FUZZY ORNESS MEASURE AND NEW ORNESS AXIOMS

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We have modified the axiomatic system of orness measures, originally introduced by Kishor in 2014, keeping altogether four axioms. By proposing a fuzzy orness measure based on the inner product of lattice operations, we compare our orness measure with Yager's one which is based on the inner product of arithmetic operations. We prove that fuzzy orness measure satisfies the newly proposed four axioms and propose a method to determine OWA operator with given fuzzy orness degree.

Keywords: aggregation function, OWA operator, orness measure

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1. INTRODUCTION

Aggregation function (or operator) [1, 6] is essential in a variety of theoretical and application areas [7, 10, 16, 17, 18, 19, 23]. Ordered Weighted Averaging (OWA) operators (proposed by Yager [25]) which generalize the or-like and and-like aggregation functions with the aggregation result lying between the Min (and) and Max (or) operators, built a well-known class of aggregation functions. OWA operators proved to be useful in numerous areas [5, 8, 9, 11, 12, 13, 14, 15, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. They can be considered as Choquet integrals with respect to symmetric capacities (see, e. g., [6]).

The corresponding orness measure plays an important role in studies of OWA operators [2, 3, 4, 8, 9, 11, 12, 13, 14, 15, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. The orness measure reflects the or-like or and-like aggregation result of an aggregation function. For an OWA operator with the weighting vector $\mathbf{w} = (w_1, \dots, w_n)$, its orness measure originally introduced by Yager [25] is defined as $\mathrm{orness}^{\mathrm{Y}}(\mathbf{w}) = \frac{1}{n-1}\sum(n-i)w_i$. This standard formula has been mostly used and studied in applications and theoretical studies since it is actually composed by an inner product of two vectors $\mathbf{w}$ and $\mathbf{s}$ using the most well-known operation pair $(+, \times)$ (where the ith coordinate of $\mathbf{s}$ is $s_i = \frac{n-i}{n-1}$). The first definition of orness was proposed by Dujmović [2, 3, 4] as the global average of $\frac{F(\mathbf{x})-\min(\mathbf{x})}{\max(\mathbf{x})-\min(\mathbf{x})}$, where F is an aggregation function. Also the forms of orness have many variations with their own practical explanation and usage. For example, Liu [13] proposed a general Yager-like orness measure

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$\text{orness}^L(\mathbf{w}, h) = \sum h\left(\frac{n-i}{n-1}\right) w_i$. Recently, Kishor et al. proposed four axioms [9] for defining various orness measures. These show that orness can have different definitions respectively its analysis and usage. Therefore, developing a possible new form of orness definition is reasonable and suitable both in applications and theoretical studies.

This paper proposes an orness measure based on a similar inner product as it is the case by Yager, but use lattice operation pair $(\vee, \wedge)$. It also proposes a new axiomatic system for orness measures consisting again of four axioms (loosing a little the fourth axiom by Kishor a strengthening the other three axioms). We show that there are many common properties of orness measures proposed by Yager and the newly introduced in this paper. And we show that these two orness measures based on $(+, \times)$ and $(\vee, \wedge)$, respectively, can (in some situations) supplement each other in order to obtain more reliable orness grades for decision makers.

The rest of this paper is organized as follows. Section 2 provides the preliminaries regarding OWA operators. Section 3 proposes a definition and analyzes the properties of fuzzy orness; also we propose a new axiomatic system of orness measures and prove that fuzzy orness satisfies all newly proposed four axioms. Section 4 discusses a method to determine OWA operator with given fuzzy orness grade. In section 5 we summarize the main results and make conclusions.

2. PRELIMINARIES

We start recalling known notions and facts with the definition of an OWA operator.

Definition 2.1. (Yager [25], Yager and Filev [34]) Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ be an unordered n-tuple to be aggregated. An OWA operator of dimension n is a mapping $F : [0,1]^n \to [0,1]$ which has an associated weighting n-tuple $\mathbf{w} = (w_1, w_2, \dots, w_n) \in [0,1]^n$ satisfying $\sum\limits_{i=1}^{n} w_i = 1$, such that

$$F(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where b_j is the jth largest value of $(a_1, a_2, \dots, a_n)$.

