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# CAUCHY-LIKE FUNCTIONAL EQUATION BASED ON A CLASS OF UNINORMS

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Commuting is an important property in any two-step information merging procedure where the results should not depend on the order in which the single steps are performed. In the case of bisymmetric aggregation operators with the neutral elements, Saminger, Mesiar and Dubois, already reduced characterization of commuting $n$-ary operators to resolving the unary distributive functional equations. And then the full characterizations of these equations are obtained under the assumption that the unary function is non-decreasing and distributive over special aggregation operators, for examples, continuous t-norms, continuous t-conorms and two classes of uninorms. Along this way of thinking, in this paper, we will investigate and fully characterize the following unary distributive functional equation $f(U(x,y)) = U(f(x), f(y))$, where $f\colon [0,1] \to [0,1]$ is an unknown function but unnecessarily non-decreasing, a uninorm $U \in \mathcal{U}_{\min}$ has a continuously underlying t-norm $T_U$ and a continuously underlying t-conorm $S_U$. Our investigation shows that the key point is a transformation from this functional equation to the several known ones. Moreover, this equation has also non-monotone solutions completely different with already obtained ones. Finally, our results extend the previous ones about the Cauchy-like equation $f(A(x,y)) = B(f(x), f(y))$, where $A$ and $B$ are some continuous t-norm or t-conorm.

*Keywords:* fuzzy connectives, distributivity functional equations, T-norms, T-conorms, uninorms

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## 1. INTRODUCTION

The aggregation of information inherent to the human thinking is viewed as the process of merging all collected data into a concrete representative value. More specifically, the aggregation process is carried out as a two-stepped procedure whereby several local fusion operations are performed in parallel and then the results are merged into a global result [21]. It may happen that in practice the two steps can be exchanged because there is no reason to perform either of the steps first [24]. Thus one would expect the two procedures yield the same results in any sensible approach, and then operations are said to be *commuting*.

In fact, early examples of commuting appear in probability theory for the merging of probability distributions. Suppose two joint probability distributions are merged by

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combining degrees of probability point-wisely. It is natural that the marginals of the resulting joint probability function are the aggregates of the marginals of the original joint probabilities. To fulfill this requirement the aggregation operation must commute with the addition operation involved in the derivation of the marginals. McConway showed that a weighted arithmetic mean is the only possible aggregation operation for probability functions [15].

After this, the commuting aggregation operators caught more and more attention. For instances, they are used to preserve the transitivity when aggregating preference matrices or fuzzy relations [8, 23, 25] or some form of additivity when aggregating set functions [9]. Specially, when Saminger, Mesiar, and Dubois [24] investigated the property of commuting for aggregation operators in connection with their relationship to bisymmetry, they gave out a full characterization of commuting operators in case that one of them is bisymmetric with some neutral element and further showed that these operators can be attained through functions distributive over the bisymmetric aggregation operator with neutral element involved. Thus the characterization of commuting $n$-ary operators is reduced to resolving the unary distributive functional equations. Note that a full characterization of all bisymmetric aggregation operators with neutral elements, in particular if the neutral elements are from the open interval, is still missing [3, 6, 7, 10] and the characterization of the set of unary functions distributing with such operators is heavily influenced by the structure of the underlying operators [17, 20]. Hence they only focused on several special subclasses of bisymmetric aggregation operators with neutral elements, namely on continuous t-norms, continuous t-conorms and two particular classes of uninorms. For these two particular classes of uninorms, they got the full characterizations of the unary distributive functional equations under the assumption that the unary function is non-decreasing. Indeed, it is very difficult to obtain the full characterization of these equations without any additional condition because they are bound up with the famous Cauchy functional equation [1, 2] which has not been completely solved so far [3, 16, 18]. Along this way of thinking, in this paper, we will investigate the following functional equation

$$f(U(x,y)) = U(f(x), f(y)), \qquad (x,y) \in [0,1]^2, \tag{1}$$

where $f\colon [0,1] \to [0,1]$ is an unknown function but unnecessarily non-decreasing, a uninorm $U \in \mathcal{U}_{\min}$ has a continuously underlying t-norm $T_U$ and a continuously underlying t-conorm $S_U$. Our investigation shows that the key point is a transformation from this functional equation to the several known ones. Moreover, this equation has also non-monotone solutions completely different with already obtained ones. Finally, our results extend the previous ones about the Cauchy-like equation $f(A(x,y)) = B(f(x), f(y))$, where $A$ and $B$ are some continuous t-norm or t-conorm.

On the other hand, it is often appropriate to amalgamate numerical statistical data into means variances, etc., by means of standard integration, so it can be appropriate to amalgamate similar data by means of monotone integrals related to monotone measures that are not necessarily additive [4, 10]. In particular, one can consider data with values in a real interval $[0,1]$ on which a pseudo-addition induces a structure characteristic for the data. Then, in order to construct an appropriate integral, one introduces a pseudo-multiplication that must be distributive with respect to the given pseudo-addition [5]. The distributivity property is expressed by the above Eq. (1).

The paper is organized as follows. In Section 2, we present some results concerning basic fuzzy logic connectives. In Section 3, some results about the Cauchy-like functional equation based on continuous t-norms or continuous t-conorms are recalled. From Section 4 to Section 6, the main sections of this paper, we will investigate and describe all solutions of Eq. (1). To illustrate structures of solutions of Eq. (1), in Section 7, an example is constructed. Finally, Conclusion is in Section 8.

## 2. PRELIMINARIES

**Definition 2.1.** (Gottwald [12], Klement et al. [14]) A binary operation $T\colon [0,1]^2 \to [0,1]$ is called a *t-norm* if it is associative, commutative, increasing and has neutral element 1, namely, it holds $T(x,1) = T(1,x) = x$ for all $x \in [0,1]$.

**Definition 2.2.** (Klement et al. [14]) A t-norm $T$ is said to be

(i) *continuous*, if for all convergent sequences $(x_n)_{n\in\mathbf{N}}, (y_n)_{n\in\mathbf{N}} \in [0,1]^{\mathbf{N}}$, we have $T(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} T(x_n, y_n)$;

(ii) *Archimedean*, if for every $x, y \in (0,1)$, there exists some $n \in \mathbf{N}$ such that $x_T^n < y$, where $x_T^1 = x,\ x_T^2 = T(x,x),\ \cdots,\ x_T^n = T(x_T^{n-1}, x)$;

(iii) *strict*, if $T$ is continuous and strictly monotone, i.e., $T(x,y) > T(x,z)$ whenever $x \in (0,1]$ and $y > z$;

(iv) *nilpotent*, if $T$ is continuous and if for each $x \in (0,1)$ there exists some $n \in \mathbf{N}$ that $x_T^n = 0$.

**Remark 2.3.** (Klement et al. [14])

(i) A continuous t-norm $T$ is Archimedean if and only if it holds $T(x,x) < x$ for all $x \in (0,1)$.

(ii) If a t-norm $T$ is strict or nilpotent, then it must be Archimedean. The converse is also true when it is continuous.

**Theorem 2.4.** (Klement et al. [14]) For a function $T\colon [0,1]^2 \to [0,1]$, the following statements are equivalent:

(i) $T$ is a continuous Archimedean t-norm.

(ii) $T$ has a continuous additive generator, i.e., there exists a continuous, strictly decreasing function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0))), \qquad x, y \in [0,1]. \tag{2}$$

**Remark 2.5.** (Gottwald [12], Grabish et al. [13])

(i) A t-norm $T$ is strict if and only if each continuous additive generator $t$ of $T$ satisfies $t(0) = \infty$.

(ii) A t-norm $T$ is nilpotent if and only if each continuous additive generator $t$ of $T$ satisfies $t(0) < \infty$.

**Theorem 2.6.** (Klement et al. [14]) $T$ is a continuous t-norm, if and only if

(i) $T = \min$, or

(ii) $T$ is continuously Archimedean, or

(iii) there exists a family $\{(a_m, b_m), T_m\}_{m\in A}$ such that $T$ is the ordinal sum of this family denoted by $T = (< a_m, b_m, T_m >)_{m\in A}$. In other words, it holds for all $x, y \in [0, 1]$,

$$T(x,y) = \begin{cases} a_m + (b_m - a_m)T_m\left(\frac{x-a_m}{b_m-a_m}, \frac{y-a_m}{b_m-a_m}\right) & \text{if } (x,y) \in [a_m, b_m]^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

where $\{(a_m, b_m)\}_{m\in A}$ is a countable family of non-overlapping, open, proper subintervals of $[0, 1]$ with each $T_m$ being a continuously Archimedean t-norm, and $A$ is a finite or countable infinite index set. For every $m \in A$, $(a_m, b_m)$ is called an open generating subinterval of $T$, and $T_m$ is called a correspondingly generating t-norm on $(a_m, b_m)$ (or $[a_m, b_m]$) of $T$.

