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POSET-VALUED PREFERENCE RELATIONS

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In decision processes some objects may not be comparable with respect to a preference relation, especially if several criteria are considered. To provide a model for such cases a poset valued preference relation is introduced as a fuzzy relation on a set of alternatives with membership values in a partially ordered set. We analyze its properties and prove the representation theorem in terms of particular order reversing involution on the co-domain poset. We prove that for every set of alternatives there is a poset valued preference whose cut relations are all relations on this domain. We also deal with particular transitivity of such preferences.

Keywords: relation, poset, order reversing involutions, weakly orthogonal poset, transitivity

Classification: 03G10, 91B08

1. INTRODUCTION

1.1. Motivation; historical background

Preference relations are a convenient tool for expressing the result of a pairwise comparison on a set of alternatives [6]. They appear in game theory [10], voting theory [17, 33], psychological studies on preference and discrimination in decision-making methods [9]. Next, in group decision making, preference relations represent collective preferences and are built from individual preferences, either by aggregation methods [16], or by consensus-reaching processes [27]. Finally, preferences are important tools in social choice theory [5, 10, 26, 29, 33].

The decisions in the mentioned processes can be either strict, in case a decision-maker is sure about her/his preference of one alternative to another, or non-strict, when the grade of preference of an alternative x to an alternative y is given by a number $R(x, y)$ from some numerical scale, usually the unit interval. However, even this may be sometimes too restrictive, namely in case when the decision statements cannot be ordered in a reasonable way.

Let us suppose we are interested in comparing products x and y in accordance to their design and functionality. Then the different possibilities are detailed in the following statements:

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- D^-F^-: x is preferred to y neither in design nor in functionality,
- D^-F^0: x is not preferred to y in design and the functionality of both is approximately equal,
- D^0F^-: the design of both is approximately equal and x is not preferred to y in functionality,
- D^-F^+: x is not preferred to y in design, but it is preferred to y in functionality,
- D^0F^0: the design and functionality of both are approximately equal,
- D^+F^-: x is preferred to y in design, but not in functionality,
- D^0F^+: the design of both is approximately equal and x is preferred to y in functionality,
- D^+F^0: x is preferred to y in design and the functionality of both is approximately equal,
- D^+F^+: x is preferred to y both in design and functionality.

These statements can be represented as elements of the ordered set (poset) given by the diagram in Figure 1. The usual order $p \leq q$ in this poset can be interpreted as "The statement p is less favorable to the preference of x towards y than the statement q".

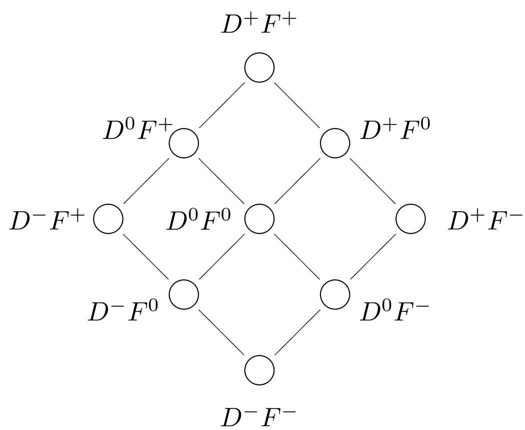


Fig. 1. Representation of the different statements associated to the comparison of two products.

In this example it is shown that there may be cases, when not all pairs of decision statements are comparable. A practical example where a total order can make no sense appears e. g, in prevention of work risks, when alternatives could be ordered from different points of view. Thus, for instance, in human reliability, it is reasonable to order the accidents in a company made by human errors in accordance to different criteria applied to the consequences: economical damage, human lives damage, company reputation damage, or environment damage. In this context, a group of experts can evaluate the risks according to these four points of view and obtain, for any pair of accidents, a vector in $\mathbb{R}^4$, which represents the intensity of the relation (one accident is more important than the other) with respect to different damage criteria. Obviously, these intensities could not be totally ordered.

The situation presented in the example above is a two-criteria problem where each criterion is linearly ordered. Such situations could be treated by usual multi-criteria methods. However, if we do not allow the option that the design and functionality of both products are approximately equal (i. e., the situation when there is no preference), then, we obtain essentially different situation, when a proper partially ordered set which is not a lattice (and not linearly ordered) naturally arises.

Now, the different options are presented in the Figure 2 (all possibilities instead of D^0F^0 above).

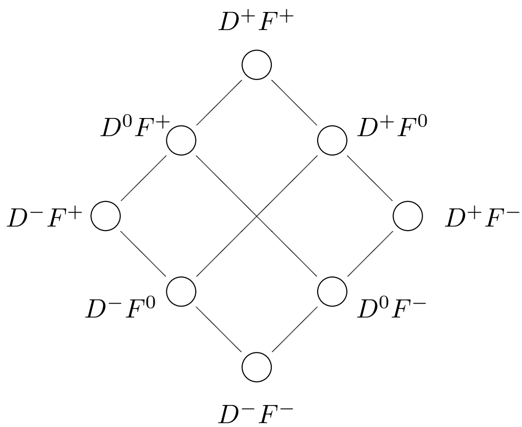


Fig. 2. Representation of different statements associated to the comparison of two products without the option of equivalent preferences.

