &= m_{x_1}(C_1) \cdot m_{x_2}(C_2) \cdot \cdots \cdot m_{x_n}(C_n) \end{aligned}$$

for all $C_1, C_2, \ldots, C_n \in \mathcal{B}(\mathbb{R})$.

**Remark 2.15.** Let $x_1, x_2, \ldots, x_n : \mathcal{B}(\mathbb{R}) \to M$ be independent observables in a probability MV-algebra $(M, m)$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function and let $h : \mathcal{B}(\mathbb{R}^n) \to M$ be the joint observable of $x_1, x_2, \ldots, x_n$. Then $g(x_1, x_2, \ldots, x_n) = h \circ g^{-1}$ is an observable.

## 3. CENTRAL LIMIT THEOREMS FOR OBSERVABLES OF MV-ALGEBRAS

In this section we present and prove the main theorems of this paper. At the beginning we recall the following theorem (see [2], Theorem 16.12) concerning the change of variable for integrals.

Let $(X, \mathcal{X})$ and $(X', \mathcal{X}')$ be measurable spaces. Assume that a function $T : X \to X'$ is $\mathcal{X}/\mathcal{X}'$ measurable. For a measure $\mu$ on $\mathcal{X}$ we define a measure $\mu T^{-1}$ on $\mathcal{X}'$ by the equality

$$\mu T^{-1}(A') = \mu\left(T^{-1}(A')\right), \; A' \in \mathcal{X}'.$$

**Theorem 3.1.** Let $f : X' \to \mathbb{R}$ be an $\mathcal{X}'$-measurable function. If $f$ is non-negative, then

$$\int_X f(Tx)\,\mu(\mathrm{d}x) = \int_{X'} f(x')\,\mu T^{-1}(\mathrm{d}x'). \tag{7}$$

A function $f$ (not necessarily non-negative) is integrable with respect to $\mu T^{-1}$ if and only if $fT$ is integrable with respect to $\mu$, in which case (7) and

$$\int_{T^{-1}(A')} f(Tx)\,\mu(\mathrm{d}x) = \int_{A'} f(x')\,\mu T^{-1}(\mathrm{d}x'), A' \in \mathcal{X}', \tag{8}$$

hold. Moreover, for any non-negative $f$, identity (8) always holds.

The following lemma, which is a consequence of the above theorem, shows the form of the expected value of a Borel function of an observable.

**Lemma 3.2.** Let $\varphi$ be an $\mathbb{R}$-valued Borel function, which domain is the whole set of real numbers $\mathbb{R}$, $x : \mathcal{B}(\mathbb{R}) \to M$ be an observable of a probability MV-algebra $(M, m)$ and $y = \varphi(x) = x \circ \varphi^{-1}$. Then $\mathbb{E}(y)$ exists if and only if $\int_{\mathbb{R}} |\varphi(t)| m_x(\mathrm{d}t) < \infty$ and then the following equality holds

$$\mathbb{E}(y) = \int_{\mathbb{R}} \varphi(t)\, m_x(\mathrm{d}t).$$

P r o o f. We apply directly Theorem 3.1 for $(X, \mathcal{X}) = (X', \mathcal{X}') = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $T = \varphi$, and $f(t) = t$. Theorem 2.12 implies that $\mu = m_x$ is a probability measure. Moreover, by direct computations we obtain the equality $\mu T^{-1} = m_{\varphi(x)} = m_y$, which ends the proof. □

The above lemma gives the possibility to compute expected values of Borel functions of observables used in MV-algebraic versions of the Lindeberg and Lyapounov conditions. Furthermore, by Lemma 3.2 we obtain the following equality for variance of any observable $x \in L_m^2$

$$\mathbb{D}^2(x) = \mathbb{E}(x - \mathbb{E}(x))^2 = \int_{\mathbb{R}} (t - \mathbb{E}(x))^2\, m_x(\mathrm{d}t).$$

Let $\{k_n\}_{n\in\mathbb{N}}$ be a fixed sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty$. To formulate general versions of central limit theorems for observables of MV-algebras we need to introduce some additional notations and definitions.

