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LOG-OPTIMAL INVESTMENT IN THE LONG RUN WITH PROPORTIONAL TRANSACTION COSTS WHEN USING SHADOW PRICES

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We consider a non-consuming agent interested in the maximization of the long-run growth rate of a wealth process investing either in a money market and in one risky asset following a geometric Brownian motion or in futures following an arithmetic Brownian motion. The agent faces proportional transaction costs, and similarly as in [17] where the case of stock trading is considered, we show how the log-optimal optimal policies in the long run can be derived when using the technical tool of shadow prices. We also provide a brief link between technical tools used in this paper and the ones used in [14,15,17].

Keywords: proportional transaction costs, logarithmic utility, shadow prices

Classification: 60H30, 60G44, 91B28

1. INTRODUCTION

The purpose of this paper is to illustrate how the notion of shadow prices can be used as a technical tool for solving the problem of investment in the presence of proportional transaction costs when we maximize the growth rate of the wealth process $(\mathcal{W}_t)_{t\geq 0}$ in the long run as follows

$$\max \liminf_{t\to\infty} \tfrac{1}{t}\, E[\ln \mathcal{W}_t], \tag{1}$$

where the maximum is taken over the set of such admissible strategies that $\ln \mathcal{W}_0$ is an integrable random variable. Here, we treat the case of stock and futures trading together with almost explicit results which may serve as illustration. We also provide a link between our tools and the ones used in some other papers also interested in the maximization problem (1). By [3] the problem (1) is a limiting case of the investment-consumption problem with logarithmic utility when we are interested in stock trading. The problem (1) is difficult to treat without passing $t \to \infty$ since the optimal policies depend also on time which is pointed out in [17]. See [25, 35] for its solution for finite discrete time. Note that the problem (1) can be understood as a maximization of the long-run growth rate of the certainty equivalent from the wealth process with logarithmic utility, and see [14] and [15] for the martingale approach to such a problem and for

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an almost explicit solution corresponding to utility functions with hyperbolic absolute risk aversion (HARA) unbounded from below.

This paper was inspired by [24] where the investment-consumption problem with logarithmic utility is treated with the help of shadow prices and where one does not have an explicit solution of the corresponding ODE.

The reader interested

- in shadow prices is referred to [4, 5, 6, 7, 16, 17, 18, 19, 24, 25, 26, 32],
- in logarithmic utility to [1, 3, 8, 9, 10, 11, 20, 27, 33, 34, 39, 40],
- in the investment-consumption problem with presence of proportional transaction costs to [2, 3, 12, 21, 22, 28, 29, 30, 36].

The reader interested in futures trading with transaction costs is referred to [14, 22]. For the investment problem with proportional transaction costs and non-constant coefficients see [13] and [26].

The paper is organized as follows. The next section is divided into subsections. In the first one, we introduce the Black-Scholes model for the stock market price and the bid-ask spread. We define an admissible strategy as a self-financing strategy such that the wealth process is positive if it is computed from the bid and ask price of the stock. For such strategies we are allowed to introduce the so called position in the market, which corresponds to the proportion of wealth invested in a stock. The second subsection is devoted to the introduction of "futures trading" with proportional transaction costs where the futures price follows an arithmetic Brownian motion.

In the third subsection, we make the first step towards the notion of shadow prices. It is based on the idea that any continuous semimartingale attaining values in the bid-ask spread can play the role of the price if we modify the value of transaction taxes in order to preserve the bid-ask spread. In Lemma 2.12, we show that the criterion (1) is not affected by such a modification of the market price under very general assumptions. The rest of the subsection is devoted to the dynamics of modified wealth and modified position presented in Lemmas 2.15, 2.18 and 2.20.

The last subsection of the second section is devoted to sufficient conditions for the modified price such that the strategy maximizing the long-run growth rate in the frictionless market is also the maximal long-run growth rate of the wealth process in the market with transaction costs.

The first subsection of the third section is devoted to the definition of shadow price and its interpretation as a dual optimizer. It is such a modified price that the maximal long-run growth rate of the wealth process in the frictionless market can be reached also in the market with transaction costs, and consequently, such a price offers the worst opportunity to maximize the long-run growth rate of the minimal wealth process, as stated in Theorem 3.11.

Subsection 3.2, namely Theorem 3.13, puts together the general theory from subsection 3.1 and the theory from section 2. In subsection 3.3, we provide an intuitive technique taken from [24] of searching for the shadow price. It is based on finding a certain ODE whose solution given by Lemma 3.18 enables us to say how the price should be modified. Part 3.4 uses calculations and arguments from stochastic analysis in order to show (Theorem 3.21) that the shadow price exists under a lot of technical assumptions denoted as (A1 – A3), and we show in Lemma 3.23 that there exists an admissible strategy keeping just the position within a certain interval.

Subsection 3.5 is preliminary and it is based on calculations and arguments from mathematical analysis. It introduces and explores the functions $\xi_{a,\varepsilon}, y_u$ helping us to switch between the nominal and modified positions.

In the last subsection, we show how the nominal price should be modified into a shadow price in Theorem 3.31, and in Corollary 3.32 we show that the derived strategy is log-optimal in the long run among a wide class of strategies. Finally, we show that it is also log-optimal in the long run among all admissible strategies up to a certain restriction on the initial wealth in Theorem 3.34.

In section 4, we compare the technical tools of this and some other papers and we also offer a brief presentation of results of the martingale approach to the maximization of the long-run growth rate of the certainty equivalent from the wealth process when considering HARA utility functions unbounded from below.

The last section is complementary and we only prove there that any admissible strategy does not lead to the bankruptcy almost surely, as stated in Remark 2.6.

2. NOTATION AND MODEL SET UP

In this section, we introduce the model of stock trading and futures trading with proportional transaction costs, the notion of modified price and cost-free strategy and we show that the cost-free strategy keeping the position process on the log-optimal proportion maximizes the long-run growth rate of the wealth process among a wide class of strategies, see Lemma 2.12 and Theorem 2.26.

For brevity of the notation, we write $(X_t, Y_t, Z_t, \ldots)_{t\ge 0}$ for a collection of processes with the index set $\mathbb{R}^+ \triangleq [0,\infty)$ and when using this notation we do not distinguish for example between three processes written one after another and between the corresponding triplet.

2.1. Stock trading with proportional transaction costs

We consider an agent who may invest in a money market with interest rate $r \ge 0$ and in one risky asset called stock with the market price $(S_t)_{t\ge 0}$ following the model of geometric Brownian motion

$$S_t \triangleq s_0 \exp\{\sigma B_t + (\mu - \tfrac{\sigma^2}{2})\, t\}, \quad t \ge 0, \tag{2}$$

driven by a standard Brownian motion $(B_t)_{t\ge 0}$ as follows

$$\mathrm{d}S_t = S_t(\mu \,\mathrm{d}t + \sigma \,\mathrm{d}B_t).$$

Here, $\mu \in \mathbb{R}\ s_0, \sigma \in (0,\infty)$ are constants similarly as r which will be assumed to be equal to zero for simplicity and also without loss of generality. We assume that the underlying probability space $(\Omega, \mathcal{F}, P)$ is complete and we consider a completed filtration $(\mathcal{F}_t)_{t\ge 0}$ generated by $(B_t)_{t\ge 0}$.

Definition 2.1. By a *(trading) strategy* we mean a pair $(\varphi_t, \psi_t)_{t\geq 0}$ of adapted right-continuous processes with (finite) left-hand limits and with locally finite variation. Then $(\varphi_t, \psi_t)_{t\geq 0}$ stands for the processes describing the number of shares held in the stock and in the bank account and the corresponding *wealth process* is defined as follows

$$\mathcal{W}_t \triangleq \psi_t + \varphi_t S_t, \quad t \geq 0. \tag{3}$$

The agent faces transaction costs that are proportional to the size of the transaction so that one pays $S_t^{\uparrow}$ for one share of the stock at time $t \geq 0$ and gets only $S_t^{\downarrow}$ for it, where

$$S_t^{\uparrow} \triangleq (1+\lambda^{\uparrow})S_t, \quad S_t^{\downarrow} \triangleq (1-\lambda^{\downarrow})S_t,$$

and where $\lambda^{\uparrow} \in (0,\infty)$ and $\lambda^{\downarrow} \in (0,1)$.

Remark 2.2. Let $(\varphi_t, \psi_t)_{t\geq 0}$ be a trading strategy. As $(\varphi_t)_{t\geq 0}$ is assumed to be rcll-process with locally finite variation, we get that there exists a pair $(\varphi_t^{\uparrow}, \varphi_t^{\downarrow})_{t\geq 0}$ of adapted non-decreasing rcll-processes starting from 0 such that

$$\varphi_t = \varphi_0 + \varphi_t^{\uparrow} - \varphi_t^{\downarrow}, \quad t \geq 0, \tag{4}$$

and such that $(\varphi_t^{\uparrow}, \varphi_t^{\downarrow})_{t\geq 0}$ do not grow at the same time, i. e. that the Lebesgue-Stieltjes measures induced by both processes are mutually singular. Then these measures restricted to $[0,T]$ represent the Hahn decomposition of a signed measure induced by $(\varphi_{t\wedge T})_{t\geq 0}$ if $T > 0$, and, as Hahn decomposition is unique, we have that the processes $(\varphi_t^{\uparrow}, \varphi_t^{\downarrow})_{t\geq 0}$ starting from zero are uniquely determined by $(\varphi_t)_{t\geq 0}$. Then $(\varphi_t^{\uparrow}, \varphi_t^{\downarrow})_{t\geq 0}$ are interpreted as the *number of shares bought and sold up to time t*, respectively, and the assumption that they do not grow at the same time is just a natural requirement that the corresponding strategy does not buy and sell the stock simultaneously.

Definition 2.3. Let $(\varphi_t, \psi_t)_{t\geq 0}$ be a trading strategy. Then the corresponding *value of transaction costs* up to $t \geq 0$ is defined as follows

$$C_t^{\varphi} \triangleq \textstyle\int_0^t S_s(\lambda^{\uparrow}\mathrm{d}\varphi_s^{\uparrow} + \lambda^{\downarrow}\mathrm{d}\varphi_s^{\downarrow}). \tag{5}$$

The above integral is understood in Lebesgue-Stieltjes sense and as the integrators are right-continuous the integral from 0 to t is understood as integral over $(0,t]$. Consequently, $(C_t^{\varphi})_{t\geq 0}$ is again a non-decreasing right-continuous adapted process.

Definition 2.4. We say that the strategy $(\varphi_t, \psi_t)_{t\geq 0}$ is *self-financing* if

$$\psi_t = \psi_0 - \textstyle\int_0^t S_s^{\uparrow}\,\mathrm{d}\varphi_s^{\uparrow} + \int_0^t S_s^{\downarrow}\,\mathrm{d}\varphi_s^{\downarrow}, \quad t \geq 0. \tag{6}$$

Obviously, the stock market price $(S_t)_{t\geq 0}$ is a continuous semimartingale with

$$\mathrm{d}S_t = S_t\,\mathrm{d}F_t, \qquad \text{where} \qquad F_t \triangleq \mu t + \sigma B_t, \quad t \geq 0. \tag{7}$$

If $(\varphi_t, \psi_t)_{t\geq 0}$ is a self-financing strategy, then the corresponding wealth process $(\mathcal{W}_t)_{t\geq 0}$ is an adapted right-continuous process with left-hand limits with

$$\mathrm{d}\mathcal{W}_t = \varphi_t\,\mathrm{d}S_t - \mathrm{d}C_t^{\varphi} = S_t[\varphi_t\,\mathrm{d}F_t - \lambda^{\uparrow}\mathrm{d}\varphi_t^{\uparrow} - \lambda^{\downarrow}\mathrm{d}\varphi_t^{\downarrow}]. \tag{8}$$

Note that this is a differential form of an integral equality. The integrals with respect to $\mathrm{d}S_t$, $\mathrm{d}F_t$ are considered in the classical sense of continuous stochastic integration and the ones with respect to $\mathrm{d}C_t^{\varphi}$, $\mathrm{d}\varphi_t^{\uparrow}$, $\mathrm{d}\varphi_t^{\downarrow}$ are considered in the Lebesgue-Stieltjes sense. Further note that verification of (8) is straightforward as it uses only *integration by parts formula* $S_t\varphi_t - S_0\varphi_0 - \int_0^t S_s \,\mathrm{d}\varphi_s \overset{\text{as}}{=} \int_0^t \varphi_s \,\mathrm{d}S_s$ for a continuous semimartingale $(S_t)_{t\geq 0}$ and a right-continuous adapted process $(\varphi_t)_{t\geq 0}$ of locally finite variation.

Definition 2.5. A self-financing strategy $(\varphi_t, \psi_t)_{t\geq 0}$ is called *admissible* if the wealth processes computed from the ask and bid prices $(S_t^{\uparrow}, S_t^{\downarrow})_{t\geq 0}$ are positive, i. e. if

$$\mathcal{W}_t^{\uparrow} \triangleq \psi_t + \varphi_t S_t^{\uparrow} > 0, \quad \mathcal{W}_t^{\downarrow} \triangleq \psi_t + \varphi_t S_t^{\downarrow} > 0, \quad t \geq 0. \tag{9}$$

Remark 2.6. If we neglect null sets, we can say that admissible strategy is such a self-financing strategy with which the investor will never get into bankruptcy. Indeed, a wealth process of an admissible strategy is positive, which can be seen as follows

$$0 < \min\{\mathcal{W}_t^{\uparrow}, \mathcal{W}_t^{\downarrow}\} = \mathcal{W}_t - S_t(\lambda^{\uparrow}\varphi_t^{-} + \lambda^{\downarrow}\varphi_t^{+}) \leq \mathcal{W}_t,$$

where we have used the following equalities

$$\mathcal{W}_t^{\uparrow} = \mathcal{W}_t + \lambda^{\uparrow}\varphi_t S_t, \quad \mathcal{W}_t^{\downarrow} = \mathcal{W}_t - \lambda^{\downarrow}\varphi_t S_t. \tag{10}$$

On the other hand, if a self-financing strategy never leads to a bankruptcy almost surely, i. e. $\tau \triangleq \{t \geq 0; \mathcal{W}_t \leq 0\} \overset{\text{as}}{=} \infty$, then it is up to a null set equal to an admissible strategy as it is proved in Lemma 5.5 in the appendix of this paper.

Definition 2.7. Whenever $(\varphi_t, \psi_t)_{t\geq 0}$ is an admissible strategy, we introduce the corresponding *position process* $(\pi_t)_{t\geq 0}$ as

$$\pi_t \triangleq \varphi_t S_t / \mathcal{W}_t. \tag{11}$$

Further, we denote $\mathcal{A} \triangleq (-1/\lambda^{\uparrow}, 1/\lambda^{\downarrow})$ and we call it the *set of admissible positions.*

Remark 2.8. The position process $(\pi_t)_{t\geq 0}$ of an admissible strategy describes the ratio of the investors wealth invested in the risky asset and it attains values in $\mathcal{A}$. To see the latter statement, it is enough to realize that the corresponding wealth process $(\mathcal{W}_t)_{t\geq 0}$ is positive and that

$$0 < \mathcal{W}_t^{\uparrow} = \mathcal{W}_t(1 + \lambda^{\uparrow}\pi_t), \quad 0 < \mathcal{W}_t^{\downarrow} = \mathcal{W}_t(1 - \lambda^{\downarrow}\pi_t), \quad t \geq 0, \tag{12}$$

as the corresponding strategy is assumed to be admissible.

2.2. Futures trading with proportional transaction costs

We are going to introduce a (theoretical) concept of *futures without expiration* based on the assumption of a zero interest rate.

We assume that there is a stock with the market price $(F_t)_{t\geq 0}$ which can be also negative due to possible expenses. Let us consider an agent taking a long position in a forward contract with maturity $T > t$ at time $t \geq 0$, for example, so that he/she agrees to pay the forward price $F(t,T)$ for the stock at time T. If we exclude arbitrage opportunities, we get that $F(t,T) = F_t$ has to hold whenever $0 \le t \le T$.

Thus, instead of stock trading, the agent may conclude and cancel forward contracts so that keeping a long position on $(t,T]$ brings $F_T - F_t$ and keeping a short position means loosing this value. As concluding and canceling of the forward contract is not immediate and as such contracts are not standardized for trading, the agent may prefer using futures, which one can imagine as a standardized forward contract with daily reevaluation, so that the agent obtains the difference $F_r - F_t$ if he/she takes a long position in a futures contract on $(t,r]$ similarly as in the previous case, however, this time, he/she obtains the corresponding increment of the futures price every day.

As futures contracts are daily reevaluated, we do not have to restrict ourselves to finite T. As only the increments of the futures price $(F_t)_{t\ge0}$ are important, we may assume that it starts from zero, i. e. $F_0 = 0$. Then trading futures without expiration, i. e. with the expiry date $T = \infty$, is nothing else but betting on the process $(F_t)_{t\ge0}$, which is modelled here similarly as in [22] by an arithmetic Brownian motion formally introduced on the right-hand side of (7).

In contrast with the stock trading, the process $(\varphi_t)_{t\ge0}$ stands for the number of concluded futures contracts. As the agent does not possess any stock, the number of shares held in the bank account is equal to the investor's wealth process, which is denoted as $(\mathcal{W}_t)_{t\ge0}$. It turns out that it is useful to denote

$$\psi_t \triangleq \mathcal{W}_t - \varphi_t F_t, \quad t \ge 0,$$

cf. (3). The agent has to pay transaction costs that are proportional to increments and decrements of $(\varphi_t)_{t\ge0}$ and this is the reason why we restrict to processes $(\varphi_t)_{t\ge0}$ satisfying (4) where the processes $(\varphi^{\uparrow}_t, \varphi^{\downarrow}_t)_{t\ge0}$ formally satisfy the same technical assumptions as in subsection 2.1. The *value of transaction costs* is here specified as follows

$$C^{\varphi}_t \triangleq \lambda^{\uparrow}\varphi^{\uparrow}_t + \lambda^{\downarrow}\varphi^{\downarrow}_t, \tag{13}$$

where $\lambda^{\uparrow}, \lambda^{\downarrow} \in (0,\infty)$. In order to be able to treat the stock and futures trading together, we also need to introduce processes

$$F^{\uparrow}_t \triangleq F_t + \lambda^{\uparrow}, \qquad F^{\downarrow}_t \triangleq F_t - \lambda^{\downarrow}, \tag{14}$$

$t \ge 0$, that will be referred to as the ask and bid price of the futures contract and we also introduce

$$\mathcal{W}^{\uparrow}_t \triangleq \mathcal{W}_t + \lambda^{\uparrow}\varphi_t = \psi_t + \varphi_t F^{\uparrow}_t, \qquad \mathcal{W}^{\downarrow}_t \triangleq \mathcal{W}_t - \lambda^{\downarrow}\varphi_t = \psi_t + \varphi_t F^{\downarrow}_t, \tag{15}$$

$t \ge 0$, cf. (9,10). Here, the *self-financing condition* is of the form

$$\psi_t = \psi_0 - \int_0^t F^{\uparrow}_s \,\mathrm{d}\varphi^{\uparrow}_s + \int_0^t F^{\downarrow}_s \,\mathrm{d}\varphi^{\downarrow}_s, \tag{16}$$

$t \ge 0$. Note that it can be easily verified by integration by parts formula that (16) is up to a null set equivalent to the condition

$$\mathcal{W}_t \overset{\text{as}}{=} \mathcal{W}_0 + \int_0^t \varphi_s \,\mathrm{d}F_s - C^{\varphi}_t \overset{\text{as}}{=} \mathcal{W}_0 + \int_0^t [\varphi_s \,\mathrm{d}F_s - \lambda^{\uparrow}\,\mathrm{d}\varphi^{\uparrow}_s - \lambda^{\downarrow}\,\mathrm{d}\varphi^{\downarrow}_s]\,\mathrm{d}s, \tag{17}$$

$t \ge 0$, cf. (8).