- D^-F^-: x is preferred to y neither in design nor in functionality,
- D^-F^0: x is not preferred to y in design and the functionality of both is approximately equal,
- D^0F^-: the design of both is approximately equal and x is not preferred to y in functionality,
- D^-F^+: x is not preferred to y in design, but it is preferred to y in functionality,
- D^+F^-: x is preferred to y in design, but not in functionality,
- D^0F^+: the design of both is approximately equal and x is preferred to y in functionality,
- D^+F^0: x is preferred to y in design and the functionality of both is approximately equal,

- D^+F^+: x is preferred to y both in design and functionality.

Other examples where preference relations are not defined on a totally ordered set appear in the case of binary relations evaluated in a linguistic scale, usually called linguistic preference relations ([21, 23, 24, 25]). A framework to reach consensus in group decision making under linguistic assessments was proposed in [23]; the linguistic ordered weighted averaging operators to aggregate linguistic preference relations was presented in [19, 21, 24]; the problem of finding a solution set of alternatives from a collective linguistic preference relation was analyzed in [20]; finally, the satisfying consistency of linguistic preference relations was discussed in [11, 12].

As it follows from the above examples, a bounded partially ordered set is worth studying as the range (set of values) for certain preference relations.

1.2. Structure of the paper

The main purpose of this paper is to introduce in a coherent way the concept of a preference relation defined on a poset. This definition should be a generalization of the preference or reciprocal relations defined on $[0, 1]$, which were introduced by Bezdek et al. [1] and later reinterpreted by Nurmi [32]. In order to do that, the paper is organized as follows. In Section 2 we recall some definitions and previous results and introduce our main order theoretic notion, *weakly ortocomplemented poset*. Section 3 is devoted to the concept of poset-valued reciprocal preference relation. Its basic properties are analyzed and a representation theorem (in terms of ordered structures) for such relations is given. In Section 4 we present a cutworthy study of these relations. We prove that every binary relation on some domain X can be a cut of particular poset valued preference on X. In Section 5 we analyze transitivity of poset-valued reciprocal preference relations.

2. PRELIMINARIES

Here we recall some relevant notions from order theory, we introduce some special posets, and we give a brief introduction to poset valued relations.

2.1. Partially ordered sets

The main notion here is a (**partially**) **ordered set**, a **poset**, $(P, \leq)$, i. e. a nonempty set P with a reflexive, antisymmetric and transitive relation. As usual, we use a notation $a < b$ for $a \leq b$ and $a \neq b$. A poset $(P, \leq)$ is **linearly**, or **totally** ordered, a **chain**, if every two elements in P are comparable ($x \leq y$ or $y \leq x$). A poset is **bounded** if it has the greatest element, the top, denoted by 1, and the smallest element, the bottom, denoted by 0.

The greatest lower bound (or the infimum) of some elements $a, b \in P$ under the ordering relation $\leq$, (if it exists) is denoted by $\wedge$. Dually, the least upper bound (or the supremum) of elements $a, b \in P$ under the ordering relation $\leq$, (if it exists) is denoted by $\vee$.

A **sub-poset** of a given poset $(P, \leq)$ is a nonempty subset Q of P, ordered by the restriction of order $\leq$ to Q. A sub-poset may be a chain if its elements are linearly ordered, and it is an **anti-chain** if it does not contain distinct comparable elements. An

atom in the the poset $(P, \leq)$ with the bottom element 0, is an element $a \in P$ such that $0 < a$ and there is no $x \in P$ such that $0 < x < a$. Dually, in a partially ordered set with the top element 1, a **co-atom** is an element $b \in P$, such that $b < 1$ and there is no $x \in P$ such that $b < x < 1$.

A unary operation $^{\perp} : P \to P$ on P is an **involution** if for all $x \in P$, $(x^{\perp})^{\perp} = x$. If $\leq$ is a partial order over P, a unary operation is called **antitone** if $x \leq y$ implies $y^{\perp} \leq x^{\perp}$ for all $x, y \in P$. An antitone involution is also called an **order reversing involution**. Obviously, for bounded posets, $0^{\perp} = 1$ and $1^{\perp} = 0$. An element $x \in P$ is a **fixed point** of the unary operation $^{\perp}$ if $x = x^{\perp}$.