Using vector form, (1) can be rewritten into $F(a_1, a_2, \dots, a_n) = \mathbf{w}\mathbf{b}^T$, where $\mathbf{b} = (b_1, b_2, \dots, b_n) = (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)})$ is the ordered (decreasing) form of inputs, i.e.,

$$b_1 \geq b_2 \geq \cdots \geq b_n \quad \left(\text{or} \quad a_{\sigma(1)} \geq a_{\sigma(2)} \geq \cdots \geq a_{\sigma(n)}\right),$$

σ is a permutation on $\{1, 2, \dots, n\}$.

Remark 2.2. Because of formula (1) we can identify an OWA operator with its weighting n-tuple. For this reason in the rest of the paper we will often write just briefly '*an OWA operator* $\mathbf{w}$'.

Definition 2.3. (Yager [25]) The measure of orness associated with an OWA operator $\mathbf{w}$ of dimension n is defined as

$$\text{orness}^Y(\mathbf{w}) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i. \tag{2}$$

The measure of andness associated with the OWA operator $\mathbf{w}$ is the complement of its orness, meaning

$$\text{andness}^Y(\mathbf{w}) = 1 - \text{orness}^Y(\mathbf{w}) = \sum_{i=1}^{n} \frac{i-1}{n-1} w_i.$$

The orness measure has the following properties (see [26]).

Proposition 2.4. (Yager [26]) Denote $\mathbf{w}^* = (1, 0, \dots, 0)$, $\mathbf{w}_* = (0, 0, \dots, 1)$ and $\mathbf{w}_A = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$ ($\mathbf{w}^*$, $\mathbf{w}_*$ and $\mathbf{w}_A$ correspond respectively to the max, min and average operators). Then $\text{orness}^Y(\mathbf{w}^*) = 1$, $\text{orness}^Y(\mathbf{w}_*) = 0$, $\text{orness}^Y(\mathbf{w}_A) = \frac{1}{2}$.

Proposition 2.5. (Yager [26]) Let us have OWA operators $\mathbf{w} = (w_1, w_2, \dots, w_n)$ and $\mathbf{w}' = (w_n, w_{n-1}, \dots, w_1)$. Assume that $\text{orness}^Y(\mathbf{w}) = \alpha$. Then $\text{orness}^Y(\mathbf{w}') = 1 - \alpha$.

Now, we recall that OWA operators are nothing else but Choquet integrals with respect to symmetric capacities (see, e. g., [6]). Actually, Choquet integrals have more general and complex forms than the OWA operators.

We denote by $N = \{1, 2, \dots, n\}$ the index set of arguments to be aggregated, and by $|A|$ the cardinality of A.

Definition 2.6. (Choquet [1], Grabisch et al. [6]) A capacity on 2^N is a function $\mu : 2^N \to [0, 1]$ satisfying

(a) $\mu(\emptyset) = 0$, $\mu(N) = 1$,

(b) $A \subset B$ implies $\mu(A) \leq \mu(B)$.

A capacity is often called also fuzzy measure (see [20]).

Definition 2.7. (Grabisch et al. [6]) A capacity $\mu : 2^N \to [0, 1]$ is called symmetric if for $A, B \in 2^N$ such that $|A| = |B|$ we have $\mu(A) = \mu(B)$.

Definition 2.8. (Choquet [1]) Let $\mu : 2^N \to [0, 1]$ be a capacity and $f : N \to [0, \infty)$. The Choquet integral of f with respect to μ is defined by

$$(C) \int f \, \mathrm{d}\mu = \int_0^\infty \mu(\{\omega \in N; f(\omega) > \alpha\}) \mathrm{d}\alpha.$$

Since we deal only with a discrete measurable space (of a fixed dimension n), the Choquet integral with respect to μ can be equivalently expressed in the following way.

Proposition 2.9. (Grabisch et al. [6]) Let $\mu : 2^N \to [0, 1]$ be a capacity and $\mathbf{x} \in [0, \infty)^n$. The Choquet integral of $\mathbf{x}$ with respect to μ can be expressed as follows:

$$C_\mu(\mathbf{x}) = \sum_{i=1}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)}) \mu(A_{\sigma(i)}), \tag{3}$$

where σ is a permutation on N such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$, and we set $x_{\sigma(0)} = 0$ and $A_{\sigma(i)} = \{\omega \in N; \omega \geq \sigma(i)\}$.