Further, we call a self-financing strategy $(\varphi_t, \psi_t)_{t\geq 0}$ *admissible* if the processes introduced in (15) are positive. Then again, we obtain, from inequalities $\mathcal{W}_t \geq \mathcal{W}_t^{\uparrow} \wedge \mathcal{W}_t^{\downarrow} > 0, t \geq 0$, that the wealth process is positive, and thus, we are allowed to define the *position process* $(\pi_t)_{t\geq 0}$ as follows

$$\pi_t \triangleq \varphi_t/\mathcal{W}_t, \quad t \geq 0. \tag{18}$$

Similarly as in Remark 2.8, we would get that the equalities in (12) hold and that the position process $(\pi_t)_{t\geq 0}$ of an admissible strategy attains values in $\mathcal{A}$.

2.3. Modified prices and modified wealth

In this subsection, we modify the nominal market price and the transaction taxes so that the bid-ask spread remains the same. We show in Lemma 2.12 that such a modification does not affect the long-run growth rate of the wealth process.

As the stock market price $(S_t)_{t\geq 0}$ enters the model only via ask and bid prices $(S_t^{\uparrow}, S_t^{\downarrow})_{t\geq 0}$, we can afford to modify the market price $(S_t)_{t\geq 0}$ if we also modify the transaction taxes $\lambda^{\uparrow}, \lambda^{\downarrow}$ so that the corresponding ask and bid prices remain the same.

Definition 2.9. Let $\tilde{\varepsilon} = (\tilde{\varepsilon}_t)_{t\geq 0}$ be a continuous semimartingale with values in $[-\lambda^{\downarrow}, \lambda^{\uparrow}]$. Then $(\tilde{S}_t)_{t\geq 0}$ from

$$\tilde{S}_t = (1 + \tilde{\varepsilon}_t) S_t, \quad t \geq 0 \tag{19}$$

is referred to as the *stock market $\tilde{\varepsilon}$-price.* Realize that it is just a continuous semimartingale attaining values in the bid-ask spread $\tilde{S}_t \in [S_t^{\downarrow}, S_t^{\uparrow}], t \geq 0$. Then the processes $(\tilde{\lambda}_t^{\uparrow}, \tilde{\lambda}_t^{\downarrow})_{t\geq 0}$ satisfying

$$(1 + \tilde{\lambda}_t^{\uparrow})\tilde{S}_t = S_t^{\uparrow}, \quad (1 - \tilde{\lambda}_t^{\downarrow})\tilde{S}_t = S_t^{\downarrow}, \tag{20}$$

$t \geq 0$, are referred to as the *$\tilde{\varepsilon}$-transaction taxes*, and we call the following process

$$\tilde{\mathcal{W}}_t \triangleq \psi_t + \varphi_t \tilde{S}_t, \quad t \geq 0, \tag{21}$$

the *$\tilde{\varepsilon}$-wealth process.* If $(\varphi_t, \psi_t)_{t\geq 0}$ is an admissible strategy, then, similarly as in Remark 2.6, we obtain that $\tilde{\mathcal{W}}_t \geq \mathcal{W}_t^{\uparrow} \wedge \mathcal{W}_t^{\downarrow} > 0, t \geq 0$, and therefore, we may introduce the *$\tilde{\varepsilon}$-position process* $(\tilde{\pi}_t)_{t\geq 0}$ as follows

$$\tilde{\pi}_t \triangleq \varphi_t \tilde{S}_t/\tilde{\mathcal{W}}_t.$$

Note that the $\tilde{\varepsilon}$-wealth process $(\tilde{\mathcal{W}}_t)_{t\geq 0}$ can be also expressed as

$$\tilde{\mathcal{W}}_t = \mathcal{W}_t + \tilde{\varepsilon}_t \varphi_t S_t. \tag{22}$$

We are going to introduce analogous notions also for the futures.

Definition 2.10. Let $\tilde\varepsilon = (\tilde\varepsilon_t)_{t\geq 0}$ be a continuous semimartingale with values in $[-\lambda^{\downarrow}, \lambda^{\uparrow}]$. Then the process $(\tilde F_t)_{t\geq 0}$ from

$$\tilde F_t = F_t + \tilde\varepsilon_t, \quad t \geq 0 \tag{23}$$

is referred to as the *futures $\tilde\varepsilon$-price* and the processes $(\tilde\lambda_t^{\uparrow}, \tilde\lambda_t^{\downarrow})_{t\geq 0}$ satisfying

$$\tilde F_t + \tilde\lambda_t^{\uparrow} = F_t^{\uparrow}, \quad \tilde F_t - \tilde\lambda_t^{\downarrow} = F_t^{\downarrow}, \tag{24}$$

are referred to as the *$\tilde\varepsilon$-transaction taxes.* The *$\tilde\varepsilon$-wealth process* $(\tilde{\mathcal W}_t)_{t\geq 0}$ is now defined analogously to (21,22) as

$$\tilde{\mathcal W}_t \triangleq \mathcal W_t + \tilde\varepsilon_t\,\varphi_t = \psi_t + \varphi_t \tilde F_t. \tag{25}$$

Then again, any admissible strategy satisfies $\tilde{\mathcal W}_t \geq \mathcal W_t^{\uparrow} \wedge \mathcal W_t^{\downarrow} > 0, t \geq 0$, and we may introduce the *$\tilde\varepsilon$-position process* $(\tilde\pi_t)_{t\geq 0}$ as follows

$$\tilde\pi_t \triangleq \varphi_t / \tilde{\mathcal W}_t.$$

Remark 2.11. In the following, we will treat both cases, the stock and the futures trading, together. We consider a binary parameter $a \in \{0, 1\}$ indicating that stock trading is considered. If $a = 0$, we are allowed to write S_t^a and $\tilde S_t^a$, which is treated as 1. If $a = 1$, we are allowed to use $(\tilde F_t)_{t\geq 0}$ standing for a continuous semimartingale such that

$$\mathrm d\tilde S_t = \tilde S_t\,\mathrm d\tilde F_t. \tag{26}$$

Such a process $(\tilde F_t)_{t\geq 0}$ exists as the stock market $\tilde\varepsilon$-price $(\tilde S_t)_{t\geq 0}$ is a positive continuous semimartingale by definition.

It turns out, as shown in the next Lemma, that the long-run growth rates of the wealth process and of the $\tilde\varepsilon$-wealth process are the same provided that the corresponding position is kept away from the boundary values $-1/\lambda^{\uparrow}, 1/\lambda^{\downarrow}$. This justifies comparing the long-run growth rates of the $\tilde\varepsilon$-wealth processes of two competing strategies.

Lemma 2.12. Let $(\mathcal W_t, \tilde{\mathcal W}_t)_{t\geq 0}$ be the wealth process and the $\tilde\varepsilon$-wealth process, respectively, corresponding to the admissible strategy $(\varphi_t, \psi_t)_{t\geq 0}$. If the position process $(\pi_t)_{t\geq 0}$ attains values in some compact subset of $\mathcal A$, then $\ln(\tilde{\mathcal W}_t/\mathcal W_t), t \geq 0$ is a bounded process. In particular, we have that

$$\tfrac{1}{t}\, E[\ln \tfrac{\tilde{\mathcal W}_t}{\mathcal W_t}] \to 0 \quad \text{and} \quad \tfrac{1}{t}\, \ln \tfrac{\tilde{\mathcal W}_t}{\mathcal W_t} \xrightarrow{\text{as}} 0, \quad \text{as} \quad t \to \infty. \tag{27}$$

Proof. As (12) holds in both cases $a \in \{0, 1\}$, we obtain that

$$\min\{1 + \lambda^{\uparrow}\pi_t, 1 - \lambda^{\downarrow}\pi_t\} = \tfrac{\mathcal W_t^{\uparrow} \wedge \mathcal W_t^{\downarrow}}{\mathcal W_t} \leq \tfrac{\tilde{\mathcal W}_t}{\mathcal W_t} \leq \tfrac{\mathcal W_t^{\uparrow} \vee \mathcal W_t^{\downarrow}}{\mathcal W_t} = \max\{1 + \lambda^{\uparrow}\pi_t, 1 - \lambda^{\downarrow}\pi_t\}.$$

The rest follows immediately from the restriction on values of $(\pi_t)_{t\geq 0}$. □

Notation 2.13. Let us denote by rcll(Ω) the set of all right-continuous processes with (finite) left-hand limits (rcll) indexed by $t \geq 0$ and defined on Ω and by $\mathbb{C}(\Omega)$ its subset of continuous processes. Note that $(\varphi_t, \varphi_t^{\uparrow}, \varphi_t^{\downarrow})_{t\geq 0} \in \mathrm{rcll}(\Omega)^3$ in general in this paper.

If $(\varphi_t, \psi_t)_{t\geq 0}$ is a self-financing strategy, then also $(\psi_t)_{t\geq 0} \in \mathrm{rcll}(\Omega)$ and as this set is closed under sums and products and as $\mathbb{C}(\Omega) \subseteq \mathrm{rcll}(\Omega)$, we get from (21,25) that also $(\tilde{\mathcal{W}}_t)_{t\geq 0} \in \mathrm{rcll}(\Omega)$.

Definition 2.14. Let $(\tilde{S}_t)_{t\geq 0}$ be a stock market $\tilde{\varepsilon}$-price (if $a = 1$) and let $(\tilde{\lambda}_t^{\uparrow}, \tilde{\lambda}_t^{\downarrow})_{t\geq 0}$ be the $\tilde{\varepsilon}$-transaction taxes. Then we define the *$\tilde{\varepsilon}$-value of transaction costs* on $(0, t]$ as

$$\tilde{C}_t^{\varphi} \triangleq \int_0^t \tilde{S}_s^a [\tilde{\lambda}_s^{\uparrow} \,\mathrm{d}\varphi_s^{\uparrow} + \tilde{\lambda}_s^{\downarrow} \,\mathrm{d}\varphi_s^{\downarrow}], \quad t \geq 0, \tag{28}$$

cf. $(5, 13)$. The next Lemma justifies the above interpretation of $(\tilde{C}_t^{\varphi})_{t\geq 0}$.

Lemma 2.15. Let $(\varphi_t, \psi_t)_{t\geq 0}$ be a self-financing strategy with the $\tilde{\varepsilon}$-wealth $(\tilde{\mathcal{W}}_t)_{t\geq 0}$ and the $\tilde{\varepsilon}$-value of transaction costs $(\tilde{C}_t^{\varphi})_{t\geq 0}$. Then $(\tilde{\mathcal{W}}_t + \tilde{C}_t^{\varphi})_{t\geq 0}$ is a continuous process and

$$\tilde{\mathcal{W}}_t \overset{\mathrm{as}}{=} \tilde{\mathcal{W}}_0 + \int_0^t \varphi_s \tilde{S}_s^a \,\mathrm{d}\tilde{F}_t - \tilde{C}_t^{\varphi}, \quad t \geq 0,$$

cf. $(8, 17)$. In the differential form with $\tilde{\varepsilon}$-transaction taxes $(\tilde{\lambda}_t^{\uparrow}, \tilde{\lambda}_t^{\downarrow})_{t\geq 0}$, we have that

$$\mathrm{d}\tilde{\mathcal{W}}_t = \tilde{S}_t^a [\varphi_t \,\mathrm{d}\tilde{F}_t - \tilde{\lambda}_t^{\uparrow} \,\mathrm{d}\varphi_t^{\uparrow} - \tilde{\lambda}_t^{\downarrow} \,\mathrm{d}\varphi_t^{\downarrow}]. \tag{29}$$

P r o o f. As $(\tilde{\mathcal{W}}_t, \tilde{C}_t^{\varphi})_{t\geq 0} \in \mathrm{rcll}(\Omega)^2$, we may show that their sum is a continuous process simply by calculating the corresponding jumps from left.

Let $a = 1$. We obtain from $(6, 21)$ and integration by parts formula that

$$\mathrm{d}\tilde{\mathcal{W}}_t = \varphi_t \,\mathrm{d}\tilde{S}_t + (\tilde{S}_t - S_t^{\uparrow}) \,\mathrm{d}\varphi_t^{\uparrow} + (S_t^{\downarrow} - \tilde{S}_t) \,\mathrm{d}\varphi_t^{\downarrow}. \tag{30}$$

Then we get by $(20, 26)$ that (29) holds. Similarly, we obtain that

$$\Delta \tilde{\mathcal{W}}_t \triangleq \tilde{\mathcal{W}}_t - \tilde{\mathcal{W}}_{t-} = (\tilde{S}_t - S_t^{\uparrow}) \,\Delta\varphi_t^{\uparrow} + (S_t^{\downarrow} - \tilde{S}_t) \Delta \,\varphi_t^{\downarrow} = -\Delta \,\tilde{C}_t^{\varphi}, \quad t \geq 0, \tag{31}$$

where $\Delta\varphi_t^{\uparrow}, \Delta\varphi_t^{\downarrow}, \Delta \,\tilde{C}_t^{\varphi}$ stand for jumps from left defined similarly as $\Delta \,\tilde{\mathcal{W}}_t$.

If $a = 0$, omit (26) and replace (6,20,21) by (16,24,25) in order to get (30,31) with $(\tilde{S}_t, S_t^{\uparrow}, S_t^{\downarrow})_{t\geq 0}$ replaced by $(\tilde{F}_t, F_t^{\uparrow}, F_t^{\downarrow})_{t\geq 0}$. □

Lemma 2.16. Let $(\varphi_t, \psi_t)_{t\geq 0}$ be an admissible strategy with the $\tilde{\varepsilon}$-wealth and the $\tilde{\varepsilon}$-position $(\tilde{\mathcal{W}}_t, \tilde{\pi}_t)_{t\geq 0}$, then $(\tilde{\mathcal{W}}_t^{-1}, \tilde{\pi}_t)_{t\geq 0} \in \mathrm{rcll}(\Omega)^2$.

In particular, the choice $\tilde{\varepsilon} = 0$ gives that $(\mathcal{W}_t^{-1}, \pi_t)_{t\geq 0} \in \mathrm{rcll}(\Omega)^2$ holds if $(\mathcal{W}_t, \pi_t)_{t\geq 0}$ are the wealth process and the position of $(\varphi_t, \psi_t)_{t\geq 0}$.

Proof. As $(\varphi_t, \tilde{S}^a_t)_{t\ge 0} \in \mathrm{rcll}(\Omega)^2$ and as $\mathrm{rcll}(\Omega)$ is closed under products, it is enough to show that $(\tilde{\mathcal{W}}_t^{-1})_{t\ge 0} \in \mathrm{rcll}(\Omega)$ in order get that also $\tilde{\pi}_t = \varphi_t \tilde{S}^a_t \tilde{\mathcal{W}}_t^{-1}, t \ge 0$, is in $\mathrm{rcll}(\Omega)$. As $(\tilde{\mathcal{W}}_t)_{t\ge 0} \in \mathrm{rcll}(\Omega)$, it is enough to show that $(\tilde{\mathcal{W}}_t^{-1})_{t\ge 0}$ has locally bounded trajectories, i. e. that $\inf_{s\le t} \tilde{\mathcal{W}}_s > 0$ holds for every $t \ge 0$.

Since $(\tilde{C}^\varphi_t)_{t\ge 0} \in \mathrm{rcll}(\Omega)$ does not decrease and $(\tilde{\mathcal{W}}_t + \tilde{C}^\varphi_t)_{t\ge 0}$ is a continuous process and as the considered strategy $(\varphi_t, \psi_t)_{t\ge 0}$ is admissible, we have that $\tilde{\mathcal{W}}_{t-} \ge \tilde{\mathcal{W}}_t > 0$ holds for every $t \ge 0$. As an infimum of a rcll-function on $[0,t]$ is attained provided that the function jumps only downwards, we get that the inequality $\inf_{s\le t} \tilde{\mathcal{W}}_s > 0$ indeed holds if $t \ge 0$. □

Remark 2.17. If the strategy $(\varphi_t, \psi_t)_{t\ge 0}$ from Lemma 2.15 is admissible, then (29) reads as follows

$$\mathrm{d}\tilde{\mathcal{W}}_t = \tilde{\mathcal{W}}_t \tilde{\pi}_t \,\mathrm{d}\tilde{F}_t - \mathrm{d}\tilde{C}^\varphi_t . \tag{32}$$

This equation has an almost surely unique solution $(\tilde{\mathcal{W}}_t)_{t\ge 0}$ given $(\tilde{\pi}_t, \tilde{F}_t, \tilde{C}^\varphi_t)_{t\ge 0}$ and the initial value $\tilde{\mathcal{W}}_0$, and it is described in the Lemma 2.18. In order to see that the solution of the equation (32) is really unique almost surely, consider the difference $(X_t)_{t\ge 0}$ of two solutions with the same initial value. Then it is an adapted rcll-process such that $X_t \overset{\mathrm{as}}{=} \int_0^t X_s \tilde{\pi}_s \,\mathrm{d}\tilde{F}_s, t \ge 0$, which is possible only if $X_t \overset{\mathrm{as}}{=} 0, t \ge 0$.

Lemma 2.18. Let $(\tilde{\mathcal{W}}_t)_{t\ge 0}$ be the $\tilde{\varepsilon}$-wealth process of an admissible strategy with the $\tilde{\varepsilon}$-position $(\tilde{\pi}_t)_{t\ge 0}$. Then

$$\tilde{\mathcal{W}}_t \overset{\mathrm{as}}{=} \tilde{\mathcal{E}}^\pi_t \cdot [\tilde{\mathcal{W}}_0 - \textstyle\int_0^t (\tilde{\mathcal{E}}^\pi_s)^{-1} \,\mathrm{d}\tilde{C}^\varphi_s] \le \tilde{\mathcal{E}}^\pi_t \,\tilde{\mathcal{W}}_0, \quad t \ge 0, \tag{33}$$

where $\tilde{\mathcal{E}}^\pi_t \triangleq \exp\{\int_0^t (\tilde{\pi}_s \,\mathrm{d}\tilde{F}_s - \frac{1}{2}\,\tilde{\pi}_s^2 \,\mathrm{d}\langle \tilde{F}\rangle_s)\}$.