**Definition 2.7.** (Klement et al. [14]) A binary operation $S\colon [0, 1]^2 \to [0, 1]$ is called a *t-conorm* if it is associative, commutative, increasing and has neutral element 0, namely, it holds $S(x, 0) = S(0, x) = x$ for all $x \in [0, 1]$.

**Definition 2.8.** (Gottwald [12]) A t-conorm $S$ is said to be

(i) *continuous*, if for all convergent sequences $(x_n)_{n\in\mathbf{N}}, (y_n)_{n\in\mathbf{N}} \in [0, 1]^{\mathbf{N}}$, we have $S(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} S(x_n, y_n)$;

(ii) *Archimedean*, if for every $x, y \in (0, 1)$, there exists some $n \in \mathbf{N}$ such that $x_S^n > y$, where $x_S^1 = x,\ x_S^2 = S(x, x),\ \cdots,\ x_S^n = S(x_S^{n-1}, x)$;

(iii) *strict*, if $S$ is continuous and strictly monotone, i.e., $S(x, y) < S(x, z)$ whenever $x \in [0, 1)$ and $y < z$;

(iv) *nilpotent*, if $S$ is continuous and if for each $x \in (0, 1)$ there exists some $n \in \mathbf{N}$ that $x_S^n = 1$.

**Remark 2.9.** (Gottwald [12], Klement et al. [14]) By the duality between t-norms and t-conorms, we easily obtain the following properties.

(i) A continuous t-conorm $S$ is Archimedean if and only if it holds $S(x, x) > x$ for all $x \in (0, 1)$.

(ii) If a t-conorm $S$ is strict or nilpotent, then it must be Archimedean. The converse is also true when it is continuous.

**Theorem 2.10.** (Klement et al. [14]) For a function $S\colon [0, 1]^2 \to [0, 1]$, the following statements are equivalent:

(i) $S$ is a continuous Archimedean t-conorm.

(ii) $S$ has a continuous additive generator, i.e., there exists a continuous, strictly increasing function $s\colon [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely determined up to a positive multiplicative constant, such that

$$S(x,y) = s^{-1}(\min(s(x)+s(y), s(1))), \qquad x,y \in [0,1]. \tag{3}$$

**Remark 2.11.** (Fodor and Roubens [10]) By the duality between t-norms and t-conorms, we easily obtain the following properties.

(i) A t-conorm $S$ is strict if and only if each continuous additive generator $s$ of $S$ satisfies $s(1) = \infty$.

(ii) A t-conorm $S$ is nilpotent if and only if each continuous additive generator $s$ of $S$ satisfies $s(1) < \infty$.

**Theorem 2.12.** (Klement et al. [14]) $S$ is a continuous t-conorm, if and only if

(i) $S = \max$, or

(ii) $S$ is continuously Archimedean, or

(iii) there exists a family $\{(a_m, b_m), S_m\}_{m\in B}$ such that $S$ is the ordinal sum of this family denoted by $S = (< a_m, b_m, S_m >)_{m\in B}$. In other words, it holds for all $x, y \in [0,1]$

$$S(x,y) = \begin{cases} a_m + (b_m - a_m) S_m\left(\frac{x-a_m}{b_m-a_m}, \frac{y-a_m}{b_m-a_m}\right) & \text{if } (x,y) \in [a_m, b_m]^2, \\ \max(x,y) & \text{otherwise,} \end{cases}$$

where $\{(a_m, b_m)\}_{m\in B}$ is a countable family of non-overlapping, open, proper subintervals of $[0,1]$ with each $S_m$ being a continuously Archimedean t-conorm, and $B$ is a finite or countable infinite index set. For every $m \in B$, $(a_m, b_m)$ is called an open generating subinterval of $S$, and $S_m$ is called a correspondingly generating t-conorm on $(a_m, b_m)$ (or $[a_m, b_m]$) of $S$.

**Definition 2.13.** (Fodor et al. [11], Ruiz and Torrens [22], Yager and Ryalkov[26]) A *uninorm* $U$ is a binary operator $U\colon [0,1]^2 \to [0,1]$, which is commutative, associative, non-decreasing in each variable and there exists some element $e \in [0,1]$ called neutral element such that $U(e,x) = x$ for all $x \in [0,1]$.

It is clear that the binary operator $U$ becomes a t-norm when $e = 1$ while $U$ a t-conorm when $e = 0$. For any other value $e \in (0,1)$ the operation works as a t-norm in the square $[0,e]^2$, and as a t-conorm in $[e,1]^2$, and its values are between min and max in the set of points $A(e)$ given by

$$A(e) = [0,e) \times (e,1] \cup (e,1] \times [0,e). \tag{4}$$

We will denote a uninorm with neutral element $e$ and a underlying t-norm $T_U$ and a underlying t-conorm $S_U$ by $U = \langle T_U, e, S_U\rangle$. In fact, it holds that

$$T_U(x,y) = \frac{U(ex,ey)}{e}, \;\; S_U(x,y) = \frac{U(e+(1-e)x, e+(1-e)y) - e}{1-e} \tag{5}$$

for all $0 < e < 1$. For any uninorm we have $U(0,1) \in \{0,1\}$ and a uninorm is called *conjunctive* when $U(0,1) = 0$ and *disjunctive* when $U(0,1) = 1$.

**Theorem 2.14.** (Fodor et al. [11]) Let $U\colon [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in (0,1)$. Then, the sections $x \mapsto (x,1)$ and $x \mapsto (x,0)$ are continuous at each point except perhaps at $e$ if and only if $U$ is given by one of the following formulas.

(i) If $U(0,1) = 0$, then

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

(ii) If $U(0,1) = 1$, then

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e} \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \max\{x,y\} & \text{otherwise.} \end{cases}$$

In the following, the set of uninorms in Case (i) be denoted by $\mathcal{U}_{\min}$ and the set of uninorms in Case (ii) by $\mathcal{U}_{\max}$.

## 3. SOME RESULTS ABOUT THE CAUCHY-LIKE FUNCTIONAL EQUATION BASED ON CONTINUOUS T-NORMS OR CONTINUOUS T-CONORMS

Note that the main results in Ref. [18], i. e., Theorem 4.17, only consider the case that $S_1$ and $S_2$ are continuous but not Archimedean t-conorms. In fact, they hold for all continuous t-conorms. Therefore, the conditions that $S_1$ and $S_2$ are not Archimedean can be dropped. Then, set $I(x,y) = f_x(y)$ and apply this theorem, we can obtain the following characterizations of the Cauchy-like functional equations based on continuous t-conorms.

**Theorem 3.1.** (Benvenuti and Vivona [5], Qin and Baczyński [18]) Consider two continuous t-conorms $S_1$ and $S_2$, and a unary function $f\colon [0,1] \to [0,1]$. The triple of functions $(S_1, S_2, f)$ satisfies

$$f(S_1(x,y)) = S_2(f(x), f(y)) \tag{6}$$

for all $x, y \in [0,1]$ if and only if $f$ is non-decreasing, preserves the idempotent property, and has the following form in every generating subinterval $(\alpha_m, \beta_m)$ of $S_1$,

(i) If $S_1$ is strict on its own generating subinterval $(\alpha_m, \beta_m)$ with the additive generator $s_m$ and $S_2$ on the generating subinterval $(c_j, d_j)$ has the additive generator $s_j$, then we have one of the following two subcases.

  (a) $f(x) = r$, if it holds that $f(x) \notin (c_j, d_j)$ for any $x \in (\alpha_m, \beta_m)$, where $r$ is idempotent and satisfies $f(\alpha_m) \le r \le f(\beta_m)$.