2.2. Weakly orthocomplemented poset

Let us recall (see [14]) that an **orthocomplemented poset** $(P, \leq, ^{\perp}, 0, 1)$ is a poset $(P, \leq)$, equipped with the top 1 and a bottom 0, and with an antitone involution $^{\perp}$ over P such that for all $x \in P$, the join $x \vee x^{\perp}$ exists and $x \vee x^{\perp} = 1$. In an orthocomplemented poset the notion of orthogonality is introduced as follows. Two elements $x, y \in P$ are called **orthogonal** if $x \leq y^{\perp}$ (for more, see [2, 35]).

We introduce the structure which is used throughout this investigation.

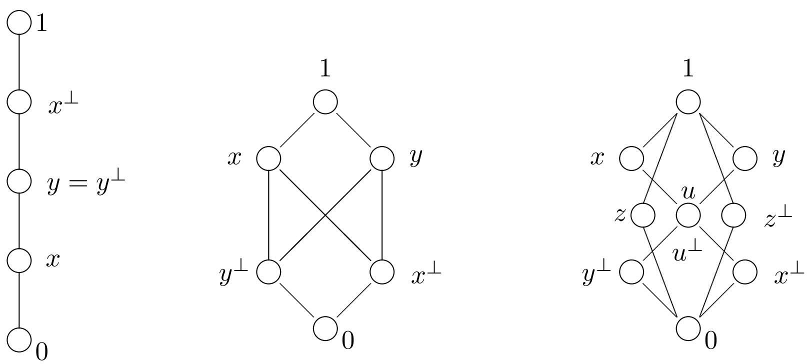
Let $(P, \leq, ^{\perp}, 0, 1)$ be a bounded poset with a unary operation $^{\perp}$, satisfying the following: for all $x, y \in P$

(i) $x^{\perp\perp} = x$;

(ii) $x \leq y$ implies $y^{\perp} \leq x^{\perp}$;

(iii) if x is not comparable with $x^{\perp}$, then the supremum $x \vee x^{\perp}$ exists and $x \vee x^{\perp} = 1$.

We call such an ordered structure a **weakly orthocomplemented poset**.



(a) (b) (c)

Fig. 3.

Obviously, the first two conditions determine the unary operation as an antitone involution. Condition (iii) is weaker then the corresponding one for orthocomplemented posets. By (i) and (ii), we have that $0^\perp = 1$ and $1^\perp = 0$. In addition, if $x \vee x^\perp$ exists and $x \vee x^\perp = 1$, then also $x \wedge x^\perp$ exists and $x \wedge x^\perp = 0$ (the proof is straightforward). By (iii), this implication holds for all non-comparable $x, x^\perp \in P$. Observe that there might exist fixed points under $^\perp$, and also that x could be comparable with $x^\perp$ and the posets presented by diagrams admit also other antitone involutions.

Let us mention that another similar generalization of an ortocomplemented poset is an *orthogonal poset* introduced by Chajda in [3].

Examples of weakly orthocoplemented posets are Boolean lattices, orthcomplemented posets (see [14]), bounded chains. Weakly orthocoplemented posets which do not belong to the mentioned classes are depicted in Figures 3 (b), (c) and Figure 4.

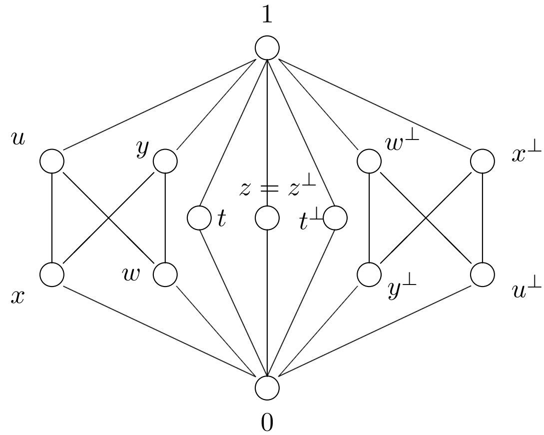


Fig. 4.

For our purposes, important weakly orthocomplemented posets are those which have at least one fixed point under involution. All posets in Figures 3 and 4 possess this property except the one in Figure 3 (b).

2.3. Poset valued relations

In our investigation we deal with mappings from a non-empty set X (domain) into a poset P (co-domain) [38]. For such a mapping, we use the term a P**-valued set**, a P**-fuzzy set** or just a **fuzzy set**, when there is no ambiguity.

Special cases of this notion are obtained when P is a complete lattice (L-fuzzy sets [18]) or the unit interval $[0, 1]$ of real numbers (classical fuzzy sets, [44]). Throughout the present text P is a poset, and any additional properties of P are explicitly stated.

If $\mu : X \to P$ is a fuzzy set on X then, for $p \in P$, the set

$$\mu_p := \{x \in X \mid \mu(x) \geq p\}$$

is said to be the p**-cut**, a **cut set** or simply a **cut** of μ.

Particular fuzzy sets are fuzzy binary relations. Namely, a P-fuzzy set $R : X \times X \to P$ is a P**-valued** (binary) **relation** [34, 39] on X.