Proof. As $(\tilde{\pi}_t)_{t\ge 0} \in \mathrm{rcll}(\Omega)$ is adapted, $(\int_0^t \tilde{\pi}_s \,\mathrm{d}\tilde{F}_s)_{t\ge 0}$ is a well defined continuous semimartingale. As $Y_t \triangleq \tilde{\mathcal{W}}_t + \tilde{C}^\varphi_t, t \ge 0$, and $(\tilde{\mathcal{E}}^\pi_t)^{-1}_{t\ge 0}$ are continuous semimartingales with

$$\mathrm{d}Y_t = \tilde{\pi}_t \tilde{\mathcal{W}}_t \,\mathrm{d}\tilde{F}_t, \qquad \mathrm{d}(\tilde{\mathcal{E}}^\pi_t)^{-1} = (\tilde{\mathcal{E}}^\pi_t)^{-1}[-\tilde{\pi}_t \,\mathrm{d}\tilde{F}_t + \tilde{\pi}_t^2 \,\mathrm{d}\langle \tilde{F}\rangle_t],$$

we obtain, with the help of calculus of continuous stochastic integration, that

$$(\tilde{\mathcal{E}}^\pi_t)^{-1} Y_t \overset{\mathrm{as}}{=} Y_0 + \textstyle\int_0^t \tilde{C}^\varphi_s \,\mathrm{d}(\tilde{\mathcal{E}}^\pi_s)^{-1}, \quad t \ge 0. \tag{34}$$

As $(\tilde{C}^\varphi_t)_{t\ge 0} \in \mathrm{rcll}(\Omega)$ is a non-decreasing adapted process and $(\tilde{\mathcal{E}}^\pi_t)^{-1}_{t\ge 0}$ a continuous semimartingale, we obtain, with the help of integration by parts formula, that

$$(\tilde{\mathcal{E}}^\pi_t)^{-1} \tilde{C}^\varphi_t \overset{\mathrm{as}}{=} \tilde{C}^\varphi_0 + \textstyle\int_0^t \tilde{C}^\varphi_s \,\mathrm{d}(\tilde{\mathcal{E}}^\pi_s)^{-1} + \int_0^t (\tilde{\mathcal{E}}^\pi_s)^{-1} \,\mathrm{d}\tilde{C}^\varphi_s , \tag{35}$$

$t \ge 0$. If we subtract (35) from (34), we obtain the equality almost surely in (33) while the right-hand inequality in (33) obviously holds as $(\tilde{C}^\varphi_t)_{t\ge 0}$ is a non-decreasing process and $(\mathcal{E}^\pi_t)_{t\ge 0}$ attains only positive values. □

Definition 2.19. We say that a strategy $(\varphi_t, \psi_t)_{t\geq 0}$ is *continuous* if the processes $(\varphi_t, \psi_t)_{t\geq 0}$ have continuous trajectories.

Lemma 2.20. Let $(\varphi_t, \psi_t)_{t\geq 0}$ be a continuous admissible strategy with the $\tilde\varepsilon$-wealth process and the $\tilde\varepsilon$-position process $(\tilde{\mathcal{W}}_t, \tilde\pi_t)_{t\geq 0}$ and let $(\tilde\lambda^{\uparrow}_t, \tilde\lambda^{\downarrow}_t)_{t\geq 0}$ be the $\tilde\varepsilon$-transaction taxes. Then $(\tilde\pi_t)_{t\geq 0}$ is a continuous semimartingale and the following holds for every $t \geq 0$

$$\begin{aligned}\tilde\pi_t \stackrel{\text{as}}{=} \tilde\pi_0 &+ \int_0^t \tilde\pi_s (a - \tilde\pi_s)[\,\mathrm{d}\tilde F_s - \tilde\pi_s \,\mathrm{d}\langle \tilde F\rangle_s] \\ &+ \int_0^t \tilde{\mathcal{W}}_s^{-1} \tilde S^a_s [(1 + \tilde\lambda^{\uparrow}_s \tilde\pi_s)\,\mathrm{d}\varphi^{\uparrow}_s - (1 - \tilde\lambda^{\downarrow}_s \tilde\pi_s)\,\mathrm{d}\varphi^{\downarrow}_s].\end{aligned}$$

Proof. By assumption, $(\tilde S_t)_{t\geq 0}$ is a continuous semimartingale and $\tilde S_t \stackrel{\text{as}}{=} \tilde S_0\, e^{\tilde F_t - \langle \tilde F\rangle_t/2}$ if $a = 1$. By a calculation using $a \in \{0, 1\}$, we obtain from Itô Lemma that

$$Z_t \triangleq \tilde S^a_t / \tilde{\mathcal{E}}^\pi_t \stackrel{\text{as}}{=} Z_0 \exp\{\int_0^t (a - \tilde\pi_s)\,\mathrm{d}\tilde F_s - \tfrac{1}{2}\int_0^t (a - \tilde\pi_s^2)\,\mathrm{d}\langle \tilde F\rangle_s\}, \quad t \geq 0,$$

is a continuous semimartingale with

$$\mathrm{d}Z_t = Z_t (a - \tilde\pi_t)[\,\mathrm{d}\tilde F_t - \tilde\pi_t\,\mathrm{d}\langle \tilde F\rangle_t]. \tag{36}$$

Further, $V_t \triangleq \tilde{\mathcal{W}}_t / \tilde{\mathcal{E}}^\pi_t \stackrel{\text{as}}{=} \tilde{\mathcal{W}}_0 - \int_0^t (\tilde{\mathcal{E}}^\pi_s)^{-1}\,\mathrm{d}\tilde C^\varphi_s, t \geq 0$, is a continuous adapted process of locally finite variation attaining only positive values with

$$\mathrm{d}V_t = -(\tilde{\mathcal{E}}^\pi_t)^{-1}\,\mathrm{d}\tilde C^\varphi_t \qquad \text{and} \qquad \mathrm{d}V_t^{-1} = \tilde{\mathcal{W}}_t^{-1} V_t^{-1}\,\mathrm{d}\tilde C^\varphi_t.$$

See (28) for the differential $\mathrm{d}\tilde C^\varphi_t$. As $(\varphi_t)_{t\geq 0}$ is also a continuous adapted process of locally finite variation, we get that $(\varphi_t V_t^{-1})_{t\geq 0}$ possesses the same property and

$$\begin{aligned}\mathrm{d}(\varphi_t V_t^{-1}) &= V_t^{-1}[\,\mathrm{d}\varphi_t + \varphi_t \tilde{\mathcal{W}}_t^{-1} \tilde S^a_t (\tilde\lambda^{\uparrow}_t\,\mathrm{d}\varphi^{\uparrow}_t + \tilde\lambda^{\downarrow}_t\,\mathrm{d}\varphi^{\downarrow}_t)] \\ &= V_t^{-1}[(1 + \tilde\lambda^{\uparrow}_t \tilde\pi_t)\,\mathrm{d}\varphi^{\uparrow}_t - (1 - \tilde\lambda^{\downarrow}_t \tilde\pi_t)\,\mathrm{d}\varphi^{\downarrow}_t].\end{aligned} \tag{37}$$

As $Z_t V_t^{-1} = \tilde S^a_t \tilde{\mathcal{W}}_t^{-1}$ holds, we obtain by (36,37) that $\tilde\pi_t = \varphi_t \tilde S^a_t \tilde{\mathcal{W}}_t^{-1} = \varphi_t Z_t V_t^{-1}, t \geq 0$, is a continuous semimartingale with

$$\begin{aligned}\mathrm{d}\tilde\pi_t &= \varphi_t V_t^{-1}\,\mathrm{d}Z_t + Z_t\,\mathrm{d}(\varphi_t V_t^{-1}) \\ &= \tilde\pi_t (a - \tilde\pi_t)[\,\mathrm{d}\tilde F_t - \tilde\pi_t\,\mathrm{d}\langle \tilde F\rangle_t] + \tilde S^a_t \tilde{\mathcal{W}}_t^{-1}[(1 + \tilde\lambda^{\uparrow}_t \tilde\pi_t)\,\mathrm{d}\varphi^{\uparrow}_t - (1 - \tilde\lambda^{\downarrow}_t \tilde\pi_t)\,\mathrm{d}\varphi^{\downarrow}_t].\end{aligned}$$

□

2.4. Long-run growth rate

In this subsection, we introduce so called cost-free strategy and log-optimal proportion, and we show that a cost-free strategy keeping its modified position on the log-optimal proportion maximizes the long-run growth rate of the modified wealth among a wide class of strategies in Theorem 2.26.

Definition 2.21. An admissible strategy $(\varphi_t, \psi_t)_{t\geq 0}$ is called an $\tilde{\varepsilon}$-*cost-free strategy* if the $\tilde{\varepsilon}$-value of transaction costs $(\tilde{C}_t^{\varphi})_{t\geq 0}$ from (28) attains only the value 0 almost surely.

Remark 2.22. Let $(\tilde{\mathcal{W}}_t, \tilde{\pi}_t)_{t\geq 0}$ be the $\tilde{\varepsilon}$-wealth process and the $\tilde{\varepsilon}$-position process of an $\tilde{\varepsilon}$-cost-free strategy $(\varphi_t, \psi_t)_{t\geq 0}$. Then we get by Lemma 2.18 that $\tilde{\mathcal{W}}_t \overset{\text{as}}{=} \tilde{\mathcal{E}}_t^{\pi} \tilde{\mathcal{W}}_0, t \geq 0$. Since $\tilde{\pi}_t \tilde{\mathcal{W}}_t = \varphi_t \tilde{S}_t^a$, we obtain that the strategy $(\varphi_t, \psi_t)_{t\geq 0}$ is, up to a null set, uniquely determined by $\tilde{\mathcal{W}}_0$ and $(\tilde{\pi}_t, \tilde{F}_t)_{t\geq 0}$.

Definition 2.23. We say that the stock market $\tilde{\varepsilon}$-price $(\tilde{S}_t)_{t\geq 0}$ (if $a = 1$) or the futures $\tilde{\varepsilon}$-price $(\tilde{F}_t)_{t\geq 0}$ (if $a = 0$) is *regular* if there exist continuous adapted processes $(\tilde{\mu}_t, \tilde{\sigma}_t)_{t\geq 0}$ such that

$$\mathrm{d}\tilde{F}_t = \tilde{\mu}_t \,\mathrm{d}t + \tilde{\sigma}_t \,\mathrm{d}B_t, \tag{38}$$

and that $(\tilde{\mu}_t, \ln \tilde{\sigma}_t)_{t\geq 0}$ are bounded processes. The processes $(\tilde{\mu}_t, \tilde{\sigma}_t)_{t\geq 0}$ are referred to as the $\tilde{\varepsilon}$-*coefficients* and given $(\tilde{F}_t, B_t)_{t\geq 0}$ they are determined uniquely up to a null set.

Definition 2.24. Let us consider a regular (stock market or futures) $\tilde{\varepsilon}$-price with $\tilde{\varepsilon}$-coefficients $(\tilde{\mu}_t, \tilde{\sigma}_t)_{t\geq 0}$. Then we introduce the $\tilde{\varepsilon}$-*log-optimal proportion* $(\tilde{\theta}_t)_{t\geq 0}$ as

$$\tilde{\theta}_t \triangleq \tilde{\sigma}_t^{-2} \tilde{\mu}_t, \quad t \geq 0. \tag{39}$$

Remark 2.25. Let $(\tilde{\mathcal{W}}_t, \tilde{\pi})_{t\geq 0}$ be the $\tilde{\varepsilon}$-wealth process and the $\tilde{\varepsilon}$-position of an $\tilde{\varepsilon}$-cost-free strategy, then

$$\tilde{\mathcal{W}}_t \overset{\text{as}}{=} \tilde{\mathcal{W}}_0 \,\tilde{\mathcal{E}}_t^{\pi} \overset{\text{as}}{=} \tilde{\mathcal{W}}_0 \exp\{\textstyle\int_0^t \tilde{\sigma}_s \tilde{\pi}_s \,\mathrm{d}B_s + \int_0^t (\tilde{\mu}_s \tilde{\pi}_s - \frac{1}{2}\,\tilde{\sigma}_s^2 \tilde{\pi}_s^2)\,\mathrm{d}s\}.$$

Obviously, the quadratic function $x \mapsto \tilde{\mu}x - \tilde{\sigma}^2 x^2/2$ attains its maximum at $x = \tilde{\sigma}^{-2}\tilde{\mu}$ provided that $\tilde{\sigma} \in (0, \infty)$. This is the reason why $(\tilde{\theta}_t)_{t\geq 0}$ from (39) is referred to as the $\tilde{\varepsilon}$-log-optimal proportion. Further, see Theorem 2.26 for the properties of an $\tilde{\varepsilon}$-cost-free strategy keeping the $\tilde{\varepsilon}$-position on the $\tilde{\varepsilon}$-log-optimal proportion.

Theorem 2.26. Let us consider a regular $\tilde{\varepsilon}$-price. Assume that $(\varphi_t^*, \psi_t^*)_{t\geq 0}$ is an $\tilde{\varepsilon}$-cost-free strategy with the $\tilde{\varepsilon}$-wealth process and $\tilde{\varepsilon}$-position $(\tilde{\mathcal{W}}_t^*, \tilde{\pi}_t^*)_{t\geq 0}$ and that the $\tilde{\varepsilon}$-position process equals to the $\tilde{\varepsilon}$-log-optimal proportion almost surely, i. e. $(\tilde{\pi}_t^*)_{t\geq 0} \overset{\text{as}}{=} (\tilde{\theta}_t)_{t\geq 0}$. Let $(\varphi_t, \psi_t)_{t\geq 0}$ be a competing admissible strategy with the $\tilde{\varepsilon}$-wealth $(\tilde{\mathcal{W}}_t)_{t\geq 0}$. Then

$$\limsup_{t\to\infty} \tfrac{1}{t}\, \ln(\tilde{\mathcal{W}}_t/\tilde{\mathcal{W}}_t^*) \overset{\text{as}}{\leq} 0. \tag{40}$$

Moreover, if $E \max\{0, \ln(\tilde{\mathcal{W}}_0/\tilde{\mathcal{W}}_0^*)\} < \infty$, then also

$$\limsup_{t\to\infty} \tfrac{1}{t}\, E \ln(\tilde{\mathcal{W}}_t/\tilde{\mathcal{W}}_t^*) \leq 0. \tag{41}$$

The proof needs the following Lemma.

Lemma 2.27. Let $(L_t)_{t\geq 0}$ be a continuous local martingale with $L_0 = 0$. Then

$$\limsup_{t\to\infty} \tfrac{1}{t}\,(L_t - \tfrac{1}{2}\,\langle L\rangle_t) \overset{\text{as}}{\leq} 0. \tag{42}$$

Proof. From the strong law of large numbers for a standard Brownian motion and from Dambis–Dubins-Schwartz Theorem, see Theorem 16.4 in [23], we get that

$$\tfrac{L_t}{\langle L\rangle_t}\, 1_{[\langle L\rangle_\infty=\infty]} \overset{\text{as}}{\longrightarrow} 0 \qquad \text{as} \qquad t\to\infty.$$

Then, obviously (even without the fraction $\frac{1}{t}$), we have that

$$\limsup_{t\to\infty} \tfrac{1}{t}\,(L_t - \tfrac{1}{2}\,\langle L\rangle_t)\cdot 1_{[\langle L\rangle_\infty=\infty]} \overset{\text{as}}{\leq} 0. \tag{43}$$

Note that Dambis–Dubins–Schwartz Theorem also gives the fact that there exists a real-valued random variable L_∞ such that $(L_t - L_\infty)\cdot 1_{[\langle L\rangle_\infty<\infty]} \overset{\text{as}}{\longrightarrow} 0$ as $t\to\infty$. Then

$$\limsup_{t\to\infty} \tfrac{1}{t}\,(L_t - \tfrac{1}{2}\,\langle L\rangle_t)\cdot 1_{[\langle L\rangle_\infty<\infty]} \leq \lim_{t\to\infty} \tfrac{1}{t}\, L_t\cdot 1_{[\langle L\rangle_\infty<\infty]} \overset{\text{as}}{=} 0. \tag{44}$$

Then (42) follows from (43, 44). □

Proof of Theorem 2.26. As the strategy $(\varphi^*_t,\psi^*_t)_{t\geq 0}$ is $\tilde\varepsilon$-cost-free, we have that the $\tilde\varepsilon$-value of the corresponding transaction costs $(\tilde C^*_t)_{t\geq 0}$ is zero almost surely. As this strategy keeps the $\tilde\varepsilon$-position $(\tilde\pi^*_t)_{t\geq 0}$ on the $\tilde\varepsilon$-log-optimal proportion $(\tilde\theta_t)_{t\geq 0}$ almost surely, we get by Lemma 2.18 that

$$\begin{aligned} \tilde{\mathcal W}^*_t &\overset{\text{as}}{=} \tilde{\mathcal E}^*_t\,\tilde{\mathcal W}^*_0, &&\text{with} && \tilde{\mathcal E}^*_t \triangleq \exp\{\textstyle\int_0^t(\tilde\theta_s\,\mathrm d\tilde F_s - \frac{1}{2}\,\tilde\theta^2_s\,\mathrm d\langle\tilde F\rangle_s)\},\\ \tilde{\mathcal W}_t &\overset{\text{as}}{\leq} \tilde{\mathcal E}_t\,\tilde{\mathcal W}_0, &&\text{with} && \tilde{\mathcal E}_t \triangleq \exp\{\textstyle\int_0^t[\tilde\pi_s\,\mathrm d\tilde F_s - \frac{1}{2}\,\tilde\pi^2_s\,\mathrm d\langle\tilde F\rangle_s]\},\end{aligned}$$

where $(\tilde\pi_t)_{t\geq 0}$ is the $\tilde\varepsilon$-position of the strategy $(\varphi_t,\psi_t)_{t\geq 0}$. Note that

$$\ln(\tilde{\mathcal E}_t/\tilde{\mathcal E}^*_t) \overset{\text{as}}{=} L_t - \tfrac{1}{2}\,\langle L\rangle_t, \qquad \text{where} \qquad L_t \triangleq \textstyle\int_0^t\tilde\sigma_s(\,\tilde\pi_s - \tilde\theta_s)\,\mathrm dB_s.$$