The following proposition shows the exact relationship between OWA operators and Choquet integrals.

Proposition 2.10. (Grabisch et al. [6]) Let $\mu : 2^N \to [0,1]$ be a capacity and F_w be an OWA operator given by a weighting n-tuple $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. Then $C_\mu = F_{\mathbf{w}}$ if and only if μ is symmetric and $w_i = \mu(A_i) - \mu(A_{i-1})$ where $A_i \subset N$ and $|A_i| = i$, $A_0 = \emptyset$.

As a direct corollary to Proposition 2.10 we have the following properties of OWA operators which were originally proven by Yager [25].

Proposition 2.11. (Yager [25]) Let F_w be an OWA operator given by a weighting n-tuple $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. Assume that $\mathbf{s} = (s_1, s_2 \ldots, s_n)$ is an arbitrary n-tuple of inputs.

1. (monotonicity) Let $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be any n-tuple of inputs such that $s_i \leq q_i$ for all $i \in N$. Then $F_{\mathbf{w}}(\mathbf{s}) \leq F_{\mathbf{w}}(\mathbf{q})$.
2. (idempotency) Assume $s_1 = s_2 = \cdots = s_n = s$. Then $F_{\mathbf{w}}(\mathbf{s}) = s$.
3. (symmetry) Let σ be any permutation on N and $\mathbf{s}_\sigma = (s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)})$. Then $F_{\mathbf{w}}(\mathbf{s}) = F_{\mathbf{w}}(\mathbf{s}_\sigma)$.
4. (boundary property) Set $\mathbf{w}^* = (1, 0, \ldots, 0)$, $\mathbf{w}_* = (0, 0, \ldots, 1)$. Then $F_{\mathbf{w}_*}(\mathbf{s}) \leq F_{\mathbf{w}}(\mathbf{s}) \leq F_{\mathbf{w}^*}(\mathbf{s})$.

3. FUZZY ORNESS DEFINITION AND PROPERTIES

Recently Kishor et al. [9] proposed an axiomatic definition of orness measures introducing four axioms. First, let us adopt the following notation for the set of all possible weighting n-tuples:

$$\mathcal{W} = \left\{ (w_1, w_2, \ldots, w_n) \in [0,1]^n ; \sum_{i=1}^{n} w_i = 1 \right\}. \tag{4}$$

Definition 3.1. (Kishor et al. [9]) Let $\mathbf{w} \in \mathcal{W}$ be an OWA operator. An orness function, denoted as ^{A}orness, is a function ${}^A\text{orness} : \mathcal{W} \to [0,1]$ satisfying the following properties.

(A1) ${}^A\text{orness}(\mathbf{w}^*) = 1$, where $\mathbf{w}^* = (1, 0, \ldots, 0)$.

(A2) ${}^A\text{orness}(\mathbf{w}_*) = 0$, where $\mathbf{w}_* = (0, 0, \ldots, 1)$.

(A3) ${}^A\text{orness}(\mathbf{w}_A) = \frac{1}{2}$, where $\mathbf{w}_A = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.

(A4) Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ $\mathbf{w}_\varepsilon = (w_1, \ldots, w_j - \varepsilon, \ldots, w_k + \varepsilon, \ldots, w_n)$, for $\varepsilon > 0$ and $j < k$, be weighting n-tuples. Then ${}^A\text{orness}(\mathbf{w}) > {}^A\text{orness}(\mathbf{w}_\varepsilon)$.

We propose another axiomatic definition of orness measure. The axioms **(A1)** – **(A3)** from Definition 3.1 will be generalized and axiom **(A4)** will be slightly weakened.

Definition 3.2. Let $\mathbf{w} \in \mathcal{W}$ be an OWA operator. An orness measure, denoted by ${}_A$orness, is a function ${}_A\text{orness} : \mathcal{W} \to [0,1]$ satisfying the following properties.