Let $\left\{\left(M_{(n)}, m_{(n)}\right)\right\}_{n\in\mathbb{N}}$ be a sequence of probability MV-algebras. For each $n \in \mathbb{N}$ and observable $x : \mathcal{B}(\mathbb{R}) \to M_{(n)}$, we denote by $\mathbb{E}_{(n)}(x)$ the expected value of $x$ and by $\mathbb{D}^2_{(n)}(x)$ the variance of $x$ with respect to $m_{(n)}$.

Let $n \in \mathbb{N}$, $x : \mathcal{B}(\mathbb{R}) \to M_{(n)}$ be an observable of $M_{(n)}$, and $x \in L^2_{m_{(n)}}$. Then for each $s > 0$, the following $\mathbb{R}$-valued function is well-defined (see Lemma 3.2):

$$l_n^x(\varepsilon, s) = \mathbb{E}_{(n)}\big(\big(x - \mathbb{E}_{(n)}(x)\big)^2 I_{|x-\mathbb{E}_{(n)}(x)|>\varepsilon s}\big).$$

**Definition 3.3.** Let, for each positive integer $n \in \mathbb{N}$, $\{x_{n1}, x_{n2}, \ldots, x_{nk_n}\}$ be a sequence of independent (with respect to $m_{(n)}$) observables of the MV-algebra $M_{(n)}$. Then $\{x_{nj} : 1 \le j \le k_n\}_{n\in\mathbb{N}}$ is called a *triangular array of independent observables.*

**Definition 3.4.** Let $\{x_{nj} : 1 \le j \le k_n\}_{n\in\mathbb{N}}$ be a triangular array of independent observables such that

$$x_{nj} \in L^2_{m_{(n)}}, 1 \le j \le k_n, n \in \mathbb{N};$$

$$s_n^2 = \sum_{j=1}^{k_n} \mathbb{D}^2_{(n)}(x_{nj}) \in (0, \infty),\ n \in \mathbb{N}. \tag{9}$$

Then $\{x_{nj} : 1 \le j \le k_n\}_{n\in\mathbb{N}}$ is said to satisfy the *Lindeberg condition* if for each $\varepsilon > 0$

$$L_n(\varepsilon) = \frac{1}{s_n^2}\sum_{j=1}^{k_n} l_n^{x_{nj}}(\varepsilon, s_n) \underset{n\to\infty}{\longrightarrow} 0. \tag{10}$$

**Definition 3.5.** Let $\left\{\left(M_{(n)}, m_{(n)}\right)\right\}_{n\in\mathbb{N}}$ be a sequence of probability MV-algebras. Let $\left\{x_n : \mathcal{B}(\mathbb{R}) \to M_{(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of observables. The sequence $\{x_n\}_{n=1}^{\infty}$ is *convergent in distribution* to a function $F : \mathbb{R} \to [0, 1]$ if for each $t \in \mathbb{R}$

$$\lim_{n\to\infty} m_{(n)}\left(x_n(-\infty, t)\right) = F(t).$$

**Definition 3.6.** Let, for each $n \in \mathbb{N}$ and $C = \prod_{j=1}^{k_n} C_j$, where $C_j \in \mathcal{B}(\mathbb{R})$, $1 \le j \le k_n$, the probability measure $\mathcal{P}_{(n)} : \mathcal{B}\left(\mathbb{R}^{k_n}\right) \to [0, 1]$ be defined by the equality

$$\mathcal{P}_{(n)}(C) = \prod_{j=1}^{k_n} \left(m_{(n)}\right)_{x_{nj}}(C_j).$$

Then $\left\{\left(\mathbb{R}^{k_n}, \mathcal{B}\left(\mathbb{R}^{k_n}\right), \mathcal{P}_{(n)}\right)\right\}_{n\in\mathbb{N}}$ is a sequence of probability spaces. For each $n \in \mathbb{N}$ and $1 \le j \le k_n$, we define random variables $\iota_j^n$ and random vectors $\bar{\iota}_j^n$ on the probability space $\left(\mathbb{R}^{k_n}, \mathcal{B}\left(\mathbb{R}^{k_n}\right), \mathcal{P}_{(n)}\right)$ by