As $(\tilde\sigma_t,\tilde\theta_t)_{t\geq 0}\in\mathbb C(\Omega)^2$ and $(\tilde\pi_t)_{t\geq 0}\in\mathrm{rcll}(\Omega)$ are adapted process, $(L_t)_{t\geq 0}\in\mathbb C(\Omega)$ is a well defined local martingale starting from $L_0=0$. Then $(\tilde{\mathcal E}_t/\tilde{\mathcal E}^*_t)_{t\geq 0}\in\mathbb C(\Omega)$ is a non-negative local martingale starting from 1, hence a supermartingale with $E(\tilde{\mathcal E}_t/\tilde{\mathcal E}^*_t)\leq 1$. Then by Jensen inequality, $E\ln(\tilde{\mathcal E}_t/\tilde{\mathcal E}^*_t)\leq \ln 1 = 0$ holds for every $t\geq 0$, and we obtain the first inequality in

$$\limsup_{t\to\infty}\tfrac{1}{t}\,E\ln(\tilde{\mathcal E}_t/\tilde{\mathcal E}^*_t)\leq 0, \qquad \limsup_{t\to\infty}\tfrac{1}{t}\ln(\tilde{\mathcal E}_t/\tilde{\mathcal E}^*_t)\overset{\text{as}}{\leq} 0, \tag{45}$$

while the second one follows from Lemma 2.27. As $\ln(\tilde{\mathcal W}_t/\tilde{\mathcal W}^*_t)-\ln(\tilde{\mathcal E}_t/\tilde{\mathcal E}^*_t)\overset{\text{as}}{\leq}\ln(\tilde{\mathcal W}_0/\tilde{\mathcal W}^*_0)$ and as the right-hand side does not depend on $t\geq 0$, we immediately obtain (40) from the second inequality in (45), while the first one gives (41) if $E\ln(\tilde{\mathcal W}_0/\tilde{\mathcal W}^*_0)<\infty$. □

Remark 2.28. Let us consider the case when $\theta \triangleq \sigma^{-2}\mu \in \{0, a\}$ and when an investor with a positive initial wealth $\mathcal{W}_0^* > 0$ keeps the position $(\pi_t^*)_{t\geq 0}$ on the log-optimal proportion θ. Then the corresponding strategy $(\varphi_t^*, \psi_t^*)_{t\geq 0}$ is of the form $(0, \psi_0^*)$ if $\theta = 0$, which corresponds to the advice "keep all the wealth in the money market", and of the form $(\varphi_0^*, 0)$ corresponding to the advice "keep all the wealth in the stock" if $\theta = a = 1$. Such strategies are obviously admissible. As the strategy does not trade, it is $\tilde{\varepsilon}$-cost-free regardless of $\tilde{\varepsilon}$ and hence, it is also $\tilde{\varepsilon}$-cost-free for the choice $\tilde{\varepsilon} \equiv 0$. Obviously, the 0-price, i. e. the nominal price itself, is regular by definition, and we get by Theorem 2.26 that $(\varphi_t^*, \psi_t^*)_{t\geq 0}$ has the wealth process $(\mathcal{W}_t^*)_{t\geq 0} = (\tilde{\mathcal{W}}_t^*)_{t\geq 0}$ with the maximal long-run growth rate among a wide class of strategies if $\ln \mathcal{W}_0^* \in \mathbb{L}_1$.

Remark 2.29. As we deal with processes from rcll(Ω), we have excluded an initial trade represented by a jump of $(\varphi_t, \psi_t)_{t\geq 0}$ at $t = 0$ from the model and we just assume that the initial trade has been executed at time $t = 0$ keeping in mind that the corresponding effect of the initial trade is negligible from the point of view of the long-run growth rate of the wealth process.

3. SHADOW PRICE

In this section, we introduce a notion of a shadow price. In the first part, we show that it can be understood as a dual optimizer in the general setting. In the second part, we customize the general setting to the considered cases of stock and futures trading. In the third part, we provide an intuitive technique of searching for the shadow price taken from [24]. In the fourth part, we provide certain technical assumptions (A1-A3) ensuring that a modified price is also a shadow price. The fifth part is supporting for the last part, in which we show how to construct a shadow price and that the corresponding strategy is log-optimal in the long run.

3.1. Shadow price as a dual optimizer

In this subsection, we introduce a notion of a shadow price in Definition 3.9 as a price that offers the same opportunity to maximize the long-run growth rate of the wealth process in the frictionless market and in the market with transaction costs. In Theorem 3.7, we provide the theoretical background for this notion with explanation of the assumptions and conclusion given in subsequent Remark 3.8. In Remark 3.10, we show that the notion of a shadow price considered in this paper is very far from being unique and, finally, in Theorem 3.11 we offer interpretation of the shadow price as the price that offers the worst opportunity to maximize the long-run growth rate of the wealth process in the frictionless market. This property of the shadow price together with its definition is the reason why it can be understood as a dual optimizer.

Notation 3.1. If $t \in [0, \infty)$, by $\mathbb{L}(\mathcal{F}_t)$ we denote the set of all real-valued $\mathcal{F}_t$-measurable random variables. Further, $FV(\mathcal{F}_t)$ will stand for the set of all $(\mathcal{F}_t)_{t\geq 0}$-adapted rcll-processes of locally finite variation, i. e.

$$FV(\mathcal{F}_t) \triangleq \{L \in \text{rcll}(\Omega); \forall\ t \in [0, \infty)\ \ L_t \in \mathbb{L}(\mathcal{F}_t),\ \text{variation of } (L_s)_{s\leq t} \text{ is finite }\}.$$

Definition 3.2. By a *market* we mean a couple of continuous semimartingales $(\mathbb{A},\mathbb{B}) = (\mathbb{A}_t,\mathbb{B}_t)_{t\geq 0}$, such that $\mathbb{A}_t \geq \mathbb{B}_t$ holds whenever $t \in [0,\infty)$. If $\mathbb{A} = \mathbb{B}$, then the market is called *frictionless*, otherwise it is called a *market with transaction costs.* The processes $\mathbb{A},\mathbb{B}$ are also referred to as *ask* and *bid prices.*

The pair $(\varphi,\psi) \in FV(\mathcal{F}_t)^2$ is called a *self-financing trading strategy* in the market $(\mathbb{A},\mathbb{B})$ if it satisfies the self-financing condition (6) with $(S^\uparrow, S^\downarrow)$ replaced by $(\mathbb{A},\mathbb{B})$ with $(\varphi^\uparrow,\varphi^\downarrow)$ specified in Remark 2.2. If the self-financing strategy (φ,ψ) in $(\mathbb{A},\mathbb{B})$ satisfies

$$\forall\ t \in [0,\infty) \quad \mathcal{W}_t^{\mathbb{A}}(\varphi,\psi) \triangleq \psi_t + \varphi_t\mathbb{A}_t > 0, \quad \mathcal{W}_t^{\mathbb{B}}(\varphi,\psi) \triangleq \psi_t + \varphi_t\mathbb{B}_t > 0,$$

cf. (9), then the strategy (φ,ψ) is call *admissible* in the market $(\mathbb{A},\mathbb{B})$.

Notation 3.3. Note that any self-financing strategy $(\varphi,\psi) = (\varphi_t,\psi_t)_{t\geq 0}$ is, by definition, uniquely determined by (φ,ψ_0) given the ask and bid price processes. Further in this section, we briefly denote the self-financing strategy (φ,ψ) in the market $(\mathbb{A},\mathbb{B})$ corresponding to (φ,ψ_0) as $\mathcal{S}_f(\varphi,\psi_0;\mathbb{A},\mathbb{B})$.

Notation 3.4. Given the ask and bid processes $(\mathbb{A},\mathbb{B}) = (\mathbb{A}_t,\mathbb{B}_t)_{t\geq 0}$, we consider the corresponding minimum of the ask and bid wealth process, further referred to as the *minimal wealth process*, defined as follows

$$\mathcal{W}_t(\varphi,\psi_0;\mathbb{A},\mathbb{B}) \triangleq \psi_0 + \min(\varphi_t\mathbb{A}_t,\varphi_t\mathbb{B}_t) - \int_0^t \mathbb{A}_s\,\mathrm{d}\varphi_s^\uparrow + \int_0^t \mathbb{B}_s\,\mathrm{d}\varphi_s^\downarrow \quad t \geq 0, \tag{46}$$

cf. (6,9,16,15), and we also consider the set of all admissible values of (φ,ψ_0) defined as

$$\mathscr{A}(\mathbb{A},\mathbb{B}) \triangleq \{(\varphi,\psi_0) \in FV(\mathcal{F}_t)\times\mathbb{L}(\mathcal{F}_0);\forall\ t\in[0,\infty)\ \ \mathcal{W}_t(\varphi,\psi_0,\mathbb{A},\mathbb{B}) > 0 \tag{47}$$

$$\ln(\psi_0 + \varphi_0\mathbb{A}_0), \ln(\psi_0 + \varphi_0\mathbb{B}_0) \in \mathbb{L}_1\}. \tag{48}$$

We are interested in maximization $\max\{\mathbb{F}(\varphi,\psi_0;\mathbb{A},\mathbb{B}); (\varphi,\psi_0) \in \mathscr{A}(\mathbb{A},\mathbb{B})\}$ where

$$\mathbb{F}(\varphi,\psi_0;\mathbb{A},\mathbb{B}) \triangleq \liminf_{t\to\infty} \tfrac{1}{t}E[\ln\mathcal{W}_t(\varphi,\psi_0;\mathbb{A},\mathbb{B})].$$

To be honest, we have to admit that the function $\mathbb{F}(\varphi,\psi_0;\mathbb{A},\mathbb{B})$ may fail to be well defined here as the corresponding expectation does not have to exist, in general. In order to ensure that $\mathbb{F}(\varphi,\psi;\mathbb{A},\mathbb{B})$ is well defined, we say that the above expectation is $-\infty$ by definition if it is not defined in the classical sense.

Remark 3.5. Note that $\mathcal{S}_f(\varphi,\psi_0;\mathbb{A},\mathbb{B})$ is an admissible strategy if $(\varphi,\psi_0) \in \mathscr{A}(\mathbb{A},\mathbb{B})$. On the other hand, let (φ,ψ) be an admissible strategy in the market $(\mathbb{A},\mathbb{B})$. Then $(\varphi,\psi_0) \in \mathscr{A}(\mathbb{A},\mathbb{B})$ holds if and only if $\ln\mathcal{W}_0^{\mathbb{A}}(\varphi,\psi), \ln\mathcal{W}_0^{\mathbb{B}}(\varphi,\psi) \in \mathbb{L}_1$, and

$$(\varphi,\psi_0) \in \mathscr{A}(\mathbb{A},\mathbb{B}) \quad \Rightarrow \quad \mathcal{W}_t(\varphi,\psi_0;\mathbb{A},\mathbb{B}) = \mathcal{W}_t^{\mathbb{A}}(\varphi,\psi) \wedge \mathcal{W}_t^{\mathbb{B}}(\varphi,\psi).$$

Lemma 3.6. Let us consider the following markets $(\mathbb{A},\mathbb{B}), (\mathbb{B},\mathbb{D})$. Then

$$\mathscr{A}(\mathbb{A},\mathbb{D}) \subseteq \mathscr{A}(\mathbb{B},\mathbb{B}). \tag{49}$$

Further, whenever $(\varphi, \psi_0) \in \mathscr{A}(\mathbb{A}, \mathbb{D})$ we have that

$$\mathbb{F}(\varphi, \psi_0; \mathbb{A}, \mathbb{D}) \leq \mathbb{F}(\varphi, \psi_0; \mathbb{B}, \mathbb{B}). \tag{50}$$

In particular, we obtain that

$$0 \leq \mathscr{F}(\mathbb{A}, \mathbb{D}) \triangleq \sup_{(\varphi,\psi_0)\in\mathscr{A}(\mathbb{A},\mathbb{D})} \mathbb{F}(\varphi, \psi_0; \mathbb{A}, \mathbb{B}) \leq \mathscr{F}(\mathbb{B}, \mathbb{B}). \tag{51}$$

P r o o f. Let $(\varphi, \psi_0) \in FV(\mathcal{F}_t) \times \mathbb{L}(\mathcal{F}_0)$. As $\mathbb{A}_t \geq \mathbb{B}_t \geq \mathbb{D}_t$ holds if $t \geq 0$, we get that

$$\varphi_t\mathbb{B}_t - \min\{\varphi_t\mathbb{A}_t, \varphi_t\mathbb{D}_t\} = \max\{\varphi_t(\mathbb{B}_t - \mathbb{A}_t), \varphi_t(\mathbb{B}_t - \mathbb{D}_t))\} \geq 0, \quad t \geq 0,$$

and as $(\varphi_t^{\uparrow}, \varphi_t^{\downarrow})_{t\geq 0}$ are non-decreasing processes, we then get, by (46), that

$$\mathcal{W}_t(\varphi, \psi_0; \mathbb{B}, \mathbb{B}) - \mathcal{W}_t(\varphi, \psi_0; \mathbb{A}, \mathbb{D}) \geq \int_0^t (\mathbb{A}_s - \mathbb{B}_s)\, \mathrm{d}\varphi_s^{\uparrow} + \int_0^t (\mathbb{B}_s - \mathbb{D}_s)\, \mathrm{d}\varphi_s^{\downarrow} \geq 0, \tag{52}$$

holds whenever $t \in [0, \infty)$. Let $(\varphi; \psi_0) \in \mathscr{A}(\mathbb{A}, \mathbb{D})$. As $\mathbb{A}_0 \geq \mathbb{B}_0 \geq \mathbb{D}_0$, we get that

$$\mathbb{L}_1 \ni \ln(\psi_0 + \min\{\varphi_0\mathbb{A}_0, \varphi_0\mathbb{D}_0\}) \leq \ln(\psi_0 + \varphi_0\mathbb{B}_0) \leq \ln(\psi_0 + \max\{\varphi_0\mathbb{A}_0, \varphi_0\mathbb{D}_0\}) \in \mathbb{L}_1,$$

i.e. the condition in (48) is satisfied with $(\mathbb{A}, \mathbb{B})$ replaced by $(\mathbb{B}, \mathbb{B})$. Then we get that $(\varphi, \psi_0) \in \mathscr{A}(\mathbb{B}, \mathbb{B})$ holds by definition (47−48), by (52) and by the assumption $(\varphi, \psi_0) \in \mathscr{A}(\mathbb{A}, D)$. Hence, the relation (49) is verified. From (52) and from the definition of $\mathbb{F}$, we get that (50) holds and then (51) follows immediately since $\mathscr{F}(\mathbb{A}, \mathbb{B}) \geq \mathbb{F}(0, 1; \mathbb{A}, \mathbb{B}) = 0$, as we always may consider a non-trading self-financing strategy with unit initial wealth invested in the money market. □

Theorem 3.7. Let $(\mathbb{A}, \mathbb{B}), (\mathbb{B}, \mathbb{D})$ be markets and $(\varphi, \psi_0) \in \mathscr{A}(\mathbb{A}, \mathbb{D})$ be such that $\mathbb{F}(\varphi, \psi_0; \mathbb{A}, \mathbb{D}) = \mathscr{F}(\mathbb{B}, \mathbb{B})$. Then also $\mathbb{F}(\varphi, \psi_0; \mathbb{B}, \mathbb{B}) = \mathscr{F}(\mathbb{B}, \mathbb{B})$.

P r o o f. By the assumption and by (49) in Lemma 3.6, we have that $(\varphi, \psi_0) \in \mathscr{A}(\mathbb{A}, \mathbb{D}) \subseteq \mathscr{A}(\mathbb{B}, \mathbb{B})$. Then by the definition of $\mathscr{F}$, by the inequality in (50) and by our assumption, we get that $\mathscr{F}(\mathbb{B}, \mathbb{B}) \geq \mathbb{F}(\varphi, \psi_0; \mathbb{B}, \mathbb{B}) \geq \mathbb{F}(\varphi, \psi_0; \mathbb{A}, \mathbb{D}) = \mathscr{F}(\mathbb{B}, \mathbb{B})$. □

Remark 3.8. The assumption $\mathbb{F}(\varphi, \psi_0; \mathbb{A}, \mathbb{D}) = \mathscr{F}(\mathbb{B}, \mathbb{B})$ from Theorem 3.7 should be read as follows. In view of (51),

1. $\mathbb{F}(\varphi, \psi_0; \mathbb{A}, \mathbb{D}) = \mathscr{F}(\mathbb{A}, \mathbb{D})$, which means that the strategy $(\varphi, \psi) = \mathcal{S}_f(\varphi, \psi_0; \mathbb{A}, \mathbb{D})$ maximizes the long-run growth rate of the minimal wealth process in the market $(\mathbb{A}, \mathbb{D})$,

2. $\mathscr{F}(\mathbb{A}, \mathbb{D}) = \mathscr{F}(\mathbb{B}, \mathbb{B})$, i.e. the maximal long-run growth rate of the minimal wealth process is the same in both markets $(\mathbb{A}, \mathbb{D})$ and $(\mathbb{B}, \mathbb{B})$.

Similarly as in the point 1, the conclusion $\mathbb{F}(\varphi, \psi_0; \mathbb{B}, \mathbb{B}) = \mathscr{F}(\mathbb{B}, \mathbb{B})$ means that the strategy (φ, ψ) maximizes the long-run growth rate of the wealth process in the frictionless market $(\mathbb{B}, \mathbb{B})$.

Definition 3.9. If the assumptions of Theorem 3.7 are satisfied, then $\mathbb{B}$ is called a *shadow price* in the market $(\mathbb{A},\mathbb{D})$, and the strategy $(\varphi,\psi) \triangleq \mathcal{S}_f(\varphi,\psi_0;\mathbb{A},\mathbb{D})$ is referred to as the corresponding *shadow strategy.*

Remark 3.10. We are going to show that the notion of shadow strategy is very far from being unique even if we prescribe its initial values. Let us consider two $(\mathbb{A},\mathbb{D})$-self-financing strategies $(\varphi,\psi),(\varphi^*,\psi^*)$ with $(\varphi_0,\psi_0)=(\varphi_0^*,\psi_0^*)$ such that $(\varphi,\psi_0),(\varphi^*,\psi_0^*)\in \mathscr{A}(\mathbb{A},\mathbb{D})$ and assume that there exist $T\in(0,\infty)$ and $L_T\in\mathbb{L}_1$ such that

$$\mathcal{W}_t(\varphi,\psi_0;\mathbb{A},\mathbb{D})=\mathcal{W}_t(\varphi^*,\psi_0^*;\mathbb{A},\mathbb{D})\,e^{L_T},\qquad t\in[T,\infty).$$

Then $\mathbb{F}(\varphi,\psi_0;\mathbb{A},\mathbb{D})=\mathbb{F}(\varphi^*,\psi_0^*;\mathbb{A},\mathbb{D})$, and therefore, $(\varphi_t,\psi_t)_{t\geq 0}$ is a shadow strategy if and only if $(\varphi_t^*,\psi_t^*)_{t\geq 0}$ is also a shadow strategy although they may differ a lot by the time T. Similarly, we could show that the shadow price is far from being unique.