(A1') Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ and $\mathbf{w}' = (w_n, w_{n-1}, \ldots, w_1)$.
Then ${}_A\text{orness}(\mathbf{w}) +_A \text{orness}(\mathbf{w}') = 1$.

(A2') Let $\mathbf{w}_H = (\alpha, 0, \ldots, 0, 1-\alpha)$ for $\alpha \in [0,1]$. Then ${}_A\text{orness}(\mathbf{w}_H) = \alpha$.

(A3') Let $\mathbf{w}_i = (w_1, w_2, \ldots, w_n)$ such that there exists $i \in N$, $w_i = 1$.
Then ${}_A\text{orness}(\mathbf{w}_i) = \frac{n-i}{n-1}$.

(A4') Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ $\mathbf{w}_\varepsilon = (w_1, \ldots, w_j - \varepsilon, \ldots, w_k + \varepsilon, \ldots, w_n)$, for $\varepsilon > 0$ and $j < k$, be OWA operators. Then ${}_A\text{orness}(\mathbf{w}) \geq_A \text{orness}(\mathbf{w}_\varepsilon)$.

Remark 3.3. (a) The axioms (A1) and (A2) from definition 3.1 are special cases of newly proposed axioms (A2') and (A3'). The axiom (A3) is a special case of (A1') (since $\mathbf{w}_A = \mathbf{w}_A{}'$). And as we have already remarked above, (A4') is a weakened form of (A4).

(b) The weighting n-tuple $\mathbf{w}_H$ from axiom (A2') corresponds to the well-known Hurowicz operator.

(c) The weighting n-tuple $\mathbf{w}_i$ corresponds to the step OWA operator introduced by Yager [26].

Remark 3.4. Axiom (A1') shows that the andness measure (corresponding to a given orness measure) of an OWA operator $\mathbf{w}$ is just the orness measure applied to the OWA operator $\mathbf{w}'$.

Obviously the Yager's orness measure defined by formula (2) satisfies the axioms (A1')–(A4') from Definition 3.2. We introduce now a new orness measure based on lattice operations $(\vee, \wedge)$. We show that also this orness measure satisfies the axioms (A1')–(A4').

Definition 3.5. (Fuzzy orness measure) For an OWA operator $\mathbf{w} \in \mathcal{W}$ of dimension n, its fuzzy orness measure is defined by

$$\text{orness}^f(\mathbf{w}) = \bigvee_{i=1}^{n} \left(\frac{n-i}{n-1} \wedge \sum_{j=1}^{i} w_i \right). \tag{5}$$

If we denote $\sum_{j=1}^{i} w_i = s_i$ then we can rewrite the fuzzy orness measure into the following form

$$\text{orness}^f(\mathbf{w}) = \bigvee_{i=1}^{n} \left(\frac{n-i}{n-1} \wedge s_i \right).$$

Remark 3.6. Definition 3.5 is derived from a very instinctive principle. The more weights are accumulated to the left end of the corresponding OWA operator the more fuzzy orness it possesses. Therefore this fuzzy orness measure is more suitable for modelling the instinct of human decision-making than for an accurate calculation which computer is adapted at. Thus, it may have a potential to be applied in artificial intelligence or fuzzy control areas.

Definition 3.7. (Sugeno [20]) Let $\mu : 2^N \to [0,1]$ be a capacity and $f : N \to [0,1]$. The Sugeno integral of f with respect to μ is defined by

$$(S)\int f\,\mathrm{d}\mu = \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu(\{\omega \in N; f(\omega) \geq \alpha\})\right). \tag{6}$$

Proposition 3.8. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be an OWA operator, $g : N \to [0,1]$ be defined as $g(i) = \frac{n-i}{n-1}$. Further, let $\mu : 2^N \to [0,1]$ be the additive measure generated by the weights w_i, i.e., $\mu(\{i\}) = w_i$. Then

1. $(C)\int g\,\mathrm{d}\mu = \mathrm{orness}^Y(\mathbf{w})$,
2. $(S)\int g\,\mathrm{d}\mu = \mathrm{orness}^f(\mathbf{w})$.

Proof of this proposition is skipped since it follows directly by Definitions 2.3, 2.8, 3.5 and 3.7.