$$\iota_j^n : \mathbb{R}^{k_n} \to \mathbb{R},\ \ \iota_j^n(u_1, u_2, \ldots, u_{k_n}) = u_j,$$
$$\bar{\iota}_j^n : \mathbb{R}^{k_n} \to \mathbb{R}^j,\ \ \bar{\iota}_j^n(u_1, u_2, \ldots, u_{k_n}) = (u_1, u_2, \ldots, u_j).$$

In what follows we denote by $E^{\mathcal{P}_{(n)}}$ and by $D^{2,\mathcal{P}_{(n)}}$ expected value and variance with respect to $\mathcal{P}_{(n)}$, respectively.

Let, for each $n \in \mathbb{N}$, $h_n : \mathcal{B}\left(\mathbb{R}^{k_n}\right) \to M_{(n)}$ be the joint observable of the sequence

$$x_{n1}, x_{n2}, \ldots, x_{nk_n},$$

$f_n : \mathbb{R}^{k_n} \to \mathbb{R}$ be the function of the form

$$f_n\left(t_1, t_2, \ldots, t_{k_n}\right) = \frac{t_1 + t_2 + \cdots + t_{k_n} - \sum_{j=1}^{k_n} \mathbb{E}_{(n)}\left(x_{nj}\right)}{s_n}$$

and $\varphi_n : \mathcal{B}(\mathbb{R}) \to M_{(n)}$ be the observable defined, according to the schema described in Remark 2.15, by the equality $\varphi_n = h_n \circ f_n^{-1}$.

We attempt to formulate an MV-algebraic version of the Lindeberg CLT. In the proof we adopt the approach applied in [28].

**Theorem 3.7.** (Lindeberg CLT) Let $\{x_{nj} : 1 \le j \le k_n\}_{n\in\mathbb{N}}$ be a triangular array of independent observables satisfying (9) and the Lindeberg condition (10). Then

$$\frac{x_{n1} - \mathbb{E}_{(n)}\left(x_{n1}\right) + x_{n2} - \mathbb{E}_{(n)}\left(x_{n2}\right) + \cdots + x_{nk_n} - \mathbb{E}_{(n)}\left(x_{nk_n}\right)}{s_n} \underset{n\to\infty}{\longrightarrow} N(0,1)$$

in distribution.

Proof. For each $n \in \mathbb{N}$, we consider the probability space $\left(\mathbb{R}^{k_n}, \mathcal{B}\left(\mathbb{R}^{k_n}\right), \mathcal{P}_{(n)}\right)$ and the random variable $\eta_n : \mathbb{R}^{k_n} \to \mathbb{R}$ defined by the equality

$$\eta_n = f_n\left(\iota_1^n, \iota_2^n, \ldots, \iota_{k_n}^n\right) = f_n \circ \bar{\iota}_{k_n}^n.$$

Then

$$\begin{aligned} m_{(n)} \circ h_n &= \left(m_{(n)}\right)_{x_{n1}} \times \left(m_{(n)}\right)_{x_{n2}} \times \cdots \times \left(m_{(n)}\right)_{x_{nk_n}}; \\ \mathcal{P}_{(n)} \circ \left(\iota_j^n\right)^{-1} &= \left(m_{(n)}\right)_{x_{nj}} \quad \text{for } 1 \le j \le k_n; \\ \mathcal{P}_{(n)} \circ \left(\bar{\iota}_{k_n}^n\right)^{-1} &= m_{(n)} \circ h_n; \\ \mathcal{P}_{(n)} \circ \eta_n^{-1} &= m_{(n)} \circ \varphi_n. \end{aligned} \tag{11}$$