For $p \in P$, a p-cut R_p of a fuzzy relation R on X is a classical relation on X: for $a, b \in X$,

$$(a, b) \in R_p \text{ if and only if } R(a, b) \geq p.$$

In the collection $\mathcal{R}_P$ of all cuts of a fuzzy relation R, the following hold (see [38, 40, 41]):

(a) If $p \leq q$, then $R_q \subseteq R_p$.

(b) For $a, b \in X$

$$R(a, b) = \bigvee \{p \in P \mid (a, b) \in R_p\}.$$

(The join on the right exists in $(P, \leq)$ for all $a, b \in X$ and is equal to $R(a, b)$.)

(c) If $Q \subseteq P$, and there exists a supremum of Q ($\bigvee\{p \mid p \in Q\}$), then

$$\bigcap \{R_p \mid p \in Q\} = R_{\bigvee\{p \mid p \in Q\}}.$$

(d) $\bigcup\{R_p \mid p \in P\} = X \times X$.

(e) For any $(a, b) \in X \times X$,

$$\bigcap \{R_p \mid (a, b) \in R_p\} \in \mathcal{R}_P.$$

3. POSET VALUED PREFERENCE RELATIONS

3.1. Definition and basic properties

As mentioned, classical preferences are mostly investigated in connection with various applications. In general, preferences are binary relations on a set of alternatives. A preference on a set of alternatives A is often investigated within the framework of a preference structure - ordered triple (P, I, J), in which P is a strict preference, I indifference and J incomparability relation on A (see e. g [37]). More about this approach to classical preferences can be found in e. g [15].

Associated to any classical preference structure without incomparable elements we could consider a three-valued binary relation R such that $R(a, b) = 1$ if $(a, b) \in P$, $R(a, b) = 1/2$ if $(a, b) \in I$ and $R(a, b) = 0$ if $(b, a) \in P$. A more realistic description of relations can be obtained if we consider R taking values on $[0, 1]$ instead of just $\{0, 1/2, 1\}$. This idea was already used in the fifties (see Menger [30]). Then, so-called probabilistic relations have been studied and applied for decision making, mathematical psychology, etc. They are often also called reciprocal or ipsodual relation. These relations are frequently used in representation of various relational preference models (see, for instance, [6, 13, 42]). In the sequel we briefly present this notion.

Let X be the set of alternatives. The mapping $R : X \times X \to [0, 1]$ is a **reciprocal relation** on X if for any $a, b \in X$

$$R(a, b) + R(b, a) = 1$$

or, equivalently, $R(a,b) = R^c(b,a)$, where R^c denotes the complement of R ($R^c(x,y) = 1 - R(x,y)$), [15].

The definition above implies that for any $a \in X$, $R(a,a) = \frac{1}{2}$. Hence the value $\frac{1}{2}$ represents indistinguishability for $[0,1]$-valued reciprocal relations. Moreover, for any $a, b \in X$, one of the numbers $R(a,b)$, $R(b,a)$ is in the interval $[0, \frac{1}{2}]$, while the other is in $[\frac{1}{2}, 1]$. Also, for $a, b, c, d \in [0,1]$, we have that $R(a,b) \leq R(c,d)$ implies $R(d,c) \leq R(b,a)$. Hence, in this approach:

- $R(a,b) = 1/2$ indicates indifference between a and b,
- $R(a,b) = 1$ indicates that a is absolutely preferred to b, and
- $R(a,b) > 1/2$ indicates that a is preferred to b in some degree.

Motivated by the above and by possibilities of wider applications (our starting example), we are interested in fuzzy relations with values in a bounded poset instead of $[0,1]$ interval. We define a new preference structure, switching to infimum and supremum in an order, instead of using addition. Our aim is to obtain preferences with properties analogue (as much as possible) to the above ones, but having additional possibilities for applications.

From now on, by $(P, \leq)$ (shortly by P) we denote a bounded poset with the bottom element 0 and the top element 1.

Definition 3.1. Let X be a nonempty set (universe of objects) and P a bounded poset. The mapping $R : X \times X \to P$ is a **poset-valued reciprocal preference relation** (a P-valued preference relation) on X if for any $a, b, c, d \in X$,

$$R(a,b) \leq R(c,d) \text{ implies } R(d,c) \leq R(b,a) \quad \text{and} \tag{1}$$

$$\text{if } R(a,b) \text{ and } R(b,a) \text{ are not comparable, then } R(a,b) \vee R(b,a) = 1. \tag{2}$$

As an immediate consequence, we get that if the mapping $R : X \times X \to P$ is a P-valued preference relation, then for any $a, b, c, d \in X$ we have the following equivalence:

$$R(a,b) \leq R(c,d) \textit{ if and only if } R(d,c) \leq R(b,a).$$

As another consequence we have the following: $R(a,b)$ is incomparable with $R(c,d)$ if and only if $R(b,a)$ is incomparable with $R(d,c)$.