(b)

$$f(x) = c_j + (d_j - c_j) \cdot s_j^{-1} \left( \min \left( c \cdot s_m \left( \frac{x - \alpha_m}{\beta_m - \alpha_m} \right), s_j(1) \right) \right) \tag{7}$$

for any $x \in (\alpha_m, \beta_m)$ and some constant $c \in (0, \infty)$, if there exists some $x_0 \in (\alpha_m, \beta_m)$ such that $f(x_0) \in (c_j, d_j)$.

(ii) If $S_1$ is nilpotent on its own generating subinterval $(\alpha_m, \beta_m)$ with the additive generator $s_m$ and $S_2$ on the generating subinterval $(c_j, d_j)$ has the additive generator $s_j$, then we have one of the following two subcases.

(a) $f(x) = \beta_m$, if it holds that $f(x) \notin (c_j, d_j)$ for any $x \in (\alpha_m, \beta_m)$.

(b) $f(x)$ has the form Eq. (7) for any $x \in (\alpha_m, \beta_m)$ and some constant $c \in [\frac{s_j(1)}{s_m(1)}, \infty)$, if there exists some $x_0 \in (\alpha_m, \beta_m)$ such that $f(x_0) \in (c_j, d_j)$.

By the duality between t-norms and t-conorms, we easily obtain the following theorem.

**Theorem 3.2.** (Saminger-Platz et al. [24]) Consider two continuous norms $T_1$ and $T_2$, and a unary function $f\colon [0,1] \to [0,1]$. The triple of functions $(T_1, T_2, f)$ satisfies

$$f(T_1(x, y)) = T_2(f(x), f(y)) \tag{8}$$

for all $x, y \in [0,1]$ if and only if $f$ is non-decreasing, preserves the idempotent property, and has the following form in every generating subinterval $(\alpha_m, \beta_m)$ of $T_1$,

(i) If $T_1$ is strict on its own generating subinterval $(\alpha_m, \beta_m)$ with the additive generator $t_m$ and $T_2$ on the generating subinterval $(c_j, d_j)$ has the additive generator $t_j$, then we have one of the following two subcases.

(a) $f(x) = r$, if it holds that $f(x) \notin (c_j, d_j)$ for any $x \in (\alpha_m, \beta_m)$, where $r$ is idempotent and satisfies $f(\alpha_m) \le r \le f(\beta_m)$.

(b)

$$f(x) = c_j + (d_j - c_j) \cdot t_j^{-1} \left( \min \left( c \cdot t_m \left( \frac{x - \alpha_m}{\beta_m - \alpha_m} \right), t_j(0) \right) \right) \tag{9}$$

for any $x \in (\alpha_m, \beta_m)$ and some constant $c \in (0, \infty)$, if there exists some $x_0 \in (\alpha_m, \beta_m)$ such that $f(x_0) \in (c_j, d_j)$.

(ii) If $T_1$ is nilpotent on its own generating subinterval $(\alpha_m, \beta_m)$ with the additive generator $t_m$ and $T_2$ on the generating subinterval $(c_j, d_j)$ has the additive generator $t_j$, then we have one of the following two subcases.

(a) $f(x) = \beta_m$, if it holds that $f(x) \notin (c_j, d_j)$ for any $x \in (\alpha_m, \beta_m)$.

(b) $f(x)$ has the form Eq. (9) for any $x \in (\alpha_m, \beta_m)$ and some constant $c \in [\frac{t_j(0)}{t_m(0)}, \infty)$, if there exists some $x_0 \in (\alpha_m, \beta_m)$ such that $f(x_0) \in (c_j, d_j)$.

Generalizing the results of Ref. [19] by means of the method of Ref. [18], we directly obtain the following theorem.

**Theorem 3.3.** Consider a continuous t-norm $T$ and a continuous t-conorm $S$, and a unary function $f\colon [0,1] \to [0,1]$. The triple of functions $(T, S, f)$ satisfies

$$f(T(x,y)) = S(f(x), f(y)) \tag{10}$$

for all $x, y \in [0,1]$ if and only if $f$ is non-increasing, preserves the idempotent property, and has the following form in every generating subinterval $(\alpha_m, \beta_m)$ of $T$,

(i) If $T$ is strict on its own generating subinterval $(\alpha_m, \beta_m)$ with the additive generator $t_m$ and $S$ on the generating subinterval $(c_j, d_j)$ has the additive generator $s_j$, then we have one of the following two subcases.

  (a) $f(x) = r$, if it holds that $f(x) \notin (c_j, d_j)$ for any $x \in (\alpha_m, \beta_m)$, where $r$ is idempotent and satisfies $f(\beta_m) \le r \le f(\alpha_m)$.

  (b)
$$f(x) = c_j + (d_j - c_j) \cdot s_j^{-1}\left(\min\left(c \cdot t_m\left(\frac{x - \alpha_m}{\beta_m - \alpha_m}\right), s_j(1)\right)\right) \tag{11}$$
  for any $x \in (\alpha_m, \beta_m)$ and some constant $c \in (0, \infty)$, if there exists some $x_0 \in (\alpha_m, \beta_m)$ such that $f(x_0) \in (c_j, d_j)$.

(ii) If $T$ is nilpotent on its own generating subinterval $(\alpha_m, \beta_m)$ with the additive generator $t_m$ and $S$ on the generating subinterval $(c_j, d_j)$ has the additive generator $s_j$, then we have one of the following two subcases.

  (a) $f(x) = \alpha_m$, if it holds that $f(x) \notin (c_j, d_j)$ for any $x \in (\alpha_m, \beta_m)$.

  (b) $f(x)$ has the form Eq. (11) for any $x \in (\alpha_m, \beta_m)$ and some constant $c \in [\frac{s_j(1)}{t_m(0)}, \infty)$, if there exists some $x_0 \in (\alpha_m, \beta_m)$ such that $f(x_0) \in (c_j, d_j)$.

By the duality between t-norms and t-conorms, we have the following theorem.

**Theorem 3.4.** Consider a continuous t-norm $T$ and a continuous t-conorm $S$, and a unary function $f\colon [0,1] \to [0,1]$. The triple of functions $(T, S, f)$ satisfies

$$f(S(x,y)) = T(f(x), f(y)) \tag{12}$$

for all $x, y \in [0,1]$ if and only if $f$ is non-increasing, preserves the idempotent property, and has the following form in every generating subinterval $(\alpha_m, \beta_m)$ of $S$,

(i) If $S$ is strict on its own generating subinterval $(\alpha_m, \beta_m)$ with the additive generator $s_m$ and $T$ on the generating subinterval $(c_j, d_j)$ has the additive generator $t_j$, then we have one of the following two subcases.

  (a) $f(x) = r$, if it holds that $f(x) \notin (c_j, d_j)$ for any $x \in (\alpha_m, \beta_m)$, where $r$ is idempotent and satisfies $f(\beta_m) \le r \le f(\alpha_m)$.

  (b)
$$f(x) = c_j + (d_j - c_j) \cdot t_j^{-1}\left(\min\left(c \cdot s_m\left(\frac{x - \alpha_m}{\beta_m - \alpha_m}\right), t_j(0)\right)\right) \tag{13}$$
  for any $x \in (\alpha_m, \beta_m)$ and some constant $c \in (0, \infty)$, if there exists some $x_0 \in (\alpha_m, \beta_m)$ such that $f(x_0) \in (c_j, d_j)$.

(ii) If $S$ is nilpotent on its own generating subinterval $(\alpha_m, \beta_m)$ with the additive generator $s_m$ and $T$ on the generating subinterval $(c_j, d_j)$ has the additive generator $t_j$, then we have one of the following two subcases.

(a) $f(x) = \beta_m$ if it holds that $f(x) \notin (c_j, d_j)$ for any $x \in (\alpha_m, \beta_m)$.

(b) $f(x)$ has the form Eq. (13) for any $x \in (\alpha_m, \beta_m)$ and some constant $c \in [\frac{t_j(1)}{s_m(0)}, \infty)$, if there exists some $x_0 \in (\alpha_m, \beta_m)$ such that $f(x_0) \in (c_j, d_j)$.

## 4. TWO KEY LEMMAS

From now on, we investigate and characterize the functional equation

$$f(U(x, y)) = U(f(x), f(y)), \tag{14}$$

where $f\colon [0,1] \to [0,1]$ is an unknown function but unnecessarily non-decreasing, a uninorm $U \in \mathcal{U}_{\min}$ has a continuously underlying t-norm $T_U$ and a continuously underlying t-conorm $S_U$. Our method are also suit for a uninorm $U \in \mathcal{U}_{\max}$ and there are similar results, but considering the limited length of the paper, they are omitted. For the sake of convenience, write $Ran(f) = \{f(x) | x \in [0,1]\}$ and $\mathbf{Id}(U) = \{x \in [0,1] | U(x,x) = x\}$.