Theorem 3.11. Let $\mathbb{B}$ be a shadow price in the market $(\mathbb{A},\mathbb{D})$ and (φ,ψ) be the corresponding shadow strategy. If $\mathbb{C}=(\mathbb{C}_t)_{t\geq 0}$ is a continuous semimartingale attaining values in the bid-ask spread, i. e. $\mathbb{C}_t\in[\mathbb{D}_t,\mathbb{A}_t]$ holds for every $t\in[0,\infty)$, then

$$\mathscr{F}(\mathbb{C},\mathbb{C})\geq \mathscr{F}(\mathbb{A},\mathbb{D})=\mathscr{F}(\mathbb{B},\mathbb{B}),$$

i. e. the shadow price $\mathbb{B}$ offers the worst opportunity to maximize the long-run growth rate of the minimal wealth process among all continuous semimartingales attaining values within the bid-ask spread.

Proof. See (51) with $\mathbb{B}$ replaced by $\mathbb{C}$ in order to get the corresponding inequality and look at Remark 3.8, point 2, for the equality. □

3.2. Shadow price and cost-free strategy

In this and the following subsections, we consider the following market

$$(\mathbb{A},\mathbb{D})=\begin{cases}(S^{\uparrow},S^{\downarrow}) & \text{if}\quad a=1,\\ (F^{\uparrow},F^{\downarrow}) & \text{if}\quad a=0.\end{cases}\tag{53}$$

In this subsection, namely in Theorem 3.13, we show that the $\tilde{\varepsilon}$-price and the strategy (φ^*,ψ^*) from Theorem 2.26 are the shadow price and the corresponding shadow strategy, respectively, under very general assumptions.

Lemma 3.12. Let (φ,ψ) be an admissible strategy with the wealth process $(\mathcal{W}_t)_{t\geq 0}$, with the ask and bid wealth processes $(\mathcal{W}_t^{\uparrow},\mathcal{W}_t^{\downarrow})_{t\geq 0}$ and with the position process $(\pi_t)_{t\geq 0}$ attaining values in a compact subset of $\mathcal{A}$, then $\ln(\mathcal{W}_t^{\uparrow}\wedge\mathcal{W}_t^{\downarrow}/\mathcal{W}_t)_{t\geq 0}$ is a bounded process.

Moreover, if $\mathbb{B}$ is an $\tilde{\varepsilon}$-price and (φ,ψ) is an $\tilde{\varepsilon}$-cost-free strategy with an $\tilde{\varepsilon}$-wealth process $\tilde{\mathcal{W}}$ and with the initial wealth $\mathcal{W}_0$ satisfying $\ln\mathcal{W}_0\in\mathbb{L}_1$, then $\ln\tilde{\mathcal{W}}_0\in\mathbb{L}_1$ and

$$(\varphi,\psi_0)\in\mathscr{A}(\mathbb{A},\mathbb{D}),\qquad \mathbb{F}(\varphi,\psi_0;\mathbb{A},\mathbb{D})=\mathbb{F}(\varphi,\psi_0;\mathbb{B},\mathbb{B}).\tag{54}$$

Proof. By Lemma 2.12, $\ln(\mathcal{W}^{\uparrow}/\mathcal{W}), \ln(\mathcal{W}^{\downarrow}/\mathcal{W}), \ln(\tilde{\mathcal{W}}/\mathcal{W})$ are bounded processes, and then the first part of the statement follows immediately. Moreover, we get that the variables $\ln \mathcal{W}_0^{\uparrow}, \ln \mathcal{W}_0^{\downarrow}, \ln \tilde{\mathcal{W}}_0$ differ from $\ln \mathcal{W}_0 \in \mathbb{L}_1$ not more than by a constant, and therefore, these variables are also integrable. Further, as the strategy (φ, ψ) is admissible, the minimum of its ask and bid wealth processes is positive, i. e.

$$\mathcal{W}_t(\varphi, \psi_0; \mathbb{A}, \mathbb{D}) = \mathcal{W}_t^{\uparrow} \wedge \mathcal{W}_t^{\downarrow} > 0 \quad \text{if} \quad t \geq 0.$$

Then we get that $(\varphi, \psi_0) \in \mathscr{A}(\mathbb{A}, \mathbb{D})$ really holds. Further, as $\tilde{\mathcal{W}}_t = \mathcal{W}_t(\varphi, \psi_0; \mathbb{B}, \mathbb{B})$ holds if $t \geq 0$ and as $\ln(\mathcal{W}_t^{\uparrow} \wedge \mathcal{W}_t^{\downarrow}/\tilde{\mathcal{W}}_t)_{t\geq 0}$ is a bounded process, we get that also

$$\mathbb{F}(\varphi, \psi_0; \mathbb{A}, \mathbb{D}) = \liminf_{t\to\infty} \tfrac{1}{t} E \ln(\mathcal{W}_t^{\uparrow} \wedge \mathcal{W}_t^{\downarrow}) = \liminf_{t\to\infty} \tfrac{1}{t} E \ln \tilde{\mathcal{W}}_t = \mathbb{F}(\varphi, \psi_0; \mathbb{B}, \mathbb{B}).$$

□

Theorem 3.13. Let $\mathbb{B}$ be a regular $\tilde{\varepsilon}$-price and (φ, ψ) be an $\tilde{\varepsilon}$-cost-free strategy s.t.

1. the $\tilde{\varepsilon}$-position equals to the $\tilde{\varepsilon}$-log-optimal proportion almost surely, i. e. $\tilde{\pi} \overset{\text{as}}{=} \tilde{\theta}$,
2. the position process π attains values in a compact subset of $\mathcal{A}$,
3. the initial wealth $\mathcal{W}_0$ is such that $\ln \mathcal{W}_0 \in \mathbb{L}_1$.

Then $\mathbb{B}$ is a shadow price and (φ, ψ) is the corresponding shadow strategy.

Proof. By Lemma 3.12, the relations in (54) hold and the $\tilde{\varepsilon}$-wealth process $\tilde{\mathcal{W}}$ satisfies $\ln \tilde{\mathcal{W}}_0 \in \mathbb{L}_1$. Let $(\varphi^\circ, \psi_0^\circ) \in \mathscr{A}(\mathbb{A}, \mathbb{D})$. Then $(\varphi^\circ, \psi^\circ) = \mathcal{S}_f(\varphi^\circ, \psi_0^\circ)$ is an admissible strategy with $\tilde{\varepsilon}$-wealth process denoted as $\tilde{\mathcal{W}}^\circ$, and

$$\ln(\tilde{\mathcal{W}}_0/\tilde{\mathcal{W}}_0^\circ) \leq \ln \tilde{\mathcal{W}}_0 - \ln(\psi_0^\circ + \min\{\varphi_0^\circ \mathbb{A}_0, \varphi_0^\circ \mathbb{D}_0\}) \in \mathbb{L}_1.$$

By (54) and by (49) in Lemma 3.6, we get that $(\varphi, \psi_0), (\varphi^\circ, \psi_0^\circ) \in \mathscr{A}(\mathbb{A}, \mathbb{D}) \subseteq \mathscr{A}(\mathbb{B}, \mathbb{B})$. Further, as $\tilde{\mathcal{W}}_t = \mathcal{W}_t(\varphi, \psi_0; \mathbb{B}, \mathbb{B})$ and $\tilde{\mathcal{W}}_t^\circ = \mathcal{W}_t(\varphi^\circ, \psi_0^\circ; \mathbb{B}, \mathbb{B})$ hold if $t \geq 0$, we get by Theorem 2.26 that

$$\mathbb{F}(\varphi, \psi_0; \mathbb{B}, \mathbb{B}) = \liminf_{t\to\infty} \tfrac{1}{t} E \ln \tilde{\mathcal{W}}_t \geq \liminf_{t\to\infty} \tfrac{1}{t} E \ln \tilde{\mathcal{W}}_t^\circ = \mathbb{F}(\varphi^\circ, \psi_0^\circ; \mathbb{B}, \mathbb{B}). \tag{55}$$

Then we obtain from (54,55) and the definition of $\mathscr{F}$ in (51) that $\mathbb{F}(\varphi, \psi_0; \mathbb{A}, \mathbb{D}) = \mathbb{F}(\varphi, \psi_0; \mathbb{B}, \mathbb{B}) = \mathscr{F}(\mathbb{B}, \mathbb{B})$, i. e. that the assumptions of Theorem 3.7 are satisfied, and we get by definition that $\mathbb{B}$ is a shadow price and (φ, ψ) is the corresponding shadow strategy. □

Definition 3.14. We say that (φ, ψ) is an *$\tilde{\varepsilon}$-shadow strategy* if the $\tilde{\varepsilon}$-price given by (19,23) is a shadow price in the market (53) and if (φ, ψ) is the corresponding shadow strategy.

Remark 3.15. If $\theta \in \{0, a\}$, then we get by Theorem 3.13 that, for example, $(\varphi, \psi) = (\theta, 1 - \theta)$ is a 0-shadow strategy, cf. Remark 2.28. It means that the nominal price itself (as 0-price) is a shadow price and the non-trading strategy (φ, ψ) is the corresponding shadow strategy.

Remark 3.16. We are going to show, with the help of Remark 3.10, that an $\tilde{\varepsilon}$-shadow strategy does not have to be $\tilde{\varepsilon}$-cost-free. Let (φ, ψ) be an $\tilde{\varepsilon}$-cost-free $\tilde{\varepsilon}$-shadow strategy, obtained from Theorem 3.13, for example. Then obviously there exists $T \in (0, \infty)$ and an admissible strategy (φ^*, ψ^*) as in Remark 3.10, which is not $\tilde{\varepsilon}$-cost-free, because it can be almost arbitrary on a nondegenerate interval. As mentioned in Remark 3.10, (φ^*, ψ^*) is also an $\tilde{\varepsilon}$-shadow strategy but it is not $\tilde{\varepsilon}$-cost-free as we assume here.

3.3. Searching for a shadow price

In this subsection, we consider a method for searching for a shadow price, taken from [24], based on our ability to solve a certain ODE (61) with a solution (60) in Lemma 3.18.

Remark 3.17. Further in this paper we will assume that $\theta \in \mathbb{R}\backslash\{0, a\}$. Note that, if $\theta \in \{0, a\}$, it is not difficult to obtain assertions similar to the main statements of this section such as Theorems 3.31, 3.34 and Corollary 3.32, with the help of Remark 3.15 and Theorem 2.26.

The shadow price will be obtained as the nominal price modified as follows

$$\begin{array}{llll} \tilde{S}_t & \overset{\text{as}}{=} S_t e^{f(\tilde{\pi}_t)} & \text{if} & a = 1, \\ \tilde{F}_t & \overset{\text{as}}{=} F_t + f(\tilde{\pi}_t) & \text{if} & a = 0, \end{array} \tag{56}$$

where $f \in C^2$, i. e. we consider a multiplicative correction (if $a = 1$) and an additive correction (if $a = 0$) smoothly depending on the modified position process $(\tilde{\pi}_t)_{t \geq 0}$. By Lemma 2.20

$$\tilde{\pi}_t \overset{\text{as}}{=} \tilde{\pi}_0 + \int_0^t \tilde{\sigma}_s z_a(\tilde{\pi}_s)\, \mathrm{d}B_s + \int_0^t \tilde{\mathcal{W}}_s^{-1} \tilde{S}_s^a \, \mathrm{d}\varphi_s, \qquad \text{where} \qquad z_a(x) \triangleq x(a - x),$$

whenever $(\tilde{\mathcal{W}}_t, \tilde{\pi}_t)_{t \geq 0}$ are the $\tilde{\varepsilon}$-wealth process and the $\tilde{\varepsilon}$-position process of a continuous $\tilde{\varepsilon}$-cost-free strategy $(\varphi_t, \psi_t)_{t \geq 0}$, where $(\tilde{S}_t)_{t \geq 0}$ is the $\tilde{\varepsilon}$-stock market price if $a = 1$. If we take the logarithm of both sides of the first equality in (56) in case $a = 1$ and when we compare the corresponding coefficients at $\mathrm{d}B_t$, $\mathrm{d}t$ and $\mathrm{d}\varphi_t$ in both cases $a \in \{0, 1\}$, we obtain the following requirements

$$\tilde{\mu}_t - \tfrac{1}{2}\,\tilde{\sigma}_t^2 a \overset{\text{as}}{=} \mu - \tfrac{1}{2}\,\sigma^2 a + \tfrac{1}{2}\,\tilde{\sigma}_t^2 z_a^2(\tilde{\pi}_t) f''(\tilde{\pi}_t), \tag{57}$$

$$\tilde{\sigma}_t \overset{\text{as}}{=} \sigma + \tilde{\sigma}_t z_a(\tilde{\pi}_t) f'(\tilde{\pi}_t), \tag{58}$$

$$0 \overset{\text{as}}{=} f'(\tilde{\pi}_t)\, \mathrm{d}\varphi_t. \tag{59}$$

From (58), we can express $\tilde{\sigma}_t \overset{\text{as}}{=} \sigma / [1 - z_a(\tilde{\pi}_t) f'(\tilde{\pi}_t)]$ and then from (57) we get that

$$\tilde{\theta}_t - \tfrac{a}{2} \overset{\text{as}}{=} (\theta - \tfrac{a}{2})[1 - z_a(\tilde{\pi}_t) f'(\tilde{\pi}_t)]^2 + \tfrac{1}{2} z_a^2(\tilde{\pi}_t) f''(\tilde{\pi}_t).$$

Then the requirement that the $\tilde{\varepsilon}$-position should be almost surely equal to the log-optimal proportion $(\tilde{\pi}_t)_{t \geq 0} \overset{\text{as}}{=} (\tilde{\theta}_t)_{t \geq 0}$ from Theorem 3.13 gives us an ODE (61) with boundary conditions coming from (59).

The following Lemma shows how the corresponding function f looks. Note that the parameter ω in the statement is connected with the level of transaction taxes.

Lemma 3.18. Let $\omega \in (0, |\theta| \wedge |a-\theta|)$. Then the function

$$f(x) \triangleq \int \left[\tfrac{1}{q_\omega(x)} + \tfrac{1}{z_a(x)} \right] \mathrm{d}x, \qquad \text{where} \qquad q_\omega(x) \triangleq (2\theta - a)x - (\theta^2 - \omega^2), \tag{60}$$

is strictly decreasing on $[\theta-\omega, \theta+\omega]$ and it satisfies the following ODE

$$x - \tfrac{a}{2} = (\theta - \tfrac{a}{2})[1 - z_a(x) f'(x)]^2 + \tfrac{1}{2}\, z_a^2(x) f''(x), \quad |x - \theta| \le \omega \tag{61}$$

with the boundary conditions $f'(\theta-\omega) = 0 = f'(\theta+\omega)$. Moreover, the function

$$x \mapsto -\sigma q_\omega(x)/z_a(x)$$

defined on $[\theta-\omega, \theta+\omega]$ is continuous and it attains only positive values there.

Proof. First we show that both functions q_ω, z_a do not attain the value zero on $I \triangleq [\theta-\omega, \theta+\omega]$ and then we obtain that the function f is by (60) defined correctly on an open superset of I and uniquely up to some additive constant.

As I is disjoint with $\{0, a\}$, we get that z_a is non-zero on I. As q_ω is an affine function and as it attains values with the same non-zero sign at the extreme points of I

$$q_\omega(\theta \pm \omega) = (\theta \pm \omega)(\theta \pm \omega - a) = -z_a(\theta \pm \omega), \tag{62}$$

we get that the function $x \mapsto -\sigma q_\omega(x)/z_a(x)$ is continuous on I and it attains only positive values there as it does not change the sign on I and as it attains the value $\sigma > 0$ at the extreme points of I. Similarly, we get that $x \mapsto q_\omega(x) z_a(x)$ attains only negative values on I. We easily obtain that

$$f'(x) = \tfrac{1}{q_\omega(x)} + \tfrac{1}{z_a(x)} = \tfrac{\omega^2 - (x-\theta)^2}{q_\omega(x)\, z_a(x)}, \qquad f''(x) = \tfrac{2x - a}{z_a(x)^2} - \tfrac{2\theta - a}{q_\omega(x)^2}, \tag{63}$$

and we also obtain that $f'(x) < 0$ holds if $|x - \theta| < \omega$. Further, simple calculations using (63) verify that (61) holds, and taking into consideration (62) and the expression for $f'(x)$ in (63) we obtain that the boundary conditions are also satisfied. □

3.4. Existence of a shadow price I

In this subsection, we show that the shadow price exists in Theorem 3.21 under assumptions (A1-A3). In Theorem 3.31 we will show that (A1) implies (A2) and (A3) if (roughly speaking) the considered strategy just keeps the position between the log-optimal policies introduced in the Definition 3.27 and if $\tilde{\varepsilon} = (\tilde{\varepsilon}_t)_{t\ge 0}$ is given by (105).

Definition 3.19. (Skorokhod problem) Let $(B_t)_{t\ge0}$ be a standard $\mathcal{F}_t$-Brownian motion and let Y be a real-valued random variable with values in $[\alpha, \beta] \subseteq \mathbb{R}$, where $\alpha < \beta$. If $\mathbb{B}, \mathbb{S} : [\alpha, \beta] \to \mathbb{R}$ are Lipschitz continuous functions, see [37] and [38] that there exists an almost surely unique triple of continuous $\mathcal{F}_t$-semimartingales $(X_t, X_t^{\uparrow}, X_t^{\downarrow})_{t\ge0}$ s.t.

$$[\alpha, \beta] \ni X_t \overset{\text{as}}{=} Y + \textstyle\int_0^t \mathbb{B}(X_s)\, \mathrm{d}s + \int_0^t \mathbb{S}(X_s)\, \mathrm{d}B_s + X_t^{\uparrow} - \mathrm{d}X_t^{\downarrow}, \quad t \ge 0, \tag{64}$$

and that $(X_t^\uparrow, X_t^\downarrow)_{t\ge 0}$ are non-decreasing processes starting from zero at 0 satisfying

$$\int_0^\infty 1_{[X_t\neq\alpha]}\,\mathrm{d}X_t^\uparrow = 0, \quad \int_0^\infty 1_{[X_t\neq\beta]}\,\mathrm{d}X_t^\downarrow = 0.$$

Then the process $(X_t)_{t\ge 0}$ is called a B_t*-diffusion process* with reflective barriers at $\{\alpha,\beta\}$ and coefficients $\mathbb{B},\mathbb{S}$ and we extend this definition even to the case when the requirement $X_t\in[\alpha,\beta], t\ge 0$, from (64) is satisfied only almost surely.