Using the fact that $R(a,b) = R(c,d)$ is equivalent with $R(a,b) \leq R(c,d)$ and $R(c,d) \leq R(a,b)$, directly from the definition we have the following property for poset-valued preference relations.

Lemma 3.2. Let X be a nonempty set and let P be a poset. If the mapping $R : X \times X \to P$ is a P-valued preference relation on X, then for all $a, b, c, d \in X$, we have that

$$R(a,b) = R(c,d) \Longleftrightarrow R(d,c) = R(b,a).$$

For P-valued preference relations in general, there is no single value representing indistinguishability, corresponding to the value $\frac{1}{2}$ from the unit interval case. However, from the definition of a P-valued preference relation, we obtain a kind of a set of equilibria, as follows:

Lemma 3.3. Let X be a nonempty set and P a poset. If the mapping $R : X \times X \to P$ is a P-valued preference relation on X, then for any $a, b \in X$ the values $R(a,a)$ and $R(b,b)$ are either equal or incomparable.

By the definition, for $a, b \in X$, if $R(a,b) \leq R(c,c)$ for some $c \in X$, then

$$R(a,b) \leq R(c,c) \leq R(b,a).$$

Obviously, by transitivity, it follows that $R(a,b) \leq R(b,a)$. Using the contraposition we obtain also the following.

Proposition 3.4. Let R be a poset valued preference relation on X. For any $a, b \in X$, if $R(a,b)$ and $R(b,a)$ are incomparable, then for every $c \in X$, both $R(a,b)$ and $R(b,a)$ are incomparable with $R(c,c)$.

Further, if $R(a,b) \leq R(c,c)$ and $R(b,a) \leq R(d,d)$ for some $c, d \in X$, then $R(a,b) = R(b,a) = R(c,c) = R(d,d)$.

By the above analysis, whenever $R(a,b)$ and $R(c,c)$ are comparable, then $R(c,c)$ is between $R(a,b)$ and $R(b,a)$, with respect to the order in P.

Thus, the class $\{R(a,a) \mid a \in P\}$ is a kind of equilibrium for R.

In particular, if the poset of membership values is a chain, then all this "medium" values coincide, as follows.

Corollary 3.5. Let R be a P-valued preference relation on X, where P is a bounded chain. Then, for all $a, b, c \in X$, $a \neq b$,

$$R(a,b) \leq R(c,c) \leq R(b,a) \text{ or } R(b,a) \leq R(c,c) \leq R(a,b) \text{ and}$$

$$R(a,a) = R(b,b).$$

The proof follows directly from the definition of a poset-valued preference relation, and from the fact that in the chain all elements are comparable.

An example for the statement in Corollary 3.5 is the case when P is a linguistic term set, that is, an ordered structure providing the term set distributed on a scale with a total order [22, 43]. E.g., a set of seven terms, P, could be given as follows:

$P = \{p_0$: none, p_1 : very low, p_2 : low, p_3 : medium, p_4 : high, p_5 : very high, p_6 : perfect$\}$.

In this case the order is total, hence $R(a,a) = R(b,b)$ for all $a, b \in X$, therefore the equilibrium is just one point. Obviously, we have the same situation if the membership values structure is the real interval $P = [0,1]$.

Remark 3.6. When dealing with fuzzy notions, one can analyze the classical case, taking the membership values structure to be the two-element poset. In our case of poset valued preferences, the crisp version of formula (1) is a simplified version of the notion that we intend to model, as explained in our motivating example. Namely, for the classical preferences, usually either aRb or bRa but not both, while in the poset valued case, $R(a,b)$ and $R(b,a)$ could be two different values, which is much more appropriate way to deal with preference structures. In addition, we can compare P-valued preferences with fuzzy preferences having numerical values.

3.2. Properties of the set of membership values

A P-valued preference has a particular impact on the structure of membership values.

For a P-valued relation $R : X \times X \to P$, by $\mathrm{Ran}(R)$ we denote the range of R, i. e., the sub-poset of P, consisting of membership values under R:

$$\mathrm{Ran}(R) = \{p \in P \mid p = R(a,b) \text{ for some } a,b \in X\}.$$

Proposition 3.7. Let R be a P-valued preference relation on X with values in a partially ordered set P. For $a,b \in X$, define

$$R(a,b)^{\perp} := R(b,a). \tag{3}$$

Then, $^{\perp}$ is an order reversing involution on a sub-poset $\mathrm{Ran}(R)$ of P. In addition, for any $a,b \in X$, $R(a,b) \vee R(a,b)^{\perp}$ exists and for non-comparable $R(a,b)$ and $R(a,b)^{\perp}$,

$$R(a,b) \vee R(a,b)^{\perp} = 1. \tag{4}$$

Proof. If $p \in \mathrm{Ran}(R)$, then $p = R(a,b)$, for some $a,b \in X$. Then $p^{\perp} = R(b,a)$. If in addition $p = R(c,d)$ for some $c,d \in X$, then by Lemma 3.2 we have $R(d,c) = R(b,a)$, and $^{\perp}$ is well defined as a unary operation on $\mathrm{Ran}(R)$.