Our aim is to prove that, for each $t \in \mathbb{R}$,

$$\lim_{n\to\infty} m_{(n)}\left(\frac{x_{n1} + x_{n2} + \cdots + x_{nk_n} - \sum_{j=1}^{k_n} \mathbb{E}_{(n)}\left(x_{nj}\right)}{s_n}(-\infty, t)\right) = \Phi(t),$$

where $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^2}{2}}\, \mathrm{d}u$. Let $t \in \mathbb{R}$. Clearly,

$$m_{(n)}\left(\frac{x_{n1} + x_{n2} + \cdots + x_{nk_n} - \sum_{j=1}^{k_n} \mathbb{E}_{(n)}\left(x_{nj}\right)}{s_n}(-\infty, t)\right)$$

$$= m_{(n)}\left(f_n\left(x_{n1}, x_{n2}, \ldots, x_{nk_n}\right)(-\infty, t)\right)$$

$$= m_{(n)}\left(h_n \circ f_n^{-1}(-\infty, t)\right)$$

$$= \left(\left(m_{(n)}\right)_{x_{n1}} \times \left(m_{(n)}\right)_{x_{n2}} \times \cdots \times \left(m_{(n)}\right)_{x_{nk_n}}\right)\left(f_n^{-1}(-\infty, t)\right)$$

$$= \mathcal{P}_{(n)}\left(\left\{u \in \mathbb{R}^{k_n} : \frac{u_1 + u_2 + \cdots + u_{k_n} - \sum_{j=1}^{k_n} \mathbb{E}_{(n)}\left(x_{nj}\right)}{s_n} < t\right\}\right)$$

$$= \mathcal{P}_{(n)}\left(\left\{u \in \mathbb{R}^{k_n} : \frac{\iota_1^n(u) + \iota_2^n(u) + \cdots + \iota_{k_n}^n(u) - \sum_{j=1}^{k_n} \mathbb{E}_{(n)}\left(x_{nj}\right)}{s_n} < t\right\}\right). \tag{12}$$

From (11) it follows that, for $n \in \mathbb{N}$ and $1 \leq j \leq k_n$, the random variables $\iota_j^n$ are independent, have distribution $\left(m_{(n)}\right)_{x_{nj}}$, and

$$\mathbb{E}_{(n)}\left(x_{nj}\right) = E^{\mathcal{P}_{(n)}} \iota_j^n, \quad \mathbb{D}_{(n)}^2\left(x_{nj}\right) = D^{2,\mathcal{P}_{(n)}} \iota_j^n, \; s_n^2 = \sum_{j=1}^{k_n} D^{2,\mathcal{P}_{(n)}} \iota_j^n.$$

Moreover,

$$l_n^{x_{nj}}\left(\varepsilon, s_n\right) = E^{\mathcal{P}_{(n)}}\left(\left(\iota_j^n - E^{\mathcal{P}_{(n)}} \iota_j^n\right)^2 I_{\left|\iota_j^n - E^{\mathcal{P}_{(n)}} \iota_j^n\right| > \varepsilon s_n}\right).$$

Finally, Theorem 2.3 implies convergence of (12) to $\Phi(t)$ as $n \to \infty$. □

We now introduce the Lyapunov condition for observables of MV-algebras.

**Definition 3.8.** Let $\left\{x_{nj} : 1 \leq j \leq k_n\right\}_{n \in \mathbb{N}}$ be a triangular array of independent observables (described by Definition 3.3 ), fulfilling condition (9). Then the array $\left\{x_{nj} : 1 \leq j \leq k_n\right\}$ satisfies the *Lyapunov condition* if there exists $\delta > 0$ such that

$$\frac{1}{s_n^{2+\delta}} \sum_{j=1}^{k_n} \mathbb{E}_{(n)}\left(\left|x_{nj} - \mathbb{E}_{(n)}\left(x_{nj}\right)\right|^{2+\delta}\right) \underset{n \to \infty}{\longrightarrow} 0. \tag{13}$$

**Theorem 3.9.** (Lyapounov CLT) Let $\left\{x_{nj} : 1 \leq j \leq k_n\right\}_{n \in \mathbb{N}}$ be a triangular array of independent observables satisfying (9) and Lyapounov's condition (13). Then

$$\frac{x_{n1} - \mathbb{E}_{(n)}\left(x_{n1}\right) + x_{n2} - \mathbb{E}_{(n)}\left(x_{n2}\right) + \cdots + x_{nk_n} - \mathbb{E}_{(n)}\left(x_{nk_n}\right)}{s_n} \underset{n \to \infty}{\longrightarrow} N(0,1)$$

in distribution.