Notation 3.20. In order to deal with transaction taxes in both cases $a\in\{0,1\}$ together, we introduce a function

$$\Lambda_a(x) \triangleq \begin{cases} x & \text{if } \ a=0,\\ \ln(1+x) & \text{if } \ a=1, \end{cases} \tag{65}$$

expressing how the bid and ask transaction taxes contribute to the entire level of transaction taxes as follows $\lambda \triangleq \Lambda_a(\lambda^\uparrow)-\Lambda_a(-\lambda^\downarrow)$. If $(\tilde\lambda_t^\uparrow,\tilde\lambda_t^\downarrow)_{t\ge 0}$ are the $\tilde\varepsilon$-transaction taxes (see Definitions 2.9 and 2.10), then

$$\Lambda_a(\tilde\varepsilon_t) = \Lambda_a(\lambda^\uparrow)-\Lambda_a(\tilde\lambda_t^\uparrow) = \Lambda_a(-\lambda^\downarrow)-\Lambda_a(-\tilde\lambda_t^\downarrow), \tag{66}$$

and note that also $\lambda = \Lambda_a(\tilde\lambda_t^\uparrow)-\Lambda_a(-\tilde\lambda_t^\downarrow), t\ge 0$.

Theorem 3.21. (A1) Let $\omega\in(0,|\theta|\wedge|a-\theta|)$ and f from (60) be such that

$$f(\theta-\omega)=\Lambda_a(\lambda^\uparrow), \qquad f(\theta+\omega)=\Lambda_a(-\lambda^\downarrow). \tag{67}$$

(A2) Let $\tilde\varepsilon=(\tilde\varepsilon_t)_{t\ge 0}$ be a $[-\lambda^\downarrow,\lambda^\uparrow]$-valued continuous semimartingale and $(\varphi_t,\psi_t)_{t\ge 0}$ be a continuous admissible strategy with the $\tilde\varepsilon$-position $(\tilde\pi_t)_{t\ge 0}$ that is a B_t-diffusion process with reflective barriers at $\{\theta-\omega,\theta+\omega\}$ and coefficients $\mathbb{B}(x)=0, \mathbb{S}(x)=-\sigma q_\omega(x)$.

(A3) Let us assume that

$$\Lambda_a(\tilde\varepsilon_t) \overset{\text{as}}{=} f(\tilde\pi_t), \quad t\ge 0. \tag{68}$$

Then

1. the $\tilde\varepsilon$-price given by (19,23) is regular, say with coefficients $(\tilde\mu_t,\tilde\sigma_t)_{t\ge 0}$,
2. $\tilde\pi_t \overset{\text{as}}{=} \tilde\theta_t \triangleq \tilde\mu_t\tilde\sigma_t^{-2}$ holds whenever $t\in[0,\infty)$,
3. (φ,ψ) is an $\tilde\varepsilon$-cost-free strategy.

In particular, (φ,ψ) is an $\tilde\varepsilon$-shadow strategy if its position $(\pi_t)_{t\ge 0}$ attains values in a compact subset of $\mathcal{A}$ and if its initial wealth $\mathcal{W}_0$ satisfies the condition $\ln\mathcal{W}_0\in\mathbb{L}_1$.

P r o o f. By assumption (A2), $(\tilde\pi_t)_{t\ge 0}$ attains values in $[\theta-\omega,\theta+\omega]$ almost surely and

$$\tilde\pi_t \overset{\text{as}}{=} \tilde\pi_0 - \sigma\int_0^t q_\omega(\tilde\pi_s)\,\mathrm{d}B_s + \tilde\pi_t^\uparrow - \tilde\pi_t^\downarrow, \tag{69}$$

where $(\tilde{\pi}^{\uparrow}_t, \tilde{\pi}^{\downarrow}_t)_{t\geq 0}$ are non-decreasing continuous adapted processes starting from zero and growing only on sets $[\tilde{\pi}_t = \theta - \omega]$ and $[\tilde{\pi}_t = \theta + \omega]$, respectively, i. e.

$$\int_0^\infty 1_{[\tilde{\pi}_t \neq \theta-\omega]} \, \mathrm{d}\tilde{\pi}^{\uparrow}_t = 0, \qquad \int_0^\infty 1_{[\tilde{\pi}_t \neq \theta+\omega]} \, \mathrm{d}\tilde{\pi}^{\downarrow}_t = 0. \tag{70}$$

As the considered filtration is assumed to be complete, there exists $A \in \mathcal{F}_0$ with $P(A) = 1$ such that $(\tilde{\pi}_t)_{t\geq 0}$ attains values in $[\theta - \omega, \theta + \omega]$ on A.

First, we show that the $\tilde{\varepsilon}$-stock market price $(\tilde{S}_t)_{t\geq 0}$ given by (19) (if $a = 1$) and the $\tilde{\varepsilon}$-futures price $(\tilde{F}_t)_{t\geq 0}$ given by (23) (if $a = 0$) is regular with coefficients

$$\tilde{\sigma}_t \triangleq \sigma[1 - q_\omega(\tilde{\pi}_t) f'(\tilde{\pi}_t) 1_A], \qquad \tilde{\mu} \triangleq \tilde{\sigma}_t^2 \tilde{\pi}_t 1_A \overset{\text{as}}{=} \tilde{\sigma}_t^2 \tilde{\pi}_t. \tag{71}$$

Note that $(f(\tilde{\pi}_t), f'(\tilde{\pi}_t), f''(\tilde{\pi}_t))_{t\geq 0}$ are defined correctly on A, i. e. up to a null set. If it is not enough, multiply each corresponding usage by $1_A \overset{\text{as}}{=} 1$ in the rest of the proof.

As $f'(\theta + \omega) = 0 = f'(\theta - \omega)$ holds by Lemma 3.18 and as $(\tilde{\pi}^{\uparrow}_t)_{t\geq 0}$ grows only on $[\tilde{\pi}_t = \theta - \omega]$ and $(\tilde{\pi}^{\downarrow}_t)_{t\geq 0}$ grows only on $[\tilde{\pi}_t = \theta + \omega]$, we obtain by Itô Lemma that

$$\mathrm{d}f(\tilde{\pi}_t) = -\sigma q_\omega(\tilde{\pi}_t) f'(\tilde{\pi}_t) \, \mathrm{d}B_t + \tfrac{1}{2} \sigma^2 q_\omega(\tilde{\pi}_t)^2 f''(\tilde{\pi}_t) \, \mathrm{d}t. \tag{72}$$

As we consider a continuous semimartingale $(\tilde{F}_t)_{t\geq 0}$ such that $\mathrm{d}\tilde{S}_t = \tilde{S}_t \, \mathrm{d}\tilde{F}_t$ if $a = 1$, i. e. $\mathrm{d}\ln \tilde{S}_t = \mathrm{d}\tilde{F}_t - \frac{1}{2} \, \mathrm{d}\langle \tilde{F} \rangle_t$, we get from (56) following from assumption (68) that

$$\mathrm{d}f(\tilde{\pi}_t) = \begin{cases} \mathrm{d}\ln(\tilde{S}_t/S_t) = \mathrm{d}(\tilde{F}_t - \frac{1}{2} \, \mathrm{d}\langle \tilde{F} \rangle_t) - \mathrm{d}(F_t - \frac{1}{2} \, \mathrm{d}\langle F \rangle_t) & \text{if} \quad a = 1, \\ \mathrm{d}(\tilde{F}_t - F_t) & \text{if} \quad a = 0. \end{cases} \tag{73}$$

Then we obtain, from (72, 73), that

$$\mathrm{d}\tilde{F}_t - \tfrac{a}{2} \, \mathrm{d}\langle \tilde{F} \rangle_t = \big[\mu - \sigma^2 \tfrac{a}{2} + \tfrac{\sigma^2}{2} q_\omega(\tilde{\pi}_t)^2 f''(\tilde{\pi}_t)\big] \, \mathrm{d}t + \sigma\big[1 - q_\omega(\tilde{\pi}_t) f'(\tilde{\pi}_t)\big] \, \mathrm{d}B_t \tag{74}$$

holds in both cases $a \in \{0, 1\}$. See the expressions of f', f'' in (63) in order to get that

$$1 - q_\omega(x) f'(x) = -\tfrac{q_\omega(x)}{z_a(x)}, \qquad (\theta - \tfrac{a}{2}) + \tfrac{1}{2} \, q_\omega(x)^2 f''(x) = (x - \tfrac{a}{2}) \tfrac{q_\omega^2(x)}{z_a^2(x)}. \tag{75}$$

Then, from the expression of $\tilde{\sigma}_t$ in (71) and from (74, 75), we get that

$$\mathrm{d}\tilde{F}_t - \tfrac{a}{2} \, \mathrm{d}\langle \tilde{F} \rangle_t = \tilde{\sigma}_t^2 (\tilde{\pi}_t - \tfrac{a}{2}) \, \mathrm{d}t + \tilde{\sigma}_t \, \mathrm{d}B_t. \tag{76}$$

Further, use (71, 75) to get that

$$\tilde{\sigma}_t = -\sigma \tfrac{q_\omega(\tilde{\pi}_t)}{z_a(\tilde{\pi}_t)} \, 1_A + \sigma \, 1_{\Omega\setminus A} \overset{\text{as}}{=} -\sigma \tfrac{q_\omega(\tilde{\pi}_t)}{z_a(\tilde{\pi}_t)}. \tag{77}$$

Note that $(\tilde{\mu}_t, \tilde{\sigma}_t)_{t\geq 0}$ are obviously bounded continuous adapted processes and that $(\ln \tilde{\sigma}_t)_{t\geq 0}$ is also bounded as $(\tilde{\sigma}_t)_{t\geq 0}$ attains values in $\{-\sigma q_\omega(x)/z_a(x); |x - \theta| \leq \omega\} \cup \{\sigma\}$ which is a compact subset of $(0, \infty)$ by the moreover part of Lemma 3.18. If $a = 0$, we immediately obtain from (76) that the $\tilde{\varepsilon}$-futures price $(\tilde{F}_t)_{t\geq 0}$ is regular with coefficients $(\tilde{\mu}_t, \tilde{\sigma}_t)_{t\geq 0}$ from (71). If $a = 1$, we have the expression of $\mathrm{d}\ln \tilde{S}_t$ in (76) and we obtain from Itô formula that

$$\mathrm{d}\tilde{S}_t = \tilde{S}_t[\, \mathrm{d}\ln \tilde{S}_t + \tfrac{1}{2} \, \mathrm{d}\langle \ln \tilde{S} \rangle_t] = \tilde{S}_t[\tilde{\mu}_t \, \mathrm{d}t + \tilde{\sigma}_t \, \mathrm{d}B_t]$$

holds, i. e. $(\tilde{S}_t)_{t\geq 0}$ is a regular stock market $\tilde{\varepsilon}$-price with coefficients $(\tilde{\mu}_t, \tilde{\sigma}_t)_{t\geq 0}$ if $a = 1$. Thus, the first point from the statement is verified and as the coefficients of a regular $\tilde{\varepsilon}$-price are determined uniquely up to a null set, we get from (71) that the second point is also satisfied.

Further, we will show that $(\varphi_t, \psi_t)_{t\geq 0}$ is an $\tilde{\varepsilon}$-cost-free strategy. Let $(\tilde{\lambda}^{\uparrow}_t, \tilde{\lambda}^{\downarrow}_t)_{t\geq 0}$ be the $\tilde{\varepsilon}$-transaction taxes. We get from (66, 67, 68) that

$$f(\theta - \omega) - f(\tilde{\pi}_t) \overset{\text{as}}{=} \Lambda_a(\lambda^{\uparrow}) - \Lambda_a(\tilde{\varepsilon}_t) = \Lambda_a(\tilde{\lambda}^{\uparrow}_t)$$
$$f(\theta + \omega) - f(\tilde{\pi}_t) \overset{\text{as}}{=} \Lambda_a(-\lambda^{\downarrow}) - \Lambda_a(\tilde{\varepsilon}_t) = \Lambda_a(-\tilde{\lambda}^{\downarrow}_t)$$

hold. Hence, up to a null set, $\tilde{\lambda}^{\uparrow}_t = 0$ whenever $\tilde{\pi}_t = \theta - \omega$ and $\tilde{\lambda}^{\downarrow}_t = 0$ whenever $\tilde{\pi}_t = \theta + \omega$. Then we get from (70) that

$$\textstyle\int_0^\infty 1_{[\tilde{\lambda}^{\uparrow}_t \neq 0]}\,\mathrm{d}\tilde{\pi}^{\uparrow}_t \overset{\text{as}}{=} 0, \quad \int_0^\infty 1_{[\tilde{\lambda}^{\downarrow}_t \neq 0]}\,\mathrm{d}\tilde{\pi}^{\downarrow}_t \overset{\text{as}}{=} 0. \tag{78}$$

Use (76) to see that $\mathrm{d}\langle \tilde{F}\rangle_t = \tilde{\sigma}_t^2\,\mathrm{d}t$ which also implies that $\mathrm{d}\tilde{F}_t - \tilde{\pi}_t\,\mathrm{d}\langle \tilde{F}\rangle_t = \tilde{\sigma}_t\,\mathrm{d}B_t$. Then we get from (77) that

$$\textstyle\int_0^t z_a(\tilde{\pi}_s)[\,\mathrm{d}\tilde{F}_s - \tilde{\pi}_s\,\mathrm{d}\langle \tilde{F}\rangle_s] \overset{\text{as}}{=} -\sigma\int_0^t q_\omega(\tilde{\pi}_s)\,\mathrm{d}B_s, \quad t \geq 0,$$

and we obtain from (69) and Lemma 2.20 that

$$\tilde{\pi}^{\uparrow}_t - \tilde{\pi}^{\downarrow}_t \overset{\text{as}}{=} \tilde{\pi}_t - \tilde{\pi}_0 + \textstyle\int_0^t q_\omega(\tilde{\pi}_s)\,\mathrm{d}B_s \overset{\text{as}}{=} \int_0^t \tilde{\mathcal{W}}_s^{-1}\tilde{S}^a_s[(1+\tilde{\lambda}^{\uparrow}_s\tilde{\pi}_s)\,\mathrm{d}\varphi^{\uparrow}_s - (1-\tilde{\lambda}^{\downarrow}_s\tilde{\pi}_s)\,\mathrm{d}\varphi^{\downarrow}_s].$$

Note that the expressions on the left-hand and right-hand sides are continuous processes of locally finite variation. See (70) to realize that $(\tilde{\pi}^{\uparrow}_t, \tilde{\pi}^{\downarrow}_t)_{t\geq 0}$ do not grow at the same time similarly as $(\varphi^{\uparrow}_t, \varphi^{\downarrow}_t)_{t\geq 0}$. Then we obtain, from the uniqueness of Hahn decomposition of a sign measure, that

$$\tilde{\pi}^{\uparrow}_t \overset{\text{as}}{=} \textstyle\int_0^t \tilde{\mathcal{W}}_s^{-1}\tilde{S}^a_s(1+\tilde{\lambda}^{\uparrow}_s\tilde{\pi}_s)\,\mathrm{d}\varphi^{\uparrow}_s, \qquad \tilde{\pi}^{\downarrow}_t \overset{\text{as}}{=} \int_0^t \tilde{\mathcal{W}}_s^{-1}\tilde{S}^a_s(1-\tilde{\lambda}^{\uparrow}_s\tilde{\pi}_s)\,\mathrm{d}\varphi^{\downarrow}_s, \quad t \geq 0.$$

Then we obtain from (78) that

$$\tilde{C}^{\varphi}_t \triangleq \textstyle\int_0^t \tilde{S}^a_s(\tilde{\lambda}^{\uparrow}_s\,\mathrm{d}\varphi^{\uparrow}_s + \tilde{\lambda}^{\downarrow}_s\,\mathrm{d}\varphi^{\downarrow}_s) \overset{\text{as}}{=} \int_0^t \tilde{\mathcal{W}}_s\big[\frac{\tilde{\lambda}^{\uparrow}_s\,\mathrm{d}\tilde{\pi}^{\uparrow}_s}{1+\tilde{\lambda}^{\uparrow}_s\tilde{\pi}_s} - \frac{\tilde{\lambda}^{\downarrow}_s\,\mathrm{d}\tilde{\pi}^{\downarrow}_s}{1-\tilde{\lambda}^{\downarrow}_s\tilde{\pi}_s}\big] = 0, \quad t \geq 0,$$

i. e. $(\varphi_t, \psi_t)_{t\geq 0}$ is an $\tilde{\varepsilon}$-cost-free strategy and we get that the third point from the statement is also verified. Then the remaining part of the statement follows from Theorem 3.13. □

Definition 3.22. Let $\alpha < \beta$ be such that $[\alpha, \beta] \subseteq \mathcal{A}\backslash\{0, a\}$. A continuous admissible strategy $(\varphi_t, \psi_t)_{t\geq 0}$ with position $(\pi_t)_{t\geq 0}$ is called an $[(\alpha, \beta)]_A$*-strategy* if $(\pi_t)_{t\geq 0}$ attains values in $[\alpha, \beta]$ on $A \in \mathcal{F}_0$ with $P(A) = 1$ and $(\varphi_t)_{t\geq 0} \equiv 0$ on $N \triangleq \Omega\backslash A$, and if

$$\textstyle\int_0^\infty 1_{[\pi_t \neq \alpha]}\,\mathrm{d}\varphi^{\uparrow}_t = 0, \quad \int_0^\infty 1_{[\pi_t \neq \beta]}\,\mathrm{d}\varphi^{\downarrow}_t = 0. \tag{79}$$

Recall that $(\varphi^{\uparrow}_t, \varphi^{\downarrow}_t)_{t\geq 0}$ from (79) are uniquely determined by $(\varphi_t)_{t\geq 0}$ as the Hahn-decomposition of a sign measure is unique. On the other hand, if $(\varphi^{\uparrow}_t, \varphi^{\downarrow}_t)_{t\geq 0}$ are non-decreasing rcll-processes satisfying (79) then they grow on disjoint sets.

Lemma 3.23. Let $\alpha < \beta$ be real numbers such that $[\alpha, \beta] \subseteq \mathcal{A}\backslash\{0, a\}$ and let $\mathcal{W}_0 > 0$ and $\pi_0 \in [\alpha, \beta]$ be $\mathcal{F}_0$-measurable random variables. Then there exist $A \in \mathcal{F}_0$ with $P(A) = 1$ and an $[(\alpha, \beta)]_A$-strategy $(\varphi_t, \psi_t)_{t\geq 0}$ with initial wealth $\mathcal{W}_0$ and with the initial position equal to π_0 on A.