Moreover, by (3) and by the definition of a poset-valued preference relation, for any $p,q \in \mathrm{Ran}(R)$, the following hold:

$$p^{\perp\perp} = p \quad \text{and} \quad p \le q \implies q^{\perp} \le p^{\perp}.$$

Hence $^{\perp}$ is an order reversing involution on $\mathrm{Ran}(R)$.

The existence of the supremum of $R(x,y)$ and $R(x,y)^{\perp}$ as well as property (4) follow directly by the definition of a P-valued preference relation. □

Corollary 3.8. If R is a P-valued preference relation on X, then the sub-poset $\mathrm{Ran}(R) \cup \{0,1\}$ of P is a weakly orthocomplemented poset under the unary operation $^{\perp}$, defined by $R(a,b)^{\perp} = R(b,a)$, and with constant 0 and 1 being respectively the bottom and the top of P, so that $0^{\perp} = 1$.

Remark 3.9. By the definition of the operation $^{\perp}$ on $\mathrm{Ran}(R)$, for every $a \in X$ we have $R(a,a)^{\perp} = R(a,a)$. Hence, for each $p \in \mathrm{Ran}(R)$ which is a value of $R(a,a)$ for some $a \in X$, we have $p^{\perp} = p$, i. e., p should be a *fixed point* of this operation on $\mathrm{Ran}(R)$.

3.3. Representation theorem

In the following we present a representation theorem for poset-valued preferences that generalize Proposition 3.7. Actually, we show that the essential property of poset valued preferences is the order reversing involution defined on its range.

Theorem 3.10. Let $(P, \leq)$ be a bounded poset with the top 1 and the bottom 0, and $X \neq \emptyset$. Let $R : X \times X \to P$ be a P-valued relation.

Then, R is a P-valued preference relation on X if and only if there is an order reversing involution ${}^{\perp}$ on $Q = \mathrm{Ran}(R) \cup \{0, 1\}$, such that the $(Q, \leq, {}^{\perp}, 0, 1)$ is a weakly orthocomplemented poset, fulfilling the following:

$$\text{If } R(x, y) = p \text{ for some } x, y \in X, \text{ then } R(y, x) = p^{\perp}. \tag{5}$$

Proof. Suppose that a relation $R : X \times X \to P$ is a P-valued preference relation. Then, we define a unary operation ${}^{\perp}$ on Q as follows:

If $q \in Q$ and $q = R(a, b)$, then $q^{\perp} := R(b, a)$, and $0^{\perp} = 1$, $1^{\perp} = 0$.

By Proposition 3.7, ${}^{\perp}$ is a well defined order reversing involution on Q, and axioms of weakly orthocomplemented poset are satisfied on Q.

To prove the converse, suppose that R is a P-valued relation on X, such that an order reversing unary operation ${}^{\perp}$ is defined on $Q = \mathrm{Ran}(R) \cup \{0, 1\}$, so that $(Q, \leq, {}^{\perp}, 0, 1)$ is a weakly orthocomplemented poset. Then, R is a poset valued preference relation. Indeed, if $R(a, b) = p$ for some $a, b \in X$, $a \neq b$, then $R(b, a) = p^{\perp}$. Since ${}^{\perp}$ is an order reversing involution, we get

$$R(a, b) \leq R(c, d) \text{ implies } R(d, c) \leq R(b, a).$$

In the case $a = b$, by (5) we have that $R(a, a) = R(a, a)^{\perp}$, which proves (1). Further, suppose that for some $a, b \in X$, $R(a, b)$ and $R(b, a)$ are not comparable. By property (iii) of weakly ortocomplemented posets, we get that (2) holds. □

As a consequence, we have a characterization theorem for posets having an order reversing involution with fixed points, in terms of poset valued preferences.

Corollary 3.11. Let P be a bounded poset such that there exists a P-valued preference relation R with $\mathrm{Ran}(R) = P$. Then, there is a unary operation ${}^{\perp}$ on P under which P is a weakly orthocomplemented poset with a nonempty set of fixed points.

Following the case of $[0, 1]$-valued relations (see [9]), we introduce a particular P-valued preference relation.

Definition 3.12. Let X be a nonempty set and let $(P, \leq, {}^{\perp}, 0, 1)$ be a weakly orthocomplemented poset. A mapping $R : X \times X \to P$ is a P**-valued probabilistic preference relation** on X if for any $a, b \in X$ we have that

$$R(b, a) = R(a, b)^{\perp}. \tag{6}$$

From Corollary 3.8 and Theorem 3.10, it is straightforward that this notion is equivalent to the previous one. Namely, we have the following.