P r o o f. We use the Lindeberg CLT. For each observable $y : \mathcal{B}(\mathbb{R}) \to M$, $y \in L_m^2$, of a probability MV-algebra $(M, m)$ and $\theta > 0$, we define an observable $\varphi_\theta(y)$ by the formula: $\varphi_\theta(y) = y^2 I_{|y|>\theta}$. Moreover, if additionally $y \in L_m^{2+\delta}$ for $\delta > 0$, then, by Lemma 3.2,

$$\mathbb{E}\left(\varphi_\theta(y)\right) = \int_{\mathbb{R}} t^2 I_{|\frac{t}{\theta}|>1} m_y(\mathrm{d}t) \leq \int_{\mathbb{R}} \frac{t^2|t|^\delta}{\theta^\delta} I_{|t|>\theta} m_y(\mathrm{d}t) \leq \frac{1}{\theta^\delta}\mathbb{E}\left(|x|^{2+\delta}\right). \tag{14}$$

We will show that the triangular array of observables $\{x_{nj} : 1 \leq j \leq k_n\}_{n\in\mathbb{N}}$ satisfies the Lindeberg condition. Since, for each $\varepsilon > 0$,

$$l_n^{x_{nj}}(\varepsilon, s_n) = \mathbb{E}_{(n)}\varphi_{\varepsilon s_n}\left(x_{nj} - \mathbb{E}_{(n)}(x_{nj})\right),$$

applying the inequality (14), we obtain

$$\begin{aligned} L_n(\varepsilon) &= \frac{1}{s_n^2}\sum_{j=1}^{k_n}\mathbb{E}_{(n)}\varphi_{\varepsilon s_n}\left(x_{nj} - \mathbb{E}_{(n)}(x_{nj})\right) \\ &\leq \frac{1}{s_n^2(\varepsilon s_n)^\delta}\sum_{j=1}^{k_n}\mathbb{E}_{(n)}\left(|x_{nj} - \mathbb{E}_{(n)}(x_{nj})|^{2+\delta}\right). \end{aligned}$$

Since the right side of the above inequality converges to 0 as $n$ tends to infinity, $L_n(\varepsilon) \underset{n\to\infty}{\longrightarrow} 0$, which ends the proof. □

Similarly as the Feller theorem in the classical probability theory, its following MV-algebraic version shows that under some conditions on a triangular array of observables, the Lindeberg condition is necessary for the convergence of their scaled row sums to the standard normal distribution.

**Theorem 3.10.** (Feller) Let $\{x_{nj} : 1 \leq j \leq k_n\}_{n\in\mathbb{N}}$ be a triangular array of independent observables satisfying (9) and such that, for each $\varepsilon > 0$,

$$\lim_{n\to\infty}\max_{1\leq j\leq k_n}\left(m_{(n)}\right)_{x_{nj}}\left((-\infty, -\varepsilon s_n) \cup (\varepsilon s_n, \infty)\right) = 0.$$

If

$$\frac{x_{n1} - \mathbb{E}_{(n)}(x_{n1}) + x_{n2} - \mathbb{E}_{(n)}(x_{n2}) + \cdots + x_{nk_n} - \mathbb{E}_{(n)}(x_{nk_n})}{s_n} \underset{n\to\infty}{\longrightarrow} N(0,1)$$

in distribution, then the Lindeberg condition (10) is satisfied.