Proof. Let $(\pi_t, \pi_t^{\uparrow}, \pi_t^{\downarrow})_{t\geq 0}$ be a solution to Skorokhod problem with coefficients

$$\mathbb{B}(x) = z_a(x)(\mu - \sigma x), \quad \mathbb{S}(x) = \sigma z_a(x),$$

and reflective barriers at $\{\alpha, \beta\}$ satisfying $\pi_t \in [\alpha, \beta], t \geq 0$, i. e. let $(\pi_t)_{t\geq 0}$ be a continuous semimartingale attaining values in $[\alpha, \beta]$ and $(\pi_t^{\uparrow}, \pi_t^{\downarrow})_{t\geq 0}$ be non-decreasing continuous adapted processes starting from 0 such that

$$\pi_t \overset{\text{as}}{=} \pi_0 + \textstyle\int_0^t z_a(\pi_s)[\sigma \,\mathrm{d}B_s + (\mu - \sigma^2\pi_s)\,\mathrm{d}s] + \pi_t^{\uparrow} - \pi_t^{\downarrow}, \tag{80}$$

$$\textstyle\int_0^\infty 1_{[\pi_t\neq\alpha]}\,\mathrm{d}\pi_t^{\uparrow} = 0, \qquad \int_0^\infty 1_{[\pi_t\neq\beta]}\,\mathrm{d}\pi_t^{\downarrow} = 0. \tag{81}$$

Put

$$D_t \triangleq \textstyle\int_0^t \big(\frac{\lambda^{\uparrow}}{1+\lambda^{\uparrow}\pi_s}\,\mathrm{d}\pi_s^{\uparrow} + \frac{\lambda^{\downarrow}}{1+\lambda^{\downarrow}\pi_s}\,\mathrm{d}\pi_s^{\downarrow}\big), \qquad t \geq 0, \tag{82}$$

$$\mathcal{W}_t \triangleq \mathcal{W}_0 \exp\{\textstyle\int_0^t(\pi_s\,\mathrm{d}F_s - \frac{1}{2}\,\sigma^2\pi_s^2\,\mathrm{d}s) - D_t\}, \quad t > 0. \tag{83}$$

Then $(\mathcal{W}_t)_{t\geq 0}$ is a continuous semimartingale with $\mathrm{d}\mathcal{W}_t = \mathcal{W}_t[\pi_t\,\mathrm{d}F_t - \mathrm{d}D_t]$ and

$$\mathrm{d}\pi_t\mathcal{W}_t = a\,\pi_t\mathcal{W}_t\,\mathrm{d}F_t + \mathcal{W}_t\big(\tfrac{1}{1+\lambda^{\uparrow}\pi_t}\,\mathrm{d}\pi_t^{\uparrow} - \tfrac{1}{1-\lambda^{\downarrow}\pi_t}\,\mathrm{d}\pi_t^{\downarrow}\big). \tag{84}$$

Further, we put

$$\varphi_t \triangleq \pi_t\mathcal{W}_tS_t^{-a}, \quad \varphi_t^{\uparrow} \triangleq \textstyle\int_0^t\mathcal{W}_sS_s^{-a}\frac{1}{1+\lambda^{\uparrow}\pi_s}\,\mathrm{d}\pi_s^{\uparrow}, \quad \varphi_t^{\downarrow} \triangleq \int_0^t\mathcal{W}_sS_s^{-a}\frac{1}{1-\lambda^{\downarrow}\pi_s}\,\mathrm{d}\pi_s^{\downarrow}. \tag{85}$$

If $a = 0$, we immediately get from $(84, 85)$ that

$$\varphi_t \overset{\text{as}}{=} \varphi_0 + \varphi_t^{\uparrow} - \varphi_t^{\downarrow}, \quad t \geq 0. \tag{86}$$

If $a = 1$, we first get from $\mathrm{d}S_t = S_t\,\mathrm{d}F_t$ that $\mathrm{d}S_t^{-1} = S_t^{-1}(\sigma^2\,\mathrm{d}t - \mathrm{d}F_t)$ and then we use $(84, 85)$ in order to get (86). Further, notice that

$$\mathrm{d}\mathcal{W}_t = \mathcal{W}_t[\pi_t\,\mathrm{d}F_t - \mathrm{d}D_t] = \varphi_tS_t^a\,\mathrm{d}F_t - \mathcal{W}_t\,\mathrm{d}D_t \tag{87}$$

in order to verify the equality almost surely in the following

$$\psi_t \triangleq \begin{cases} \mathcal{W}_t - \varphi_tF_t \overset{\text{as}}{=} \psi_0 - \int_0^t F_s\,\mathrm{d}\varphi_s - \lambda^{\uparrow}\varphi_t^{\uparrow} - \lambda^{\downarrow}\varphi_t^{\downarrow} & \text{if} \quad a = 0, \\ \mathcal{W}_t - \varphi_tS_t \overset{\text{as}}{=} \psi_0 - (1+\lambda^{\uparrow})\int_0^t S_s\,\mathrm{d}\varphi_s^{\uparrow} + (1-\lambda^{\downarrow})\int_0^t S_s\,\mathrm{d}\varphi_s^{\downarrow} & \text{if} \quad a = 1. \end{cases} \tag{88}$$

If $a = 0$, (88) follows immediately from (87,82,85) and the integration by parts formula. If $a = 1$, realize that $\varphi_tS_t = \pi_t\mathcal{W}_t$ and subtract (84) from (87) before looking at (82,85).

As $\pi_t \in [\alpha, \beta] \subseteq \mathcal{A} = (-1/\lambda^{\uparrow}, 1/\lambda^{\downarrow})$ and $\mathcal{W}_t > 0$ hold whenever $t \geq 0$, we obtain that

$$\mathcal{W}_t^{\uparrow} = \mathcal{W}_t + \lambda^{\uparrow}\varphi_tS_t^a = \mathcal{W}_t(1 + \lambda^{\uparrow}\pi_t) > 0, \tag{89}$$

$$\mathcal{W}_t^{\downarrow} = \mathcal{W}_t - \lambda^{\downarrow}\varphi_tS_t^a = \mathcal{W}_t(1 - \lambda^{\downarrow}\pi_t) > 0 \tag{90}$$

hold whenever $t \geq 0$.

Note that the considered filtration $(\mathcal{F}_t)_{t\geq 0}$ is assumed to be complete. Hence, we may afford to redefine the strategy $(\varphi_t, \psi_t)_{t\geq 0}$ on a null set. Let $A \in \mathcal{F}_0$ with $P(A) = 1$ be the set, where the equalities in (86, 88) hold for every $t \geq 0$ and consider a continuous strategy $(\varphi_t^*, \psi_t^*)_{t\geq 0}$ defined as follows

$$(\varphi_t^*, \psi_t^*) \triangleq (\varphi_t, \psi_t) \cdot 1_A + (0, \mathcal{W}_0) \cdot 1_{\Omega \setminus A} \overset{\text{as}}{=} (\varphi_t, \psi_t), \quad t \geq 0, \tag{91}$$

with the wealth process denoted by $(\mathcal{W}_t^*)_{t\geq 0}$. Obviously, $(\varphi_t^*, \psi_t^*)_{t\geq 0}$ is a self-financing strategy and $\mathcal{W}_0^* = \mathcal{W}_0$. Further, $(\varphi_t^*, \psi_t^*)_{t\geq 0}$ is also an admissible strategy as

$$\mathcal{W}_t^* + \lambda^{\uparrow}\varphi_t^* S_t^a \geq \min\{\mathcal{W}_t^{\uparrow}, \mathcal{W}_0\} > 0, \quad \mathcal{W}_t^* - \lambda^{\downarrow}\varphi_t^* S_t^a \geq \min\{\mathcal{W}_t^{\downarrow}, \mathcal{W}_0\} > 0$$

hold whenever $t \geq 0$. Then the corresponding position

$$\pi_t^* \triangleq \varphi_t^* S_t^a / \mathcal{W}_t^* = \pi_t 1_A, \quad t \geq 0,$$

attains values in $[\alpha, \beta]$ on A and it starts from π_0 there. As

$$\varphi_t^* = 1_A \varphi_t = 1_A(\varphi_0 + \varphi_t^{\uparrow} - \varphi_t^{\downarrow}), \quad t \geq 0,$$

we only have to verify that the equalities in (79) hold with $(\pi_t, \varphi_t^{\uparrow}, \varphi_t^{\downarrow})_{t\geq 0}$ replaced by $1_A(\pi_t, \varphi_t^{\uparrow}, \varphi_t^{\downarrow})_{t\geq 0}$ in order to get that $(\varphi_t^*, \psi_t^*)_{t\geq 0}$ is really an $[(\alpha, \beta)]_A$-strategy and also in order to complete the proof. First, the equalities in (79) follow immediately from (81, 85) and then the validity of the equalities in (79) after the above-mentioned replacement is obvious. □

3.5. Preparation

In this subsection, we introduce new objects and we show their basic properties in order to be prepared to prove Theorem 3.31 in the next subsection.

Notation 3.24. Let us consider the following function

$$\xi_{a,\varepsilon} : x \mapsto \xi_{a,\varepsilon}(x) \triangleq \tfrac{1+a\varepsilon}{1+x\varepsilon} x, \tag{92}$$

if $a \in \{0, 1\}$ and $1 + x\varepsilon \neq 0$, where $\varepsilon \in \mathbb{R}$ is a parameter. See the following Lemma for its interpretation.

Lemma 3.25. Let $\tilde{\varepsilon} = (\tilde{\varepsilon}_t)_{t\geq 0}$ be a continuous semimartingale with values in $[-\lambda^{\downarrow}, \lambda^{\uparrow}]$ and let $(\varphi_t, \psi_t)_{t\geq 0}$ be an admissible strategy with position $(\pi_t)_{t\geq 0}$ and $\tilde{\varepsilon}$-position $(\tilde{\pi}_t)_{t\geq 0}$. Then

$$\tilde{\pi}_t = \xi_{a,\tilde{\varepsilon}_t}(\pi_t), \quad t \geq 0. \tag{93}$$

Proof. Let $(\mathcal{W}_t, \tilde{\mathcal{W}}_t)_{t\geq 0}$ be the wealth and the $\tilde{\varepsilon}$-wealth of $(\varphi_t, \psi_t)_{t\geq 0}$. As $(\varphi_t, \psi_t)_{t\geq 0}$ is admissible, both processes $(\mathcal{W}_t, \tilde{\mathcal{W}}_t)_{t\geq 0}$ attain only positive values. If $a = 1$, then

$$\begin{aligned} \tilde{\pi}_t \tilde{\mathcal{W}}_t &= \varphi_t \tilde{S}_t = (1 + \tilde{\varepsilon}_t)\varphi_t S_t = (1 + \tilde{\varepsilon}_t)\pi_t \mathcal{W}_t, \\ \tilde{\mathcal{W}}_t &= \psi_t + \varphi_t \tilde{S}_t = (1 - \pi_t)\mathcal{W}_t + (1 + \tilde{\varepsilon}_t)\pi_t \mathcal{W}_t = \mathcal{W}_t(1 + \tilde{\varepsilon}_t \pi_t), \end{aligned}$$

give that $\tilde{\pi}_t = \frac{1+\tilde{\varepsilon}_t}{1+\tilde{\varepsilon}_t\pi_t}\pi_t = \xi_{1,\tilde{\varepsilon}_t}(\pi_t), t \geq 0$. If $a = 0$, we get from the equalities

$$\tilde{\pi}_t\tilde{\mathcal{W}}_t = \varphi_t = \pi_t\mathcal{W}_t, \quad \tilde{\mathcal{W}}_t = \mathcal{W}_t + \tilde{\varepsilon}_t\varphi_t = \mathcal{W}_t(1+\tilde{\varepsilon}_t\pi_t),$$

that $\tilde{\pi}_t = \frac{\pi_t}{1+\tilde{\varepsilon}_t\pi_t} = \xi_{0,\tilde{\varepsilon}_t}(\pi_t)$ holds whenever $t \geq 0$. □

Lemma 3.26. Let $x \in \mathcal{A}\backslash\{0, a\}$ and $\varepsilon \in [-\lambda^{\downarrow}, \lambda^{\uparrow}]$, then

$$\Lambda_a(\varepsilon) = \int_x^{\xi_{a,\varepsilon}(x)} \frac{\mathrm{d}u}{z_a(u)},$$

and $\mathrm{sign}(x) = \mathrm{sign}(\xi_{a,\varepsilon}(x))$ and $\mathrm{sign}(a-x) = \mathrm{sign}(a - \xi_{a,\varepsilon}(x))$.

Proof. As $x \in \mathcal{A} = (-1/\lambda^{\uparrow}, 1/\lambda^{\downarrow})$ and $\varepsilon \in [-\lambda^{\downarrow}, \lambda^{\uparrow}]$, we get that $1+x\varepsilon > 0$. Note that we assume that $\lambda^{\downarrow} \in (0,1)$, i.e. $1/\lambda^{\downarrow} > 1$ if $a = 1$. Hence, we get that $a \in \mathcal{A}$ and also that the inequality $1 + a\varepsilon > 0$ holds in both cases $a \in \{0,1\}$. Then we get the equalities of signs in the statement of the lemma from the definition of $\xi_{a,\varepsilon}(x)$ in (92) and from

$$a - \xi_{a,\varepsilon}(x) = \frac{a-x}{1+x\varepsilon}.$$

Further, we have that

$$\int_x^{\xi_{a,\varepsilon}(x)} \frac{\mathrm{d}u}{z_a(u)} = \begin{cases} \frac{1}{\xi_{a,\varepsilon}(x)} - \frac{1}{x} = \frac{1+x\varepsilon}{x} - \frac{1}{x} = \varepsilon = \Lambda_a(\varepsilon) & \text{if} \quad a = 0, \\ \ln\left|\frac{\xi_{a,\varepsilon}(x)}{1-\xi_{a,\varepsilon}(x)}\right| - \ln\left|\frac{x}{1-x}\right| = \ln(1+\varepsilon) = \Lambda_a(\varepsilon) & \text{if} \quad a = 1. \end{cases}$$

□

Definition 3.27. Let $\lambda^{\uparrow} \in (0,\infty), \lambda^{\downarrow} \in (0,1)$ if $a = 1$ and $\lambda^{\uparrow}, \lambda^{\downarrow} \in (0,\infty)$ if $a = 0$. Put

$$\lambda \triangleq \Lambda_a(\lambda^{\uparrow}) - \Lambda_a(-\lambda^{\downarrow}). \tag{94}$$

Then we get by Lemma 6.2 in [15] and Lemma 11.2 in [14] that there exists just one $\omega = \omega_\lambda \in (0, |\theta| \wedge |a-\theta|)$ such that

$$0 = \lambda + \int_{\theta-\omega}^{\theta+\omega} \left[\frac{1}{q_\omega(x)} + \frac{1}{z_a(x)}\right] \mathrm{d}x \tag{95}$$

holds. Then the following values

$$\underline{\pi} \triangleq \xi^{-1}_{a,\lambda^{\uparrow}}(\theta-\omega), \qquad \bar{\pi} \triangleq \xi^{-1}_{a,-\lambda^{\downarrow}}(\theta+\omega), \tag{96}$$

are called the *log-optimal policies*.

Remark 3.28. See [15] for $a = 1$ and [14] for $a = 0$ that the above introduced values $\underline{\pi}, \bar{\pi}$ really play the role of log-optimal policies in the long run. Also see Lemma 6.4 in [15] that $\underline{\pi} < \bar{\pi}$ and that $[\underline{\pi}, \bar{\pi}] \subseteq \mathcal{A}\backslash\{0,1\}$ if $a = 1$. If $a = 0$, see Lemma 11.3 in [14] that $\underline{\pi} < \bar{\pi}$ and that $\underline{\pi}, \bar{\pi} \in \mathcal{A}$. Further, $\underline{\pi}, \bar{\pi}$ have the same non-zero sign if $a = 0$ by the second part of Lemma 3.26 as $\omega \in (0, |\theta|)$, i.e. as $\theta \pm \omega$ have the same non-zero sign. Hence, we have that $[\underline{\pi}, \bar{\pi}] \subseteq \mathcal{A}\backslash\{0, a\}$ holds in both cases $a \in \{0,1\}$.

Notation 3.29. Let $\omega = \omega_\lambda \in (0, |\theta| \wedge |a-\theta|)$ be such that (95) holds. Put

$$Q^{\uparrow}_{a,\omega}(x) \triangleq \Lambda_a(\lambda^{\uparrow}) + \int_{\theta-\omega}^{x} \frac{\mathrm{d}v}{z_a(v)}, \qquad Q^{\downarrow}_{a,\omega}(x) \triangleq \Lambda_a(-\lambda^{\downarrow}) + \int_{\theta+\omega}^{x} \frac{\mathrm{d}v}{z_a(v)}$$

whenever the integrals converge, i. e. if $\operatorname{sign}(x) = \operatorname{sign}(\theta)$ and $\operatorname{sign}(a-x) = \operatorname{sign}(a-\theta)$. Then we have two following expressions of a newly introduced function

$$y_u(x) \triangleq \begin{cases} \frac{1}{2\theta-a}\,\left(\theta^2-\omega^2+q_\omega(\theta-\omega)\,\exp\left\{(a-2\theta)Q^{\uparrow}_{a,\omega}(x)\right\}\right) & \text{if } 2\theta \neq a, \\ \frac{1}{2}-\omega+(\frac{1}{4}-\omega^2)Q^{\uparrow}_{a,\omega}(x) & \text{if } 2\theta = a, \end{cases} \tag{97}$$

$$= \begin{cases} \frac{1}{2\theta-a}\,\left(\theta^2-\omega^2+q_\omega(\theta+\omega)\,\exp\left\{(a-2\theta)Q^{\downarrow}_{a,\omega}(x)\right\}\right) & \text{if } 2\theta \neq a, \\ \frac{1}{2}+\omega+(\frac{1}{4}-\omega^2)Q^{\downarrow}_{a,\omega}(x) & \text{if } 2\theta = a. \end{cases} \tag{98}$$

To see that both expressions are equivalent, it is enough to realize that

$$Q^{\uparrow}_{a,\omega}(x) = Q^{\downarrow}_{a,\omega}(x) - \int_{\theta-\omega}^{\theta+\omega} \frac{\mathrm{d}v}{q_\omega(v)} = Q^{\downarrow}_{a,\omega}(x) + \begin{cases} \frac{1}{a-2\theta}\ln\frac{q_\omega(\theta+\omega)}{q_\omega(\theta-\omega)} & \text{if } 2\theta \neq a, \\ \frac{2\omega}{1/4-\omega^2} & \text{if } 2\theta = a. \end{cases}$$

Note that the first equality is nothing else but assumption (95). Further, see the end of Remark 4.1 for the justification of the notation y_u.