**Lemma 4.1.** (Saminger-Platz et al. [24]) Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0,1)$, and a unary function $f\colon [0,1] \to [0,1]$. If $f$ satisfies Eq. (14), then all of the following statements hold.

(i) If $x \in \mathbf{Id}(U)$, then $f(x) \in \mathbf{Id}(U)$.

(ii) If $x \in [0,1]$, then $U(f(e), f(x)) = f(x)$.

(iii) If $e \in Ran(f)$, then $f(e) = e$.

**Remark 4.2.** Ref. [24] has already shown Lemma 4.1 holds under hypothesis that $f$ is monotone. But monotonicity is not used in its proof progress. Hence these results hold without any restriction on $f$ w.r.t. its monotonicity. That is, Lemma 4.1 is right without its proof. Moreover, it also shows that $f$ preserves idempotency and $f(e)$ plays the neutral element in $Ran(f)$.

Now, let us investigate Eq. (14) under the assumption that there exists some $x_0 \in [0, e)$ such that $f(x_0) < e$, but as seen later this assumption is not essential.

**Lemma 4.3.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0,1)$, a unary function $f\colon [0,1] \to [0,1]$, and there exists some $x_0 \in [0, e)$ such that $f(x_0) < e$. If $f$ satisfies Eq. (14), then it holds that $f(x) < e$ for all $x \in [0, x_0]$.

*Proof.* Take any $x \in [0, x_0]$, then we know from hypotheses and structure of $U$ that there exists some $x' \in [0, e]$ such that $x = U(x', x_0)$. Note that $f(x_0) < e$, then we have from structure of $U$ and Eq. (14) that $f(x) = f(U(x', x_0)) = U(f(x'), f(x_0)) \leq \min(f(x'), f(x_0)) \leq f(x_0) < e$. □

Due to Lemma 4.3, define $E = \{x \in [0, e) | f(x) < e\}$, then we stipulate

$$\alpha = \sup E. \tag{15}$$

It is obvious that $\alpha \le e$. Now, we claim that $\alpha$ is an idempotent element of $U$, namely, $U(\alpha, \alpha) = \alpha$. In fact, the result is clearly right when $\alpha = e$. Now, we assume $\alpha < e$ and take $x \in (\alpha, e)$. It follows from definition of $\alpha$ that $f(x) \ge e$. Thus we imply from Eq. (14) that $f(U(x, x)) = U(f(x), f(x)) \ge f(x) \ge e$, which means $U(x, x) \notin E$, i. e., $U(x, x) \ge \alpha$. Let $x \to \alpha$ and apply continuity of $U$ on the region $[0, e]^2$, then we obtain $U(\alpha, \alpha) \ge \alpha$. On the other hand, we know from $\alpha < e$ that $U(\alpha, \alpha) \le \alpha$. Therefore it holds that $U(\alpha, \alpha) = \alpha$. It is here pointed out that the case $\alpha = 0$ do include the above excluded case that $f(x_0) \ge e$ for all $x_0 \in [0, e)$. Hence, in next discussion, we will cancel this restriction that there exists some $x_0 \in [0, e)$ such that $f(x_0) < e$.

Next, depending on the order relation between $\alpha$ and $e$, we need to consider two cases: (I) $\alpha < e$ and (II) $\alpha = e$. We firstly consider the case $\alpha < e$.

## 5. CASE (I)

**Lemma 5.1.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0, 1)$, a unary function $f\colon [0, 1] \to [0, 1]$, the symbol $\alpha$ defined above by Eq. (15) and fulfilling $\alpha < e$. If $f$ satisfies Eq. (14), then all of the following statements hold.

(i) $f|_{[0,\alpha)}$ is increasing, $Ran(f|_{[0,\alpha)}) \subseteq [0, e)$.

(ii) $f|_{(\alpha,e)}$ is decreasing, $Ran(f|_{(\alpha,e)}) \subseteq [f(1), 1]$.

(iii) $f|_{[e,1]}$ is increasing, $Ran(f|_{(e,1]}) \subseteq [e, f(1)]$.

Proof. (i) Suppose $x, y \in [0, \alpha)$ and $x < y$, then we have from definition of $\alpha$ that $f(x) < e$ and $f(y) < e$. Applying structure of $U$, then there exists some $x' \in [0, e)$ such that $x = U(x', y)$. Hence it follows from Eq. (14) that $f(x) = f(U(x', y)) = U(f(x'), f(y)) \le \min(f(x'), f(y)) \le f(y)$. Thus we have just proven that $f|_{[0,\alpha)}$ is increasing. Further, we get from definition of $\alpha$ that $Ran(f|_{[0,\alpha)}) \subseteq [0, e)$.

(ii) At first, let us show $f(x) \ge f(1) \ge e$ for $x \in (\alpha, e)$. Obviously, then we know from structure of $U$ and $x \in (\alpha, e)$ that $x = U(x, 1)$. Thus it follows from Eq. (14) that $f(x) = f(U(x, 1)) = U(f(x), f(1))$. Now we claim $f(1) \ge e$. Otherwise we assume $f(1) < e$, then we obtain $f(x) = U(f(x), f(1)) = \min(f(x), f(1)) \le f(1) < e$. On the other hand, it follows from definition of $\alpha$ that $f(x) \ge e$, this is a contradiction. So $f(1) \ge e$, applying again structure of $U$, we must obtain $f(x) = U(f(x), f(1)) \ge \max(f(x), f(1)) \ge f(1) \ge e$.

Suppose $x, y \in (\alpha, e)$ and $x < y$, then we know from definition of $\alpha$ that $f(x) \ge e$ and $f(y) \ge e$. And then, by means of structure of $U$ and idempotent elements $\alpha$ and $e$, there exists some $x' \in (\alpha, e)$ such that $x = U(x', y)$. Further we have from Eq. (14) that $f(x) = f(U(x', y)) = U(f(x'), f(y)) \ge \max(f(x'), f(y)) \ge f(y)$, which means that $f|_{(\alpha,e)}$ is decreasing. Finally we obtain that $Ran(f|_{(\alpha,e)}) \subseteq [f(1), 1]$.

(iii) To finish proof, we must show that $f(x) \ge e$ holds for all $x \in (e, 1]$. Otherwise we assume that there exists some $x_0 \in (e, 1]$ such that $f(x_0) < e$. Note that $\alpha < e$, then we take $y_0 \in (\alpha, e)$ and then we obtain from definition of $\alpha$ that $f(y_0) \geq e$. Using structure of $U$, we get $y_0 = U(x_0, y_0)$. Hence it follows from Eq. (14) and structure of $U$ that $f(y_0) = f(U(x_0, y_0)) = U(f(x_0), f(y_0)) = \min(f(x_0), f(y_0)) \leq f(x_0) < e$, this is a contradiction.

Suppose $x, y \in (e, 1]$ and $x < y$, then we know from the previous proof that $f(x) \geq e$ and $f(y) \geq e$. By virtue of structure of $U$, there exists some $x' \in (e, 1]$ such that $y = U(x', x)$. Hence we know from Eq. (14) and structure of $U$ that $f(y) = f(U(x', x)) = U(f(x'), f(x)) \geq \max(f(x'), f(x)) \geq f(x)$. This shows that $f|_{(e,1]}$ is increasing, and then we know that $Ran(f|_{(e,1]}) \subseteq [e, f(1)]$. □

**Lemma 5.2.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0, 1)$, a unary function $f\colon [0, 1] \to [0, 1]$, the symbol $\alpha$ defined above by Eq. (15) and fulfilling $\alpha < e$. If $f$ satisfies Eq. (14), then one of the following three statements hold.

(i) If $f(\alpha) < e$, then $f(\alpha) = \max(Ran(f|_{[0,\alpha]}))$.

(ii) If $f(\alpha) > e$, then $f(\alpha) = \max(Ran(f|_{[0,1]}))$.

(iii) $f(\alpha) = e$.