P r o o f. We use the same notation as in the proof of Theorem 3.7. For each positive integer $n$, the random variables

$$\iota_1^n, \iota_2^n, \ldots, \iota_{k_n}^n$$

are independent, for each $1 \leq j \leq k_n$ $\iota_j^n$ has distribution $\left(m_{(n)}\right)_{x_{nj}}$, and

$$\mathbb{E}_{(n)}(x_{nj}) = E^{\mathcal{P}_{(n)}}\iota_j^n, \quad \mathbb{D}_{(n)}^2(x_{nj}) = D^{2,\mathcal{P}_{(n)}}\iota_j^n.$$

Clearly, $s_n^2 = \sum_{j=1}^{k_n} D^{2,\mathcal{P}_{(n)}} \iota_j^n$ and furthermore,

$$\mathcal{P}_{(n)}\left(|\iota_j^n| > \varepsilon s_n\right) = \left(m_{(n)}\right)_{x_{nj}}\left((-\infty, -\varepsilon s_n) \cup (\varepsilon s_n, \infty)\right).$$

Since, for each $t \in \mathbb{R}$,

$$\mathcal{P}_{(n)}\left(\left\{u \in \mathbb{R}^{k_n} : \frac{\iota_1^n(u) + \iota_2^n(u) + \cdots + \iota_{k_n}^n(u) - \sum_{j=1}^{k_n} \mathbb{E}_{(n)}(x_{nj})}{s_n} < t\right\}\right)$$

$$= m_{(n)}\left(\frac{x_{n1} + x_{n2} + \cdots + x_{nk_n} - \sum_{j=1}^{k_n} \mathbb{E}_{(n)}(x_{nj})}{s_n}(-\infty, t)\right) \xrightarrow[n\to\infty]{} \Phi(t),$$

by the classical Feller theorem, the Lindeberg condition is satisfied, i. e.,

$$\frac{1}{s_n^2}\sum_{j=1}^{k_n} E^{\mathcal{P}_{(n)}}\left(\left(\iota_j^n - E^{\mathcal{P}_{(n)}}\iota_j^n\right)^2 I_{|\iota_j^n - E^{\mathcal{P}_{(n)}}\iota_j^n| > \varepsilon s_n}\right) \xrightarrow[n\to\infty]{} 0.$$

The equality

$$l_n^{x_{nj}}(\varepsilon, s_n) = E^{\mathcal{P}_{(n)}}\left(\left(\iota_j^n - E^{\mathcal{P}_{(n)}}\iota_j^n\right)^2 I_{|\iota_j^n - E^{\mathcal{P}_{(n)}}\iota_j^n| > \varepsilon s_n}\right)$$

also implies the convergence

$$L_n(\varepsilon) = \frac{1}{s_n^2}\sum_{j=1}^{k_n} l_n^{x_{nj}}(\varepsilon, s_n) \xrightarrow[n\to\infty]{} 0,$$

which ends the proof. □

## 4. EXAMPLES OF APPLICATIONS

As it was shown in the proof of Theorem 3.9, the Lyapunov condition implies the Lindeberg condition. In this section we present two examples of arrays of observables with convergent scaled row sums. The CLT version proved in [29] cannot be applied for them, since they are not identically distributed. In the first case the considered observables have discrete distributions and the Lyapunov condition is satisfied. In the second case the distributions of observables are continuous, defined by a series of functions. We show that the Lindeberg condition is satisfied and simultaneously the Lyapounov condition fails. In both cases we consider observables taking values in the same probability MV-algebra $(M, m)$, where $M = [0, 1]$ is the real unit interval equipped with the operations $\neg, \oplus, \odot$ given by formulas:

$$\neg a = 1 - a,\ a \oplus b = (a + b) \wedge 1,\ a \odot b = (a + b - 1) \vee 0$$

and $m$ is the faithful state of the form $m(t) = t$. Moreover, we assume that $x_{nj} = x_j$ and $k_n = n$ for each $n \in \mathbb{N}$.