Let us give some examples to illustrate the computation of the fuzzy orness measure.

Example 3.9. Calculate the fuzzy orness grades of OWA operators

$$\mathbf{w}_1 = (0.9, 0.1, 0, 0, 0), \quad \mathbf{w}_2 = (0, 0, 0.05, 0.15, 0.8), \quad \mathbf{w}_3 = (0.4, 0.4, 0.2, 0, 0),$$
$$\mathbf{w}_4 = (0.25, 0.2, 0.1, 0, 0, 0.2, 0.05, 0.15, 0, 0, 0.05).$$

Solution. We can calculate the corresponding fuzzy orness grades of $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ from Table 1 where the first row represents the values of the function $g(i) = \frac{n-i}{n-1}$ and the second up to fourth rows the measure $\mu_k(i) = \sum_{j=1}^{i} w_k(j)$ for $k = 1, 2, 3$, where $w_k(i)$ is the ith coordinate of $\mathbf{w}_k$ (similarly we can calculate the fuzzy orness grade of $\mathbf{w}_4$ from Table 2). Since the values in the first row are decreasing and in the subsequent rows increasing, we can simply find the maximum of minima of $(g(i), \mu_k(i))$ for $k = 1, 2, 3$ (and similarly also for $k = 4$ in Table 2).

$g(i)$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\mu_1(i)$	0.9	1	1	1	1
$\mu_2(i)$	0	0	0.05	0.2	1
$\mu_3(i)$	0.4	0.8	1	1	1

Tab. 1. Table corresponding to OWA operators $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

$g(i)$	1	$\frac{8}{9}$	$\frac{7}{9}$	$\frac{2}{3}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$	0
$\mu_4(i)$	0.25	0.45	0.55	0.55	0.55	0.75	0.8	0.95	0.95	1

Tab. 2. Table corresponding to OWA operator $\mathbf{w}_4$.

We get the following values of fuzzy orness grades:
$\text{orness}^f(\mathbf{w}_1) = 0.9$, $\text{orness}^f(\mathbf{w}_2) = 0.2$, $\text{orness}^f(\mathbf{w}_3) = 0.75$, $\text{orness}^f(\mathbf{w}_4) = 0.55$.

Note that

$$\text{orness}^Y(\mathbf{w}_1) = 0.975, \quad \text{orness}^Y(\mathbf{w}_2) = 0.0625, \quad \text{orness}^Y(\mathbf{w}_3) = 0.8,$$
$$\text{orness}^Y(\mathbf{w}_4) = 0.644.$$

We find that the results of these two orness measures (Yager's orness and fuzzy orness) are generally different, but the difference of the results is usually small.

Next, we show that these two orness measures sometimes supplement each other. And this can help decision makers to judge which OWA operator is more an or-like one. Also, in some context lattice operations based aggregation may suit better then Yager's orness measure (based on arithmetic operations).

Example 3.10. Let $\mathbf{w}_1 = (0.25, 0, 0, 0.75)$ and $\mathbf{w}_2 = (0.05, 0.15, 0.3, 0.5)$ be OWA operators. Then for their orness grades we get $\text{orness}^Y(\mathbf{w}_1) = \text{orness}^Y(\mathbf{w}_2) = 0.25$. We cannot distinguish the orness of these two OWA operators.
Let us compute their fuzzy orness grades. Similarly to Example 3.9 we write Table 3 with the function $g(i) = \frac{4-i}{3}$ in the first and in the second and third rows the measure $\mu_k(i) = \sum_{j=1}^{i} w_k(j)$ for $k = 1, 2$, where $w_k(i)$ is the ith coordinate of $\mathbf{w}_k$.

$g(i)$	1	$\frac{2}{3}$	$\frac{1}{3}$	0
$\mu_1(i)$	0.25	0.25	0.25	1
$\mu_2(i)$	0.05	0.2	0.5	1

Tab. 3. Table corresponding to OWA operators $\mathbf{w}_1$ and $\mathbf{w}_2$.

For the fuzzy orness grades we have $\text{orness}^f(\mathbf{w}_1) = 0.25$ and $\text{orness}^f(\mathbf{w}_2) = \frac{1}{3}$.
We can see that the fuzzy orness measure bears some additional information to our decision which OWA operator is more or-like.