Corollary 3.13. Let X be a nonempty set (universe of objects) and let $(P, \leq, ^\perp, 0, 1)$ be a bounded poset with a unary operation $^\perp$. Then, the mapping $R : X \times X \to P$ is a poset-valued probabilistic preference relation on $Ran(R) \cup \{0, 1\}$ if and only if it is a poset-valued preference relation on $Ran(R) \cup \{0, 1\}$.

Thus, a reciprocal fuzzy relation on $[0, 1]$ is a $[0, 1]$-valued preference relation with the order reversing involution: $x^\perp = 1 - x$, that is, the complement. Therefore, the concept introduced here over poset generalizes the classical fuzzy notion.

4. CUTS OF PREFERENCES

Cut sets and relations are known as a useful tool for investigation of fuzzy structures.

A poset valued preference is not a cut-worthy notion, i. e., the complexity of its fuzzy properties is not preserved by cuts. In Example 4.1 we show that the crisp version of property (1), need not be true for cut relations.

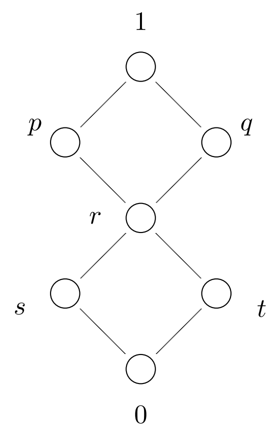


Fig. 5.

Example 4.1. Let P be a poset represented in Figure 5. An order reversing involution $^\perp$ on P is given by

x	1	p	q	r	s	t	0
$x^\perp$	0	s	t	r	p	q	1

Let $X = \{a, b, c, d\}$ and let $R : X \times X \to P$ be defined by the table.

R	a	b	c	d
a	r	p	p	p
b	s	r	1	1
c	s	0	r	q
d	s	0	t	r

This relation is a P-valued preference by Theorem 3.10.

Now, we consider the cut relation R_s as a characteristic function $R_s : X \times X \to \{0, 1\}$, where

$$R_s(a,b) = \begin{cases} 1 & \text{for } R(a,b) \geq s, \\ 0 & \text{otherwise.} \end{cases}$$

We can easily check that the implication

$$R_s(x,y) \leq R_s(z,t) \implies R_s(t,z) \leq R_s(y,x)$$

is not true in general, since
$R_s(c,d) = R_s(a,b) = R_s(b,a) = 1$ and $R_s(d,c) = 0$. □

Next we prove that for every finite set of alternatives there exists a poset valued preference relation whose cuts are all crisp relations on this domain.

Theorem 4.2. Let X be a finite nonempty set. Then, there is a poset P and a poset valued fuzzy preference relation $R : X \times X \to P$, such that every relation on X is a cut of R.

P r o o f. Let n be a cardinality of X. Let P be a Boolean lattice with n^2 atoms, and let C be the set of co-atoms of P. Further, let $f : X \times X \to C$ be a bijection from $X \times X$ to C. Now, we define a poset valued relation $R : X \times X \to P$, with $R(x,y) := f(x,y)$. This relation is a preference relation, since all the elements from the range of the function are incomparable, and for any $a, b, c, d \in X$, the implication

$$R(a,b) \leq R(c,d) \implies R(d,c) \leq R(b,a)$$

is trivially true. In addition,since all the images are co-atoms in P, we have also that for all $a, b \in X$

$$R(a,b) \vee R(b,a) = 1.$$

Here the range of the relation is an anti-chain C, and since R is a one to one function, we can define an order reversing unary operation on $C \cup \{0, 1\}$ as in Theorem 3.10.

Now, we analyze the cuts of this poset valued relation.

Let $Q \subseteq X \times X$ be an arbitrary non-empty relation on $X \times X$. We consider an element

$$\alpha = \bigwedge_{(a,b) \in Q} R(a,b).$$

It is straightforward to check that $R_\alpha = Q$. For $Q = \emptyset$, we have $R_1 = \emptyset$, where 1 is the top element of P. □

5. TRANSITIVITY

As already mentioned, transitivity is a frequent property of preference structures which are used for modeling in different fields. In particular, in the framework of $[0, 1]$-valued preferences, some types of transitivity have been developed and investigated, specifically for reciprocal relations (e. g, various types of stochastic transitivity [13, 31, 36]). In this part we mention some possible approaches to transitivity for poset valued preferences.

Recall that a two variable mapping g is said to be **commutative** if for all x, y from the domain, we have that $g(x,y) = g(y,x)$.