Proof. (i) Suppose $f(\alpha) < e$ and take $x \in [0, \alpha)$, then we have from definition and the idempotent element $\alpha$ that $f(x) < e$ and $U(x, \alpha) = x$. Hence it follows from Eq. (14) that $f(x) = f(U(x, \alpha)) = U(f(x), f(\alpha)) \leq \min(f(x), f(\alpha)) \leq f(\alpha)$. Further we have $f(\alpha) = \max(Ran(f|_{[0,\alpha]}))$.

(ii) Suppose $f(\alpha) > e$ and take $x \in (\alpha, e)$, then we have from definition and the idempotent element $\alpha$ that $f(x) \geq e$ and $U(x, \alpha) = \alpha$. Hence it follows from Eq. (14) that $f(\alpha) = f(U(x, \alpha)) = U(f(x), f(\alpha)) \geq \max(f(x), f(\alpha)) \geq f(x)$. Further we have from Lemma 5.1 that $f(\alpha) = \max(Ran(f|_{[0,1]}))$. □

Suppose $x, y < \alpha$, define two functions $\phi\colon [0, \alpha] \to [0, 1]$ and $\varphi\colon [0, e] \to [0, 1]$ by the formulas $\phi(x) = \frac{x}{\alpha}$ and $\varphi(x) = \frac{x}{e}$, respectively. Then there exists some continuous t-norm $T_1$ such that two sides of Eq. (14) are respectively written as $U(x, y) = \phi^{-1}T_1(\phi(x), \phi(y))$ and $U(f(x), f(y)) = \varphi^{-1}T_U(\varphi(f(x)), \varphi(f(y)))$. Therefore, for any $x, y < e$, Eq. (14) can be rewritten as $f(\phi^{-1}T_1(\phi(x), \phi(y))) = \varphi^{-1}T_U(\varphi(f(x)), \varphi(f(y)))$, from which we get $(\varphi_1 \circ f \circ \phi^{-1})(T_1(\phi(x), \phi(y))) = T_U(\varphi(f(x)), \varphi(f(y)))$. By routine substitutions

$$g = \varphi \circ f \circ \phi^{-1},\ a = \phi(x),\ b = \phi(y), \tag{16}$$

we have the Cauchy like functional equation

$$g(T_1(a, b)) = T_U(g(a), g(b)) \quad \text{for } a,\ b \in [0, 1], \tag{17}$$

where $g\colon [0, 1] \to [0, 1]$ is an unknown function. This means that resolving of Eq. (14) when $x,\ y < \alpha$ is reduced to characterize all solutions of Eq. (17). Fortunately, the full characterization of this case can be found in Theorem 3.2.

Suppose $\alpha < x, y < e$, define yet two functions $\phi'\colon [\alpha, e] \to [0, 1]$ and $\varphi'\colon [f(1), 1] \to [0, 1]$ by the formulas $\phi'(x) = \frac{x-\alpha}{e-\alpha}$ and $\varphi'(x) = \frac{x-f(1)}{1-f(1)}$, respectively. Then there exists a continuous t-norm $T_2$ and a continuous t-conorm $S_1$ such that two sides of Eq.(14) are respectively written as $U(x,y) = (\phi')^{-1}T_2(\phi'(x), \phi'(y))$ and $U(f(x), f(y)) = (\varphi')^{-1}S_1(\varphi'(f(x)), \varphi'(f(y)))$. Hence, for any $\alpha < x, y < e$, Eq. (14) is rewritten as $f((\phi')^{-1}T_2(\phi'(x), \phi'(y))) = (\varphi')^{-1}S_1(\varphi'(f(x)), \varphi'(f(y)))$, from which we get $(\varphi' \circ f \circ (\phi')^{-1})(T_2(\phi'(x), \phi'(y))) = S_1(\varphi'(f(x)), \varphi'(f(y)))$. By routine substitutions

$$g' = \varphi' \circ f \circ (\phi')^{-1},\ a' = \phi'(x),\ b' = \phi'(y), \tag{18}$$

we have the Cauchy like functional equation

$$g'(T_2(a', b')) = S_1(g'(a'), g'(b')), \quad \text{ for } a',\ b' \in [0,1], \tag{19}$$

where $g' \colon [0,1] \to [0,1]$ is an unknown function. This means that resolving of Eq. (8) when $\alpha < x, y < e$ is reduced to characterize all solutions of Eq. (19). Fortunately, the full characterization of this case can be found in Theorem 3.3.

Suppose $x, y > e$, define also two functions $\psi \colon [e,1] \to [0,1]$ and $\omega \colon [e, f(1)] \to [0,1]$ by the formulas $\phi(x) = \frac{x-e}{1-e}$ and $\omega(x) = \frac{x-e}{f(1)-e}$ respectively. Then there exists some continuous t-norm $T_2'$ such that two sides of Eq. (14) are respectively written as $U(x,y) = \psi^{-1}S_U(\psi(x), \psi(y))$ and $U(f(x), f(y)) = \omega^{-1}T_2'(\omega(f(x)), \omega(f(y)))$. Therefore, for any $(x,y) \in (e,1]^2$, Eq. (14) can be rewritten as $f(\psi^{-1}S_U(\psi(x), \psi(y))) = \omega^{-1}T_2'(\omega(f(x)), \omega(f(y)))$, from which we get $(\omega \circ f \circ \psi^{-1})(S_U(\psi(x), \psi(y))) = T_2'(\omega(f(x)), \omega(f(y)))$. By routine substitutions,

$$h = \omega \circ f \circ \psi^{-1},\ c = \psi(x),\ d = \psi(y), \tag{20}$$

we have the Cauchy like functional equation

$$h(S_U(c,d)) = T_2'(h(c), h(d)), \quad \text{ for } c,\ d \in [0,1], \tag{21}$$

where $h \colon [0,1] \to [0,1]$ is an unknown function. This means that resolving of Eq. (8) when $(x,y) \in [e,1]^2$ is reduced to characterize all solutions of Eq. (21). Fortunately, this case can be found in Theorem 3.4.

According to the above analyse and lemmas, we have the following theorem.

**Theorem 5.3.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0,1)$, a unary function $f \colon [0,1] \to [0,1]$, and the symbols $\alpha,\ g,\ g',\ h$ defined above by Eq.(15), (16), (18) and (20) respectively and fulfilling $\alpha < e$. Then $f$ satisfies Eq. (14) if and only if all of the following statements hold.

(i) It holds that $f(x) \in \mathbf{Id}(U)$ for all $x \in \mathbf{Id}(U)$.

(ii) It holds that $U(f(e), f(x)) = f(x)$ for all $x \in [0,1]$.

(iii) $f|_{[0,\alpha)}$ is increasing, $g$ satisfies Eq. (17), $Ran(f|_{[0,\alpha)}) \subseteq [0,e)$.

(iv) $f|_{(\alpha,e)}$ is decreasing, $g'$ satisfies Eq. (19), $Ran(f|_{(\alpha,e)}) \subseteq [f(1),1]$.

(v) $f|_{(e,1]}$ is increasing, $h$ satisfies Eq. (21), $Ran(f|_{(e,1]}) \subseteq [e, f(1)]$.

(vi) One of the following three statements hold:

(a) If $f(\alpha) < e$, then $f(\alpha) = \max(Ran(f|_{[0,\alpha]}))$,
(b) If $f(\alpha) > e$, then $f(\alpha) = \max(Ran(f|_{[0,1]}))$,
(c) $f(\alpha) = e$.

Proof. According to Lemmas 5.1, 5.2 and the above analyses, it is easy to show necessity. To finish proof, we need only check sufficiency. In virtue of hypotheses (i) and (ii), we know easily that all of $f(0), f(\alpha)$ and $f(1)$ are idempotent elements of $U$. Next, there are the following several cases to consider.

(A) Suppose either $x = e$ or $x = \alpha$ or $y = e$ or $y = \alpha$, it holds that Eq. (14) since hypothesis (iv), $e$ is neutral element of $U$ and $\alpha$ is idempotent.

(B) Suppose $x, y < e$, then we obtain from hypothesis (iii) that Eq. (14).

(C) Suppose $\alpha < x, y < e$, then we have from hypothesis (iv) that Eq. (14).

(D) Suppose $e < x, y < 1$, then we get from hypothesis (v) that Eq. (14).