### 4.1. Array of observables satisfying the Lyapunov condition

Let $E = \{e_1, e_2, e_3\}$, $e_1 = -1$, $e_2 = 0$, $e_3 = 1$, and

$$p_1^j = p_3^j = \frac{1}{2}\left(1 - \frac{1}{j^2}\right), \; p_2^j = \frac{1}{j^2}, \; j = 1, 2, 3, \ldots$$

We define the sequence of observables

$$x_j : \mathcal{B}(\mathbb{R}) \to M, \; j = 1, 2, 3, \ldots$$

for each $A \in \mathcal{B}(\mathbb{R})$ by the formula

$$x_j(A) = \sum_{e_i \in A} p_i^j.$$

We assume that $\{x_j\}_{j\in\mathbb{N}}$ are independent, and for each $n \in \mathbb{N}$, the joint observable $h_n : \mathcal{B}(\mathbb{R}^n) \to M$ of $\{x_j\}_{j=1}^n$, as the product measure, is described, for each $A \in \mathcal{B}(\mathbb{R}^n)$, by the formula

$$h_n(A) = \sum_{\left(e_{i_1}, e_{i_2}, \ldots, e_{i_n}\right) \in A} p_{i_1}^1 p_{i_2}^2 \ldots p_{i_n}^n, \;\; i_1, i_2, \ldots, i_n \in \{1, 2, 3\}.$$

Then for $j \in \mathbb{N}$ and $\delta > 0$,

$$\mathbb{E}(x_j) = 0 \text{ and } \mathbb{E}\left(x_j^2\right) = \mathbb{E}\left(|x_j|^{2+\delta}\right) = 1 - \frac{1}{j^2}.$$

Therefore, for $n \geq 2$,

$$s_n > \sqrt{n - \frac{\pi^2}{6}}, \quad \sum_{j=1}^n \mathbb{E}\left(|x_j|^{2+\delta}\right) < n$$

and

$$\frac{1}{s_n^{2+\delta}} \sum_{j=1}^n \mathbb{E}\left(|x_j|^{2+\delta}\right) \leq \frac{n}{\left(n - \frac{\pi^2}{6}\right)^{1+\frac{\delta}{2}}} \xrightarrow[n\to\infty]{} 0.$$

Finally, by Theorem 3.9,

$$\frac{x_1 + x_2 + \cdots + x_n}{s_n} \xrightarrow[n\to\infty]{} N(0,1)$$

in distribution.

### 4.2. Array of observables with not well-defined Lyapunov's condition

Let, for each $a \in \left[\frac{1}{2}, 1\right]$,

$$f^{(a)}(t) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i^{(a)}(t),$$

where

$$f_i^{(a)}(t) = \frac{(2i+1)\,a^{2+\frac{1}{i}}}{2i|t|^{3+\frac{1}{i}}} I_{|t|>a}(t)$$

for each positive integer $i$. Let an observable $x^{(a)} : \mathcal{B}(\mathbb{R}) \to M$ be described by the equality

$$x^{(a)}(A) = \int_A f^{(a)}(t)\,\mathrm{d}t.$$

Then $\mathbb{E}\left(x^{(a)}\right) = 0$, $\mathbb{D}^2\left(x^{(a)}\right) = \mathbb{E}\left(\left(x^{(a)}\right)^2\right) = a^2 \sum_{i=1}^{\infty} \frac{2i+1}{2^i}$, and

$$\frac{1}{4}\mathbb{E}\left(\left(x^{(1)}\right)^2\right) \le \mathbb{E}\left(\left(x^{(a)}\right)^2\right) \le \sum_{i=1}^{\infty} \frac{2i+1}{2^i} = \mathbb{E}\left(\left(x^{(1)}\right)^2\right) < \infty. \tag{15}$$

Moreover, for each $b > 1$,

$$\mathbb{E}\left(\left(x^{(a)}\right)^2 I_{|x^{(a)}|>b}\right) = a^2 \sum_{i=1}^{\infty} \frac{2i+1}{2^i}\left(\frac{a}{b}\right)^{\frac{1}{i}}$$

and therefore

$$\frac{1}{8}\mathbb{E}\left(\left(x^{(1)}\right)^2 I_{|x^{(1)}|>b}\right) \le \mathbb{E}\left(\left(x^{(a)}\right)^2 I_{|x^{(a)}|>b}\right) \le \mathbb{E}\left(\left(x^{(1)}\right)^2 I_{|x^{(1)}|>b}\right) < \infty. \tag{16}$$