Definition 5.1. Let $(P, \leq, ^{\perp}, 0, 1)$ be a weakly orthocomplemented poset and let g be a commutative increasing $P \times P \to P$ mapping. Let also R be a P-valued preference relation on X and

$$P(R) := \{p \in P \mid (\exists a \in X) R(a,a) \leq p\}.$$

Then, we say that R is g**-stochastic transitive** if for any $a, b, c \in X$

$$\left.\begin{array}{l} R(a,b) \in P(R) \\ R(b,c) \in P(R) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} R(a,c) \in P(R) \\ g(R(a,b), R(b,c)) \leq R(a,c). \end{array}\right.$$

Intuitively, $P(R)$ represents the values with a "positive" relationship between the elements of X. In particular, when R is a $[0,1]$-valued preference relation, i. e., when $P = [0,1]$, then $P(R) = [\frac{1}{2}, 1]$.

In the case of P being a bounded chain, by Corollary 3.5, the condition $R(a,b) \in P(R)$ is equivalent with requiring that $R(b,a)$ is orthogonal to itself. Finally, for $P = [0,1]$ and with $^{\perp}$ being the complement, the above condition becomes $R(b,a) \leq 1/2$, or equivalently, $R(a,b) \geq 1/2$. Thus, in the $[0,1]$-case we obtain the usual definition of a g-stochastic transitivity.

Proposition 5.2. If $g_2 \leq g_1$, then g_1-stochastic transitivity implies g_2-stochastic transitivity.

P r o o f. If R is g_1-stochastic transitive, then g_2-stochastic transitivity follows from the definition, by $g_2(R(a,b), R(b,c)) \leq g_1(R(a,b), R(b,c)) \leq R(a,c)$. □

Let us introduce particular cases of g-stochastic transitivities:

- *Strong* stochastic transitivity: $g = \vee$ (supremum).
- *Moderate* stochastic transitivity: $g = \wedge$ (imfimum).
- *Weak* stochastic transitivity: $g = 0$ (constant zero-function).

Clearly, for strong and moderate stochastic transitivity, P should be a lattice.

Proposition 5.3. A P-valued preference relation R on X is weak stochastic transitive if and only if

If $R(a,b) \in P(R)$ and $R(b,c) \in P(R)$ then $R(a,c) \in P(R)$.

P r o o f. Straightforwardly by the fact that the condition $g(R(a,b), R(b,c)) \leq R(a,c)$ is fulfilled in case g is constant zero function. □

Proposition 5.4. Strong stochastic transitivity implies moderate stochastic transitivity, and moderate stochastic transitivity implies weak stochastic transitivity.

P r o o f. The proof follows from the fact that $0 \leq R(a,b) \wedge R(b,c) \leq R(a,b) \vee R(b,c)$. □

The converse implications do not hold in general, as we can see in the following example.

Example 5.5. Let us consider again the linguistic term set P, as follows:

$$P = \{p_0 : \text{none}, p_1 : \text{very low}, p_2 : \text{low}, p_3 : \text{medium}, p_4 : \text{high}, p_5 : \text{very high}, p_6 : \text{perfect}\}$$

with the logical order, under which it is a chain. Let also the order reversing involution on P be defined by $p_i^{\perp} = p_{6-i}$. On a set $X = \{a, b, c\}$ we define the following P-valued relations:

$$\begin{array}{c|ccc} R_i & a & b & c \\ \hline a & p_3 & p_4 & p_i \\ b & p_2 & p_3 & p_5 \\ c & p_{6-i} & p_1 & p_3 \end{array}, \qquad i = 1, \ldots, 6.$$

In this case, $P(R) = \{p_3, p_4\}$, and according to the above particular cases of the operation g, we have that:

- R_6 is strong stochastic transitive, and therefore it is also moderate and weak stochastic transitive.
- R_4 is moderate stochastic transitive, but it is not strong stochastic transitive.
- R_3 is weak stochastic transitive, but it is not transitive in the sense of other two transitivities.
- R_2 is not transitive for any g, since $R(a, b) = p_4$, $R(b, c) = p_5$, hence these two values belong to $P(R)$, but $R(a, c) = p_2 \notin P(R)$.

6. CONCLUSIONS

A model for preferences with values in a partially ordered set is introduced and discussed. It turned out that to maintain reasonable properties of a preference, the values should create a weakly orthocomplemented poset (as a sub-poset of a given poset). After presenting properties of poset valued preferences, we have analyzed transitivity. This property provides a wide field for the future research. Namely, our model is aimed at comparing objects which are not necessarily linearly ordered. Thus, for a set of objects, we are able to create several poset-valued preference relations (possibly with different weakly orthocomplemented posets of the values). However, for decision processes it would be useful to aggregate these into a single structure describing the global preference on the given set. Clearly, the role of transitivity here would be crucial. In particular, we intend to generalize the concept of cycle-transitivity for reciprocal relations ([7, 8]). It could be useful in some applications, since it does not exclude cyclic behavior, as it happens usually for reciprocal relations.

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