(E) Suppose $x < \alpha < y$ and $y \neq e$, then it follows from hypotheses (iii) (iv) and (v) that $f(x) < e \leq f(y)$. We know from structure of $U$ that $U(f(x), f(y)) = \min(f(x), f(y)) = f(x)$. It follows from the idempotent element $\alpha$ and structure of $U$ that $U(x, y) = x$. Hence, we have $f(U(x, y)) = f(x) = \min(f(x), f(y)) = U(f(x), f(y))$ and then we obtain Eq. (14).

(F) Suppose $\alpha < x < e < y$, then we have from hypotheses (iv) and (v) that $f(x) \geq f(1) \geq f(y) \geq e$. And then later, applying structure of $U$, we get $U(f(x), f(y)) = \max(f(x), f(y)) = f(x)$. Further we obtain from the neutral element $e$ and structure of $U$ that $U(x, y) = x$. Therefore it holds that $f(U(x, y)) = f(x) = \max(f(x), f(y)) = U(f(x), f(y))$, thus we have Eq. (14).

The remaining cases clearly hold. □

## 6. CASE (II)

In this section, we discuss Eq. (14) under the condition $\alpha = e$ and the assumption that there exists some some $y_0 \in (e, 1]$ such that $f(y_0) < e$, but as seen later this assumption is not essential.

**Lemma 6.1.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0, 1)$, a unary function $f\colon [0, 1] \to [0, 1]$, the symbol $\alpha$ defined above by Eq.(15) and fulfilling $\alpha = e$ and there exists some $y_0 \in (e, 1]$ such that $f(y_0) < e$. If $f$ satisfies Eq. (14), then it holds that $f(y) < e$ for all $y \in [y_0, 1]$.

Proof. Take $y \in [y_0, 1]$, then we know from hypotheses and structure of $U$ that there exist some $y' \in (e, 1]$ such that $y = U(y', y_0)$. Note that hypothesis $f(y_0) < e$, then we have $f(y) = f(U(y', y_0)) = U(f(y'), f(y_0)) \leq \min(f(y'), f(y_0)) \leq f(y_0) < e$. □

Due to Lemma 6.1, define $F = \{x \in (e, 1] | f(x) < e\}$, then we stipulate

$$\beta = \inf F. \tag{22}$$

It is obvious that $e \le \beta$. Now, we claim that $\beta$ is an idempotent element of $U$, namely, $U(\beta,\beta) = \beta$. In fact, the result is clearly right when $\beta = e$. Hence, we assume $\beta > e$ and take $x \in (e,\beta)$. It follows from definition of $\beta$ that $f(x) \ge e$. Then we imply from Eq. (14) that $f(U(x,x)) = U(f(x), f(x)) \ge f(x) \ge e$, which shows $U(x,x) \notin F$, i. e., $U(x,x) \le \beta$. Let $x \to \beta$ and apply continuity of $U$ on the region $[e,1]^2$, then we obtain $U(\beta,\beta) \le \beta$. On the other hand, we know from $\beta > e$ that $U(\beta,\beta) \ge \beta$. Therefore it holds that $U(\beta,\beta) = \beta$. It is here pointed out that the case $\beta = 1$ do include the above excluded case that $f(x) \ge e$ for all $x \in (e,1]$. Hence, in next discussion, we will cancel this restriction that there exists some $y_0 \in (e,1]$ such that $f(y_0) < e$.

Next, depending on the order relation between $\beta$ and $e$, we need to consider two subcases: (i) $\beta = e$ and (ii) $\beta > e$. At first, let us consider the subcase $\beta = e$.

### 6.1. Subcase: $\beta = e$

**Lemma 6.2.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0,1)$, a unary function $f\colon [0,1] \to [0,1]$, the symbols $\alpha$ and $\beta$ defined above by Eq.(15) and Eq.(22) respectively and fulfilling $\alpha = \beta = e$. If $f$ satisfies Eq. (14), then the following two statements hold.

(i) $f|_{[0,e)}$ is increasing, $Ran(f|_{[0,e)}) \subseteq [0, f(1)]$.

(ii) $f|_{(e,1]}$ is decreasing, $Ran(f|_{(e,1]}) \subseteq [f(1), e)$.

Proof. (i) Suppose $x, y \in [0,e)$ and $x < y$, then we know from $\alpha = e$ that $f(x) < e$ and $f(y) < e$. In virtue of structure of $U$, there exists some $x' \in [0,e)$ such that $x = U(x', y)$. Hence it follows from Eq. (14) that $f(x) = f(U(x', y)) = U(f(x'), f(y)) \le \min(f(x'), f(y)) \le f(y)$. This means that $f|_{[0,e)}$ is increasing.

Take $x \in [0,e)$ and $y = 1$, then we know from $\alpha = e$ that $f(x) < e$. In virtue of structure of $U$, we have $x = U(x,1)$. Hence it follows from Eq. (14) that $f(x) = f(U(x,1)) = U(f(x), f(1)) \le \min(f(x), f(1)) \le f(1)$. We further obtain $Ran(f|_{[0,e)}) \subseteq [0, f(1)]$.

(ii) The proof is omitted since it is completely similar to Case (i). □

Suppose $x, y \in [0,e)$, define two functions $\phi_1\colon [0,e] \to [0,1]$ and $\varphi_1\colon [0, f(1)] \to [0,1]$ by the formulas $\phi_1(x) = \frac{x}{e}$ and $\varphi_1(x) = \frac{x}{f(1)}$ respectively. Then there exists some continuous t-norm $T_3$ such that both sides of Eq. (14) can be written as $U(x,y) = \phi_1^{-1} T_U(\phi_1(x), \phi_1(y))$ and $U(f(x), f(y)) = \varphi_1^{-1} T_3(\varphi_1(f(x)), \varphi_1(f(y)))$. Thus, for $x, y \in [0,e)$, Eq. (14) can be rewritten as $f(\phi_1^{-1} T_U(\phi_1(x), \phi_1(y))) = \varphi_1^{-1} T_3(\varphi_1(f(x)), \varphi_1(f(y)))$, from which we get $(\varphi_1 \circ f \circ \phi_1^{-1})(T_U(\phi_1(x), \phi_1(y))) = T_3(\varphi_1(f(x)), \varphi_1(f(y)))$. By routine substitution

$$g_1 = \varphi_1 \circ f \circ \phi_1^{-1},\ a_1 = \phi_1(x),\ b_1 = \phi_1(y), \tag{23}$$

we have the Cauchy like functional equation

$$g_1(T_U(a_1, b_1)) = T_3(g_1(a_1), g_1(b_1)), \quad \text{for } a_1,\ b_1 \in [0,1], \tag{24}$$

where $g_1\colon [0,1] \to [0,1]$ is an unknown function. This means that resolving of Eq. (14) when $x, y \in [0, e]$ is reduced to characterize all solutions of Eq. (24). Fortunately, the full characterization of this case can be found in Theorem 3.2.

Suppose $x, y > e$, define two functions $\psi_1\colon [e,1] \to [0,1]$ and $\omega_1\colon [f(1), e] \to [0,1]$ by the formulas $\psi_1(x) = \frac{x-e}{1-e}$ and $\omega_1(x) = \frac{x-f(1)}{e-f(1)}$ respectively. Then there exists some continuous t-norm $T_4$ such that two sides of Eq. (14) are respectively written as $U(x,y) = \psi_1^{-1} S_U(\psi_1(x), \psi_1(y))$ and $U(f(x), f(y)) = \omega_1^{-1} T_4(\omega_1(f(x)), \omega_1(f(y)))$. Thus, for any $(x,y) \in [e,1]^2$, Eq. (14) can be rewritten as $f(\psi_1^{-1} S_U(\psi_1(x), \psi_1(y))) = \omega_1^{-1} T_4(\omega_1(f(x)), \omega_1(f(y)))$, from which we get $(\omega_1 \circ f \circ \psi_1^{-1}) S_U(\psi_1(x), \psi_1(y))) = T_4(\omega_1 (f(x)), \omega_1(f(y)))$. By routine substitution

$$h_1 = \omega_1 \circ f \circ \psi_1^{-1},\ c_1 = \psi_1(x),\ d_1 = \psi_1(y), \tag{25}$$

we have the Cauchy like functional equation

$$h_1(S_U(c_1, d_1)) = T_4(h_1(c_1), h_1(d_1)), \qquad \text{for } c_1,\ d_1 \in [0,1], \tag{26}$$

where $h_1\colon [0,1] \to [0,1]$ is an unknown function. This means that resolving of Eq. (14) when $(x,y) \in [e,1]^2$ is reduced to characterize all solutions of Eq. (26). Fortunately, the full characterization of this case can be found in Theorem 3.4.