Theorem 3.11. Let $f : N \to [0,1]$ be a strictly decreasing function and $m : 2^N \to [0,1]$ be the additive measure generated by an n-tuple of weights $\mathbf{w} = (w_1, w_2, \dots, w_n)$. Let $\bar{m} : 2^N \to [0,1]$ be the additive measure generated by the system of weights $\mathbf{w}'$ in the reversed order. Then

$$(S)\int f \, \mathrm{d}\bar{m} = 1 - (S)\int f \, \mathrm{d}m.$$

Proof. First, observe that $(S)\int f\,\mathrm{d}\bar{m} = (S)\int (1-f)\,\mathrm{d}m$. Set $(S)\int f\,\mathrm{d}m = \alpha$. There are two possibilities.

(1) There exists $i \in N$ such that $f(i) = \alpha$. Then

$$m(\{j \in N; j \leq i\}) = m(\{j \in N; f(j) \geq \alpha\}) \geq \alpha.$$

This implies

$$m(\{j \in N; j > i\}) = m(\{j \in N; f(j) < \alpha\}) = m(\{j \in N; 1 - f(j) < 1 - \alpha\}) \leq 1 - \alpha.$$

This together with the fact that $f(i) = \alpha$ implies that $(S)\int (1-f)\,\mathrm{d}m = 1 - \alpha$.

(2) There is no $i \in N$ such that $m(\{1, 2, \ldots, i\}) = \alpha$ and $f(i) > \alpha$ and $f(i+1) < \alpha$. This implies that $1 - f(i+1) > 1 - \alpha$ and $m(\{j \in N; j > i\}) = 1 - \alpha$. Since we have $1 - f(i) < 1 - \alpha$, we get also in this case that $(S)\int (1-f)\,\mathrm{d}m = 1 - \alpha$. □

Proposition 3.12. The fuzzy orness measure $\mathrm{orness}^f : \mathcal{W} \to [0,1]$ satisfies axioms (A1') – (A4') of Definition 3.2.

Proof. By Proposition 3.8 and Theorem 3.11 we have that orness^f satisfies (A1'). Let $g : N \to [0,1]$ and $\mu : 2^N \to [0,1]$ have the same meaning as in Proposition 3.8. Then by a simple calculation of the Sugeno integral

$$(S)\int g\,\mathrm{d}\mu = \mathrm{orness}^f(\mathbf{w})$$

for an n-tuple of weights $\mathbf{w}$ we get that orness^f fulfils axioms (A2') and (A3').
Consider weighting n-tuples $\mathbf{w} = (w_1, w_2, \ldots, w_3)$ and $\mathbf{w}_\varepsilon = (w_1, \ldots, w_j - \varepsilon, \ldots, w_k + \varepsilon, \ldots, w_n)$ as in axiom (A4'). Then we have for every $i \in N$

$$\mu(\{1, 2, \ldots, i\}) = \sum_{m=1}^{i} w_m \geq \bar{\mu}(\{1, 2, \ldots, i\}) = \sum_{m=1}^{i} w_\varepsilon^m,$$

where w_ε^m denotes the mth coordinate of $\mathbf{w}_\varepsilon$ and $\bar{\mu}$ is the additive measure generated by the weighting n-tuple $\mathbf{w}_\varepsilon$. This gives

$$\begin{aligned} \mathrm{orness}^f(\mathbf{w}) &= (S)\int g\,\mathrm{d}\mu = \max\{\min\{g(i), \mu(\{1, 2, \ldots, i\})\}; i \in N\} \\ &\geq \max\{\min\{g(i), \bar{\mu}(\{1, 2, \ldots, i\})\}; i \in N\} = (S)\int g\,\mathrm{d}\bar{\mu} = \mathrm{orness}^f(\mathbf{w}_\varepsilon), \end{aligned}$$

and the proof if finished. □

4. ONE METHOD TO DETERMINE OWA OPERATOR WITH GIVEN FUZZY ORNESS GRADE

As a direct corollary to Proposition 3.8 we have the following.