Let $a_j = \frac{1}{2} + \frac{1}{2^j}$, $j = 1, 2, \ldots$The sequence of observables

$$x_j : \mathcal{B}(\mathbb{R}) \to M,\ j = 1, 2, 3, \ldots$$

for each $A \in \mathcal{B}(\mathbb{R})$ has the form

$$x_j(A) = \int_A f^{(a_j)}(t)\,\mathrm{d}t.$$

We assume that $\{x_j\}_{j\in\mathbb{N}}$ are independent. For each $n \in \mathbb{N}$, the joint observable $h_n : \mathcal{B}(\mathbb{R}^n) \to M$ of $\{x_j\}_{j=1}^{n}$ is, similarly as before, the product measure, and for each $A \in \mathcal{B}(\mathbb{R}^n)$,

$$h_n(A) = \int_A f^{(a_1)}(t_1)\, f^{(a_2)}(t_2) \ldots f^{(a_n)}(t_n)\,\mathrm{d}t_1 dt_2 \ldots \mathrm{d}t_n.$$

From (15)

$$\frac{n}{4}\mathbb{E}\left(\left(x^{(1)}\right)^2\right) \le s_n^2 \le n\mathbb{E}\left(\left(x^{(1)}\right)^2\right)$$

and from (16)

$$\frac{n}{8}\mathbb{E}\left(\left(x^{(1)}\right)^2 I_{|x^{(1)}|>b}\right) \le \sum_{j=1}^{n} \mathbb{E}\left(\left(x^{(a_j)}\right)^2 I_{|x^{(a_j)}|>b}\right) \le n\mathbb{E}\left(\left(x^{(1)}\right)^2 I_{|x^{(1)}|>b}\right).$$

Let $\varepsilon > 0$ and $n$ be large enough to fulfill the inequality $\varepsilon s_n > 1$. Then

$$\lim_{n\to\infty} L_n(\varepsilon) \leq \lim_{n\to\infty} \frac{n\mathbb{E}\left(\left(x^{(1)}\right)^2 I_{|x^{(1)}|>\varepsilon s_n}\right)}{\frac{n}{4}\mathbb{E}\left(\left(x^{(1)}\right)^2\right)} = \frac{4}{\mathbb{E}\left(\left(x^{(1)}\right)^2\right)} \lim_{n\to\infty} \mathbb{E}\left(\left(x^{(1)}\right)^2 I_{|x^{(1)}|>\varepsilon s_n}\right) = 0$$

by the Dominated Convergence Theorem. By Theorem 3.7 for the considered array of observables, their scaled row sums converge to standard normal distribution. Simultaneously, the Lyapunov condition is not well-defined.

Indeed, for each $\delta > 0$, there exists a positive integer $i$ such that $\frac{1}{i} < \delta$ and

$$\mathbb{E}\left(|x^{(a_j)}|^{2+\delta}\right) \geq C(a_j, i) \int_{a_j}^{\infty} t^{-1-\frac{1}{i}+\delta}\,\mathrm{d}t = \infty,$$

where $C(a_j, i)$ is a positive constant. Thus $\mathbb{E}\left(|x^{(a_j)}|^{2+\delta}\right)$ does not exist.

## 5. CONCLUSIONS

In the paper we formulated and proved MV-algebraic versions of the Lindeberg CLT, the Lyapunov CLT, and the Feller theorem. The mentioned theorems are essential generalizations of the known result concerning convergence in distribution of scaled sums of independent, identically distributed observables. In particular, the theory considered in the paper can be used for the MV-algebras of fuzzy sets, which are special cases of the general MV-algebras. We presented examples of applications of the obtained by us results for not identically distributed observables. Our future work will concern further development of MV-algebraic probability theory and studying its applicability to Atanassov's IFS.

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