**Theorem 6.3.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0,1)$, a unary function $f\colon [0,1] \to [0,1]$, and the symbols $\alpha,\ \beta\ g_1,\ h_1$ are defined above by Eq.(15), (22), (23) and (25) respectively and fulfilling $\alpha = \beta = e$. Then $f$ satisfies Eq. (14) if and only if all of the following statements hold.

(i) It holds that $f(x) \in \mathbf{Id}(U)$ for all $x \in \mathbf{Id}(U)$.

(ii) It holds that $U(f(e), f(x)) = f(x)$ for all $x \in [0,1]$.

(iii) $f|_{[0,e)}$ is increasing, $g_1$ satisfies Eq. (24), $Ran(f|_{[0,e)}) \subseteq [0, f(1)]$.

(iv) $f|_{(e,1]}$ is decreasing, $h_1$ satisfies Eq. (26), $Ran(f|_{(e,1]}) \subseteq [f(1), e)$.

Proof. By means of Lemma 6.2 and the above analysis, it is easy to obtain necessity. Hence it is enough to show sufficiency. Indeed, it follows from conditions (i) and (ii) that both $f(1)$ and $f(e)$ are idempotent elements of $U$. To continue our progress, there are following five cases to discuss.

(A) Suppose $x = e$ or $y = e$, then it follows from hypothesis (ii) that Eq. (14) since $e$ is neutral element of $U$.

(B) Suppose $x, y < e$, then we get from hypotheses (iii) and (iv) that $f(x), f(y) \leq f(1) < e$. Using again hypothesis (iii), we know that Eq. (14) holds.

(C) Suppose $x, y > e$, then we know from hypothesis (iv) that Eq. (8) holds.

(D) Suppose $x < e < y$, then it follows from hypotheses (iii) and (iv) that $f(x) \leq f(1) \leq f(y) < e$. Applying structure of $U$, we obtain $U(f(x), f(y)) = \min(f(x), f(y)) = f(x)$. Note that $e$ is the neutral of $U$, then we have from structure of $U$ that $U(x,y) = x$. Therefore it holds that $f(U(x,y)) = f(x) = \min(f(x), f(y)) = U(f(x), f(y))$, that is, Eq. (14) holds.

(E) Suppose $y < e < x$, similar to Case (D), it obviously holds that Eq. (14). $\square$

**Remark 6.4.** Take $f(1) = f(e)$ in Theorem 6.3, then we get a part of Proposition 33 in Ref. [24].

Next, consider the remaining case $\beta > e$.

### 6.2. Subcase: $\beta > e$

**Lemma 6.5.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0, 1)$, a unary function $f\colon [0, 1] \to [0, 1]$, the symbols $\alpha$ and $\beta$ defined above by Eq.(15) and Eq.(22) respectively and fulfilling $\alpha = e < \beta$. If $f$ satisfies Eq. (14), then all of the following statements hold.

(i) $f|_{[0,e)}$ is increasing, $Ran(f|_{[0,e)}) \subseteq [0, f(1)]$.

(ii) $f|_{(e,\beta)}$ is increasing, $Ran(f|_{(e,\beta)}) \subseteq [e, 1]$.

(iii) $f|_{(\beta,1]}$ is decreasing, $Ran(f|_{(\beta,1]}) \subseteq [f(1), e)$.

Proof. We omit proofs of monotonicity of Cases (i), (ii) and (iii) since they are similar to these of Lemma 6.2. Therefore we only prove the remaining statements.

(i) Take $x \in [0, e)$ and $y = 1$, then we know from $\alpha = e$ and structure of $U$ that $f(x) < e$ and $U(x, 1) = x$. Hence it follows from Eq. (14) that$f(x) = f(U(x, 1)) = U(f(x), f(1)) = \min(f(x), f(1)) \leq f(1)$.

(ii) By definitions of $\beta$, the result obviously holds.

(iii) By definitions of $\beta$ and monotonicity of $f|_{(\beta,1]}$, we have $Ran(f|_{(\beta,1]}) \subseteq [f(1), e)$. □

**Lemma 6.6.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0, 1)$, a unary function $f\colon [0, 1] \to [0, 1]$, the symbols $\alpha$ and $\beta$ defined above by Eq.(15) and Eq.(22) respectively and fulfilling $\alpha = e < \beta$. If $f$ satisfies Eq. (14), then one of the following statements holds.

(i) If $f(\beta) < e$, then $f(\beta) = \max(Ran(f|_{[\beta,1]}))$.

(ii) If $f(\beta) > e$, then $f(\beta) = \max(Ran(f|_{[0,1]}))$.

(iii) $f(\beta) = e$.

Proof. We omit proofs since they are similar to these of Lemma 5.2. □

Suppose $x, y \in [0, e)$, define two functions $\phi_2\colon [0, e] \to [0, 1]$ and $\varphi_2\colon [0, f(1)] \to [0, 1]$ by the formulas $\phi_2(x) = \frac{x}{e}$ and $\varphi_2(x) = \frac{x}{f(1)}$ respectively. Then there exists some continuous t-norm $T_5$ such that both sides of Eq. (14) can be written as $U(x, y) = \phi_2^{-1} T_U(\phi_2(x), \phi_2(y))$ and $U(f(x), f(y)) = \varphi_2^{-1} T_5(\varphi_2(f(x)), \varphi_2(f(y)))$. Hence, for $x, y \in [0, e)$, Eq. (14) can be rewritten as $f(\phi_2^{-1} T_U(\phi_2(x), \phi_2(y))) = \varphi_2^{-1} T_5(\varphi_2(f(x)), \varphi_2(f(y)))$, from which we get $(\varphi_2 \circ f \circ \phi_2^{-1})(T_U(\phi_2(x), \phi_2(y))) = T_5(\varphi_2(f(x)), \varphi_2(f(y)))$. By routine substitution

$$g_2 = \varphi_2 \circ f \circ \phi_2^{-1},\ a_2 = \phi_2(x),\ b_2 = \phi_2(y), \tag{27}$$

we have the Cauchy like functional equation

$$g_2(T_U(a_2, b_2)) = T_5(g_2(a_2), g_2(b_2)), \quad \text{for } a_2,\ b_2 \in [0,1], \tag{28}$$

where $g_2\colon [0,1] \to [0,1]$ is an unknown function. This means that resolving of Eq. (14) when $x, y \in [0, e]$ is reduced to characterize all solutions of Eq. (28). Fortunately, the full characterization of this case can be found in Theorem 3.2.

Suppose $x, y \in (e, \beta)$, define two functions $\phi_2'\colon [e, \beta] \to [0,1]$ and $\varphi_2'\colon [e,1] \to [0,1]$ by the formulas $\phi_2'(x) = \frac{x-e}{\beta-e}$ and $\varphi_2'(x) = \frac{x-e}{1-e}$ respectively. Then there exists some continuous t-conorm $S_3$ such that two sides of Eq. (14) can be written as $U(x,y) = (\phi_2')^{-1} S_3(\phi_2'(x), \phi_2'(y))$ and $U(f(x), f(y)) = (\varphi_2')^{-1} S_U(\varphi_2'(f(x)), \varphi_2'(f(y))$. Therefore, for $x, y \in (e, \beta)$, Eq. (14) can be rewritten as $f((\phi_2')^{-1} S_3(\phi_2'(x), \phi_2'(y))) = (\varphi_2')^{-1} S_U(\varphi_2'(f(x)), \varphi_2'(f(y))$, from which we have $(\varphi_2' \circ f \circ (\phi_2')^{-1})(S_3(\phi_2'(x), \phi_2'(y))) = S_U(\varphi_2'(f(x)), \varphi_2'(f(y)))$. By routine substitution

$$g_2' = \varphi_2' \circ f \circ (\phi_2')^{-1},\ a_2' = \phi_2'(x),\ b_2' = \phi_2'(y), \tag{29}$$

we have the Cauchy like functional equation

$$g_2'(S_3(a_2', b_2')) = S_U(g_2'(a_2'), g_2'(b_2')), \quad \text{for } a_2',\ b_2' \in [0,1], \tag{30}$$

where $g_2'\colon [0,1] \to [0,1]$ is an unknown function. This means that resolving of Eq. (14) when $x, y \in [e, \beta)$ is reduced to characterize all solutions of Eq. (30). Fortunately, the full characterization of this case can be found in Theorem 3.1.