Lemma 4.1. For every OWA operator $\mathbf{w} = (w_1, w_2, \dots, w_n)$ of dimension n $(n \geq 2)$ there exists an integer $m \in \{1, 2, \dots, n-1\}$ such that

$$\sum_{i=1}^{m} w_i \leq \frac{n-m}{n-1} \quad \text{and} \quad \sum_{i=1}^{m+1} w_i \geq \frac{n-(m+1)}{n-1}.$$

Proposition 3.8 and Lemma 4.1 have the following corollary.

Corollary 4.2. For every OWA operator $\mathbf{w} = (w_1, w_2, \dots, w_n)$ of dimension n $(n \geq 2)$ there exists an integer $m \in \{1, 2, \dots, n-1\}$ such that

$$\text{orness}^f(\mathbf{w}) = \max\left\{\frac{n-(m+1)}{n-1}, \sum_{i=1}^{m} w_i\right\}. \tag{7}$$

Remark 4.3. **(a)** The value m occurring in Lemma 4.1 and Corollary 4.2 is not necessarily given uniquely. This ambiguity of m occurs if there exists k such that $\frac{n-k}{n-1} = \sum_{i=1}^{k} w_i$. In this case we can set $m = k$ or $m = k-1$. In both cases $\text{orness}^f(\mathbf{w})$ can be computed using formula (7).

(b) We may be looking for an OWA operator $\mathbf{w}$ such that $\text{orness}^f(\mathbf{w}) = \alpha$. In this case we can set $\sum_{i=1}^{m} w_i = \alpha$, where $m = \lfloor n - \alpha(n-1) \rfloor$ due to inequalities from Lemma 4.1 ($\lfloor \cdot \rfloor$ is the floor function). Using this condition, one can add another constraint, e. g., to obtain the OWA operator with the maximal entropy or to obtain an equidifferent OWA operator [11].

Example 4.4. Determine an OWA operator $\mathbf{w}$ of dimension $n = 10$ with given fuzzy orness grade $\text{orness}^f(\mathbf{w}) = 0.75$.

Solution. First we determine $m = \lfloor n - \alpha(n-1) \rfloor = \lfloor 10 - 9 \cdot 0.75 \rfloor = 3$. This means that the weighting 10-tuple we are looking for has to fulfil condition $w_1 + w_2 + w_3 = 0.75$. This is the only condition $\mathbf{w}$ must fulfil. Hence, we can choose $\mathbf{w}$ to be an equidifferent OWA operator [11]. It means that $w_k = w_1 - (k-1)d$ if $w_1 - (k-1)d > 0$ and $w_k = 0$ otherwise. If we want to have exactly k non-zero weights we get the following system of constraints.

$$\begin{aligned}
w_1 + w_2 + w_3 = 0.75 \quad &\Rightarrow \quad w_2 = w_1 - d = 0.25, \\
\sum_{i=1}^{k} w_i = 1 \quad &\Rightarrow \quad k \cdot w_1 - \frac{1}{2}k(k-1)d = 1, \\
w_k > 0 \quad &\Rightarrow \quad w_1 - (k-1)d > 0, \\
w_{k+1} = 0 \quad &\Rightarrow \quad w_1 - k \cdot d \leq 0.
\end{aligned}$$

This system of constraints has unique solution, namely $k = 6$, $w_1 = \frac{11}{36}$, $d = \frac{1}{18}$. This gives the weighting 10-tuple $\mathbf{w} = \left(\frac{11}{36}, \frac{9}{36}, \frac{7}{36}, \frac{5}{36}, \frac{3}{36}, \frac{1}{36}, 0, 0, 0, 0\right)$.

5. CONCLUSIONS

Orness and fuzzy orness measures can be represented by Choquet and Sugeno integral, respectively. By presenting a set of four new orness axioms, we showed that both Yager's orness measure as well as the fuzzy orness measure satisfy the newly proposed orness axioms (A1') – (A4'). The fuzzy orness measure is useful to supplement Yager's orness measure when we face the problem that different OWA operators have the same orness grade. Fuzzy orness measure is also useful when lattice operations better suite to solving (or modelling) a problem of ours.

We proposed also a method to determine an OWA operator with given fuzzy orness grade.

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