Suppose $x, y \in (\beta, 1]$, define two functions $\psi_2\colon [\beta, 1] \to [0,1]$ and $\omega_2\colon [f(1), e] \to [0,1]$ by the formulas $\psi_2(x) = \frac{x-\beta}{1-\beta}$ and $\omega_2(x) = \frac{x-f(1)}{e-f(1)}$ respectively. Then there exist a continuous t-conorm $S_4$ and a continuous t-norm $T_6$ such that two sides of Eq. (14) are respectively written as $U(x,y) = \psi_2^{-1} S_4(\psi_2(x), \psi_2(y))$ and $U(f(x), f(y)) = \omega_2^{-1} T_6(\omega_2(f(x)), \omega_2(f(y)))$. Therefore, for any $(x, y) \in [\beta, 1]^2$, Eq. (14) can be rewritten as $f(\psi_2^{-1} S_4(\psi_2(x), \psi_2(y))) = \omega_2^{-1} T_6(\omega_2(f(x)), \omega_2(f(y)))$, from which we have $(\omega_2 \circ f \circ \psi_2^{-1})(S_4(\psi_2(x), \psi_2(y))) = T_6(\omega_2(f(x)), \omega_2(f(y)))$. By routine substitution

$$h_2 = \omega_2 \circ f \circ \psi_2^{-1},\ c_2 = \psi_2(x),\ b_2 = \psi_2(y), \tag{31}$$

we have the Cauchy like functional equation

$$h_2(S_4(c_2, d_2)) = T_6(h_2(c_2), h_2(d_2)), \quad \text{for } c_2,\ d_2 \in [0,1], \tag{32}$$

where $h_2\colon [0,1] \to [0,1]$ is an unknown function. This means that resolving of Eq. (14) when $(x, y) \in [\beta, 1]^2$ is reduced to characterize all solutions of Eq. (32). Fortunately, the full characterization of this case can be found in Theorem 3.4.

**Theorem 6.7.** Consider a uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e \in (0,1)$, a unary function $f\colon [0,1] \to [0,1]$, the symbols $\alpha,\ \beta,\ g_2,\ g_2',\ h_2$ defined above by Eq.(15), Eq.(22), Eq.(27), Eq.(29) and Eq.(31) respectively and fulfilling $\alpha = e < \beta$. Then $f$ satisfies Eq. (14) if and only if all of the following statements hold.

(i) It holds that $f(x) \in \mathbf{Id}(U)$ for all $x \in \mathbf{Id}(U)$.

(ii) It holds that $U(f(e), f(x)) = f(x)$ for all $x \in [0,1]$.

(iii) $f|_{[0,e)}$ is increasing, $g_2$ satisfies Eq. (28), $Ran(f|_{[0,e)}) \subseteq [0, f(1)]$.

(iv) $f|_{(e,\beta)}$ is increasing, $g_2'$ satisfies Eq. (30), $Ran(f|_{(e,\beta)}) \subseteq [e, 1]$.

(v) $f|_{(\beta,1]}$ is decreasing, $h_2$ satisfies Eq. (32), $Ran(f|_{(\beta,1]}) \subseteq [f(1), e)$.

(vi) One of the following three statements hold:

(a) If $f(\beta) < e$, then $f(\beta) = \max(Ran(f|_{[\beta,1]}))$.
(b) If $f(\beta) > e$, then $f(\beta) = \max(Ran(f|_{[0,1]}))$.
(c) $f(\beta) = e$.

Proof. The proof is omitted because it is similar to that of Theorem 5.3. □

## 7. EXAMPLE

**Example 7.1.** Consider the following uninorm $U \in \mathcal{U}_{\min}$ with neutral element $e = \frac{1}{2}$,

$$U(x,y) = \begin{cases} 8x + 8y - 8xy - 7, & \text{if } (x,y) \in [\frac{7}{8}, 1]^2, \\ 7x + 7y - 8xy - \frac{21}{4}, & \text{if } (x,y) \in [\frac{3}{4}, \frac{7}{8}]^2, \\ \max(x,y), & \text{if } (x,y) \in [\frac{1}{2}, 1]^2 \setminus ([\frac{7}{8}, 1]^2 \cup [\frac{3}{4}, \frac{7}{8}]^2), \\ \frac{1}{8}(8x-1)(8y-1) + \frac{1}{8}, & \text{if } (x,y) \in [\frac{1}{8}, \frac{1}{4}]^2, \\ 8xy, & \text{if } (x,y) \in [0, \frac{1}{8}]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Then we know

$$T_U(x,y) = \begin{cases} 4xy, & \text{if } (x,y) \in [0, \frac{1}{4}]^2, \\ \frac{1}{4}(4x-1)(4y-1) + \frac{1}{4}, & \text{if } (x,y) \in [\frac{1}{4}, \frac{1}{2}]^2, \\ \min(x,y), & \text{otherwise,} \end{cases}$$

and

$$S_U(x,y) = \begin{cases} 8(x+1) + 8(y+1) - 4(x+1)(y+1) - 15, & \text{if } (x,y) \in [\frac{3}{4}, 1]^2, \\ 7(x+1) + 7(y+1) - 4(x+1)(y+1) - \frac{23}{2}, & \text{if } (x,y) \in [\frac{1}{2}, \frac{3}{4}]^2, \\ \max(x,y), & \text{otherwise.} \end{cases}$$

In fact, $T_U$ and $S_U$ are two ordinal sums with twice the product as summands and twice the probabilistic sum as summands respectively. Let us recall that the product $T_P = xy$ has a additive generator $t(x) = -\ln x$ while the probabilistic sum $S_U = x + y - xy$ has a additive generator $s(x) = -\ln(1-x)$.

(i) Take $\alpha = \frac{1}{8}$, then we know from Theorem 5.3 that

$$f(x) = \begin{cases} \frac{1}{8}, & \text{if } x \in [0, \frac{1}{8}], \\ 1, & \text{if } x \in (\frac{1}{8}, \frac{1}{2}), \\ \frac{1}{2}, & \text{if } x = \frac{1}{2}, \\ \frac{3}{4}, & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

is a non-monotone solution of Eq. (8).

(ii) Take $\alpha = e = \beta = \frac{1}{2}$, then we know from Theorem 6.3 that

$$f(x) = \begin{cases} \frac{1}{8}, & \text{if } x \in [0, \frac{1}{2}), \\ \frac{1}{2}, & \text{if } x = \frac{1}{2}, \\ \frac{1}{4}, & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

is a non-monotone solution of Eq. (8).

(iii) Take $\alpha = e = \frac{1}{2},\ \beta = \frac{7}{8}$, then we know from Theorem 6.7 that

$$f(x) = \begin{cases} \frac{1}{8}, & \text{if } x \in [0, \frac{1}{2}), \\ \frac{3}{4}, & \text{if } x \in [\frac{1}{2}, \frac{7}{8}], \\ \frac{1}{4}, & \text{if } x \in (\frac{7}{8}, 1], \end{cases}$$

is a non-monotone solution of Eq. (8).

## 8. CONCLUSION

To investigate property of commuting for bisymmetric aggregation operators with neutral element, according to Saminger, Mesiar and Dubois's suggestion [24], in this paper, we have investigated and fully characterized the following functional equation $f(U(x, y)) = U(f(x), f(y))$, where $f\colon [0, 1] \to [0, 1]$ is an unknown function but unnecessarily non-decreasing, a uninorm $U \in \mathcal{U}_{\min}$ has a continuously underlying t-norm $T_U$ and a continuously underlying t-conorm $S_U$. Our investigation shows this equation has also non-monotone solutions completely different with already obtained ones. These results are an important step towards obtaining a complete characterization of the above-mentioned other unary distributive functional equations. Obviously, there are several unary distribute functions not to be consider in this direction. Thus, future work will be devoted to deal with $f(U(x, y)) = U(f(x, y))$, where $f\colon [0, 1] \to [0, 1]$ is an unknown function but unnecessarily non-decreasing, and $U$ comes from the other kind of special uninorms.

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