Lemma 3.30. Let $\omega \in (0, |\theta| \wedge |a-\theta|)$ and λ satisfy (94,95). Then function y_u defined in Notation 3.29 is an increasing C^2-bijection $y_u : [\underline{\pi}, \bar{\pi}] \to [\theta-\omega, \theta+\omega]$ with

$$y_u'(x) = -\frac{q_\omega(y_u(x))}{z_a(x)} > 0, \qquad y_u''(x) = 2(\theta-x)\,\frac{q_\omega(y_u(x))}{z_a(x)^2}, \tag{99}$$

if $x \in [\underline{\pi}, \bar{\pi}]$. Moreover, the values of y_u can be also uniquely defined by any of the two following equations

$$Q^{\uparrow}_{a,\omega}(x) + \int_{\theta-\omega}^{y_u(x)} \frac{\mathrm{d}v}{q_\omega(v)} = 0 = Q^{\downarrow}_{a,\omega}(x) + \int_{\theta+\omega}^{y_u(x)} \frac{\mathrm{d}v}{q_\omega(v)}. \tag{100}$$

P r o o f. By Lemma 3.26 and (96), we get that

$$\Lambda_a(\lambda^{\uparrow}) = \int_{\underline{\pi}}^{\xi_{a,\lambda^{\uparrow}}(\underline{\pi})} \frac{\mathrm{d}v}{z_a(v)} = -\int_{\theta-\omega}^{\underline{\pi}} \frac{\mathrm{d}v}{z_a(v)}, \quad \Lambda_a(-\lambda^{\downarrow}) = \int_{\bar{\pi}}^{\xi_{a,-\lambda^{\downarrow}}(\bar{\pi})} \frac{\mathrm{d}v}{z_a(v)} = \int_{\bar{\pi}}^{\theta+\omega} \frac{\mathrm{d}v}{z_a(v)}, \tag{101}$$

i. e. $Q^{\uparrow}_{a,\omega}(\underline{\pi}) = 0$ and $Q^{\downarrow}_{a,\omega}(\bar{\pi}) = 0$. Then we obtain by (97–98) that

$$y_u(\underline{\pi}) = \theta-\omega \quad \text{and} \quad y_u(\bar{\pi}) = \theta+\omega \tag{102}$$

as $q_\omega(x) = (2\theta-a)x - (\theta^2-\omega^2)$ holds by definition. Further, we easily obtain that (97–98) are equivalent to the equalities

$$Q^{\uparrow}_{a,\omega}(x) = \begin{cases} \frac{1}{a-2\theta}\ \ln\frac{q_\omega(y_u(x))}{q_\omega(\theta-\omega)} & \text{if } 2\theta \neq a \\ \frac{y_u(x)-(1/2-\omega)}{1/4-\omega^2} & \text{if } 2\theta = a, \end{cases} \qquad Q^{\downarrow}_{a,\omega}(x) = \begin{cases} \frac{1}{a-2\theta}\ \ln\frac{q_\omega(y_u(x))}{q_\omega(\theta+\omega)} & \text{if } 2\theta \neq a \\ \frac{y_u(x)-(1/2+\omega)}{1/4-\omega^2} & \text{if } 2\theta = a. \end{cases}$$

Then look at the definition of $q_\omega(x)$ recalled above in order to verify that the above equalities are equivalent to those in (100). Obviously, y_u is a C^2 function by definition and we obtain the left-hand equality in (99) from (100), for example, while the second equality in (99) can be easily obtained from the first one as follows

$$y_u''(x) = \frac{z_a'(x)\, q_\omega(y_u(x))}{z_a(x)^2} - \frac{q_\omega'(y_u(x))\, y_u'(x)}{z_a(x)} = \frac{q_\omega(y_u(x))}{z_a(x)^2}\,[(a-2x)+(2\theta-a)].$$

We have used that $[\underline{\pi},\bar{\pi}] \subseteq \mathcal{A}\backslash\{0,a\}$ pointed out in Remark 3.28 in order to get $z_a(x) \neq 0$.

It remains to show that $y_u'(x) > 0$ holds if $x \in [\underline{\pi},\bar{\pi}]$. It means to show that $q_\omega(y_u(x))$ and $z_a(x)$ have opposite signs. First, we obtain from the second part of Lemma 3.26 and definition of $\underline{\pi},\bar{\pi}$ in (96) that

$$z_a(\bar{\pi}), z_a(\underline{\pi}) \quad \text{have the same sign as} \quad z_a(\theta \pm \omega), \quad \text{i.e. as} \quad z_a(\theta), \tag{103}$$

since $\omega \in (0, |\theta| \wedge |a-\theta|)$. If $2\theta = a$, then $a = 1, \theta = 1/2$ and the sign from (103) is $+1$, i.e. $[\underline{\pi},\bar{\pi}] \subseteq (0,1)$, and we immediately have that

$$y_u'(x) = \frac{1/4-\omega^2}{x(1-x)} > 0 \qquad \text{if} \quad x \in [\underline{\pi},\bar{\pi}].$$

Let $2\theta \neq a$. See (97−98) or the equivalent equalities written above in this proof that

$$q_\omega(y_u(x)) \qquad \text{has the same sign as} \qquad q_\omega(\theta \pm \omega) = -z_a(\theta \pm \omega), \tag{104}$$

if $x \in [\underline{\pi},\bar{\pi}]$. As $[\underline{\pi},\bar{\pi}] \subseteq \mathcal{A}\backslash\{0,a\}$, we get from (103, 104) that $z_a(x)$ has the same sign as $z_a(\bar{\pi}), z_a(\underline{\pi})$ and opposite to the sign of $q_\omega(y_u(x))$. Therefore, $y_u'(x) = -\frac{q(y_u(x))}{z_a(x)} > 0$ holds if $x \in [\underline{\pi},\bar{\pi}]$ in both considered cases $a \in \{0,1\}$. □

3.6. Existence of a shadow price II

In Theorem 3.31 we will show that (A1) implies (A2) and (A3) from Theorem 3.21 if (roughly speaking) the considered strategy keeps the position between the log-optimal policies introduced in the Definition 3.27 and if $\tilde{\varepsilon} = (\tilde{\varepsilon}_t)_{t\geq 0}$ is given by (105). We show that the strategy is log-optimal in the long run among a wide class of strategies in Corollary 3.32, and in Theorem 3.34 we show that the strategy is also log-optimal in the long run among all admissible strategies satisfying a certain restriction on initial wealth.

Theorem 3.31. Let ω and f satisfy assumption (A1) from Theorem 3.21 and $\underline{\pi},\bar{\pi}$ be the log-optimal policies from Definition 3.27. Let $(\varphi_t,\psi_t)_{t\geq 0}$ be a $[(\underline{\pi},\bar{\pi})]_A$-strategy with position $(\pi_t)_{t\geq 0}$. Put

$$\tilde{\pi}_t \triangleq 1_A \cdot y_u(\pi_t), \quad \tilde{\varepsilon}_t \triangleq 1_A \cdot \Lambda_a^{-1}(f(\tilde{\pi}_t)), \quad t \geq 0. \tag{105}$$

Then assumptions (A2, A3) from Theorem 3.21 are satisfied. In particular,

1. the $\tilde{\varepsilon}$-price given by (19,23) is regular, and it is a shadow price,
2. (φ,ψ) is an $\tilde{\varepsilon}$-cost-free strategy keeping $\tilde{\varepsilon}$-position on the log-optimal proportion a.s.,

3. (φ, ψ) is an $\tilde{\varepsilon}$-shadow strategy if its initial wealth $\mathcal{W}_0$ is such that $\ln \mathcal{W}_0 \in \mathbb{L}_1$.

Proof. As $(\varphi_t, \psi_t)_{t\geq 0}$ is a continuous admissible strategy by assumption, we get by Lemma 2.20 that $(\pi_t)_{t\geq 0}$ is (as 0-position process) a continuous semimartingale with

$$\mathrm{d}\pi_t = z_a(\pi_t)[\sigma^2(\theta-\pi_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t] + \mathcal{W}_t^{-1} S_t^a[(1+\lambda^\uparrow \pi_t)\,\mathrm{d}\varphi_t^\uparrow - (1-\lambda^\downarrow \pi_t)\,\mathrm{d}\varphi_t^\downarrow], \quad (106)$$

and it attains values in $[\underline{\pi}, \bar{\pi}]$ on A by assumption. Here, $(\mathcal{W}_t)_{t\geq 0}$ stands for the wealth process of $(\varphi_t, \psi_t)_{t\geq 0}$ and if $a = 1$, $(S_t)_{t\geq 0}$ stands for the stock market price. Further, we get by Lemma 3.30, Itô Lemma and (105) that $(\tilde{\pi}_t)_{t\geq 0}$ is also a continuous semimartingale with values in $[\theta-\omega, \theta+\omega]$ on A with

$$\mathrm{d}\tilde{\pi}_t = -\sigma q_\omega(\tilde{\pi}_t)\,\mathrm{d}B_t + \mathrm{d}\tilde{\pi}_t^\uparrow - \mathrm{d}\tilde{\pi}_t^\downarrow, \quad (107)$$

where

$$\tilde{\pi}_t^\uparrow \triangleq -\int_0^t \tfrac{q_\omega(\tilde{\pi}_s)\, S_s^a}{z_a(\pi_s)\,\mathcal{W}_s}\,(1+\lambda^\uparrow \pi_s)\,\mathrm{d}\varphi_s^\uparrow, \quad \tilde{\pi}_t^\downarrow \triangleq -\int_0^t \tfrac{q_\omega(\tilde{\pi}_s)\, S_s^a}{z_a(\pi_s)\,\mathcal{W}_s}\,(1-\lambda^\downarrow \pi_s)\,\mathrm{d}\varphi_s^\downarrow. \quad (108)$$

As f is a decreasing function on $[\theta-\omega, \theta+\omega]$ by Lemma 3.18 and as it is continuous, we obtain by (67) from assumption (A1) of Theorem 3.21 that f maps $[\theta-\omega, \theta+\omega]$ onto $[\Lambda_a(-\lambda^\downarrow), \Lambda_a(\lambda^\uparrow)]$. As Λ_a^{-1} is an increasing C^2-function, we obtain by Itô Lemma that $(\tilde{\varepsilon}_t)_{t\geq 0}$ is a continuous semimartingale with values in $[-\lambda^\downarrow, \lambda^\uparrow] \ni 0$ everywhere on Ω.

Note that assumption (A3) from Theorem 3.21 is satisfied by definition of $(\tilde{\varepsilon}_t)_{t\geq 0}$ in (105) as $P(A) = 1$ holds by assumption. As $(\varphi_t)_{t\geq 0} \equiv 0$ holds on $\Omega\backslash A$ by definition, we get that $(\varphi_t^\uparrow, \varphi_t^\downarrow)_{t\geq 0}$ and subsequently also $(\tilde{\pi}_t^\uparrow, \tilde{\pi}_t^\downarrow)_{t\geq 0}$ possess the same property. Further, the left-hand inequality in (99) in Lemma 3.30 gives that

$$1_A \tfrac{q_\omega(\tilde{\pi}_t)}{z_a(\pi_t)} = 1_A \tfrac{q_\omega(y_u(\pi_t))}{z_a(\pi_t)} \leq 0, \qquad t \geq 0,$$

and we get from (108) that $(\tilde{\pi}_t^\uparrow, \tilde{\pi}_t^\downarrow)_{t\geq 0} = 1_A(\tilde{\pi}_t^\uparrow, \tilde{\pi}_t^\downarrow)_{t\geq 0}$ are really non-decreasing processes. Further, we get from Lemma 3.30 saying that $y_u : [\underline{\pi}, \bar{\pi}] \to [\theta-\omega, \theta+\omega]$ is an increasing bijection and from (105) that

$$\left.\begin{array}{l} [\pi_t = \underline{\pi}] = [\tilde{\pi}_t = \theta-\omega] \\ [\pi_t = \bar{\pi}] = [\tilde{\pi}_t = \theta+\omega] \end{array}\right\} \text{ hold on } A, \text{ i.e. } \left\{\begin{array}{l} A \cap [\pi_t = \underline{\pi}] = A \cap [\tilde{\pi}_t = \theta-\omega] \\ A \cap [\pi_t = \bar{\pi}] = A \cap [\tilde{\pi}_t = \theta+\omega]. \end{array}\right.$$

As the same holds with equalities "=" replaced by inequalities "≠" in the brackets, and as $(\tilde{\pi}_t^\uparrow, \tilde{\pi}_t^\downarrow)_{t\geq 0}$ are equal to zero on $\Omega\backslash A$, we get from (79) with $\alpha = \underline{\pi}$ and $\beta = \bar{\pi}$ and from (108) that

$$\int_0^\infty 1_{[\tilde{\pi}_t \neq \theta-\omega]}\,\mathrm{d}\tilde{\pi}_t^\uparrow = 0, \qquad \int_0^\infty 1_{[\tilde{\pi}_t \neq \theta+\omega]}\,\mathrm{d}\tilde{\pi}_t^\downarrow = 0.$$

Then we conclude that $(\tilde{\pi}_t)_{t\geq 0}$ is really a B_t-diffusion process with reflective barriers at $\{\theta-\omega, \theta+\omega\}$ and coefficients $\mathbb{B}(x) = 0, \mathbb{S}(x) = -\sigma q_\omega(x)$ according to the extended definition, i.e. the assumption (A2) of Theorem 3.21 is almost verified so that we only need to show that $(\tilde{\pi}_t)_{t\geq 0}$ is really the $\tilde{\varepsilon}$-position of $(\varphi_t, \psi_t)_{t\geq 0}$.

By Lemma 3.25, $(\xi_{a,\tilde{\varepsilon}_t}(\pi_t))_{t\geq 0}$ plays the role of the $\tilde{\varepsilon}$-position of $(\varphi_t,\psi_t)_{t\geq 0}$, and so our last step is to show that

$$(\tilde{\pi}_t)_{t\geq 0} = (\xi_{a,\tilde{\varepsilon}_t}(\pi_t))_{t\geq 0}. \tag{109}$$

As $\xi_{a,\varepsilon}(0) = 0$ holds by definition, we get from the definition of $(\tilde{\pi}_t)_{t\geq 0}$ in (105) and from the definition of $[(\underline{\pi},\bar{\pi})]_A$-strategy that the desired equality (109) holds on $\Omega\backslash A$. Let $x \in [\underline{\pi},\bar{\pi}]$, we get by Lemma 3.30 that $y_u(x) \in [\theta-\omega,\theta+\omega]$ and we get from assumption (A1) of Theorem 3.21 covering equalities (60, 67) the first equality in the following

$$f(y_u(x)) = \Lambda_a(\lambda^{\uparrow}) + \int_{\theta-\omega}^{y_u(x)}\big(\tfrac{1}{z_a(v)} + \tfrac{1}{q_\omega(v)}\big)\,\mathrm{d}v = \int_x^{y_u(x)}\tfrac{\mathrm{d}v}{z_a(v)}. \tag{110}$$

The second equality follows from the moreover part of Lemma 3.30 covering the first equality in (100). Then we get from (110), definition of $(\tilde{\pi}_t)_{t\geq 0}$ and $(\tilde{\varepsilon}_t)_{t\geq 0}$ in (105) and from Lemma 3.26 that

$$\int_{\pi_t}^{\tilde{\pi}_t}\tfrac{\mathrm{d}v}{z_a(v)} = \int_{\pi_t}^{y_u(\pi_t)}\tfrac{\mathrm{d}v}{z_a(v)} = f(y_u(\pi_t)) = f(\tilde{\pi}_t) = \Lambda_a(\tilde{\varepsilon}_t) = \int_{\pi_t}^{\xi_{a,\tilde{\varepsilon}_t}(\pi_t)}\tfrac{\mathrm{d}v}{z_a(v)}, \qquad t\geq 0$$

hold on A, which gives the desired equality (109) on A. Note that $x_1 = x_2$ holds whenever $\int_{x_1}^{x_2}\frac{\mathrm{d}v}{z_a(v)} = 0$ as the function $v\mapsto 1/z_a(v) = z^{-1}(a-z)^{-1}$ does not change the sign on the interval $[x_1,x_2]$ (or $[x_2,x_1]$ respectively) since otherwise it would not be an integrable function on the considered interval.

The remaining part of the proof follows from Theorem 3.21. If $\ln\mathcal{W}_0 \in \mathbb{L}_1$, we consider the original strategy (φ,ψ) and if $\ln\mathcal{W}_0 \notin \mathbb{L}_1$, we consider a strategy $(\varphi,\psi)/\mathcal{W}_0$ instead in order to obtain that the $\tilde{\varepsilon}$-price is a shadow price also in this case. □

Corollary 3.32. Given the transaction taxes $\lambda^{\uparrow},\lambda^{\downarrow}$, let $\lambda > 0, \omega = \omega_\lambda \in (0,|\theta| \wedge |a-\theta|)$ and let $\underline{\pi},\bar{\pi}$ be the log-optimal policies from Definition 3.27. *Let* $(\varphi_t,\psi_t)_{t\geq 0}$ *be a* $[(\underline{\pi},\bar{\pi})]_A$*-strategy with the wealth process* $(\mathcal{W}_t)_{t\geq 0}$*, where* $A\in\mathcal{F}_0$ *is a set of probability* 1.

(i) If $(\hat{\mathcal{W}}_t)_{t\geq 0}$ is the wealth process of an admissible strategy keeping the position within a compact subset of $\mathcal{A}$, then

$$\limsup_{t\to\infty}\tfrac{1}{t}\,\ln(\hat{\mathcal{W}}_t/\mathcal{W}_t) \overset{\text{as}}{\leq} 0.$$

Moreover, if $E\max\{0,\ln(\hat{\mathcal{W}}_0/\mathcal{W}_0)\} < \infty$, then also

$$\limsup_{t\to\infty}\tfrac{1}{t}\,E[\ln(\hat{\mathcal{W}}_t/\mathcal{W}_t)] \leq 0.$$

(ii) If $\ln\mathcal{W}_0 \in \mathbb{L}_1$, then $(\varphi_t,\psi_t)_{t\geq 0}$ is a shadow strategy and the corresponding shadow price can be obtained from (19) if $a = 1$ and from (23) if $a = 0$, where $\tilde{\varepsilon}$ comes from (105), where $(\pi_t)_{t\geq 0}$ is the position process of $(\varphi_t,\psi_t)_{t\geq 0}$ and where

$$f(x) \triangleq \Lambda_a(\lambda^{\uparrow}) + \int_{\theta-\omega}^{x}\big[\tfrac{1}{q_\omega(u)} + \tfrac{1}{z_a(u)}\big]\,\mathrm{d}u. \tag{111}$$

(iii) Generally, (ii) holds with (φ,ψ) replaced by $(\varphi,\psi)/\mathcal{W}_0$, even if $\ln\mathcal{W}_0 \notin \mathbb{L}_1$.

Proof. It follows from the definition of $\omega = \omega_\lambda$ and the definition of function f in (111) that assumption (A1) of Theorem 3.21 is satisfied. Then we obtain from Theorem 3.31 that (ii) holds and that the the assumptions of Theorem 2.26 are satisfied so that we get from the Theorem 2.26 together with Lemma 2.12 that (i) holds. In order to verify (iii), it is sufficient to realize that the strategy $(\varphi, \psi)/\mathcal{W}_0$ has unit initial wealth and that it satisfies the assumptions of this Corollary. It is also helpful to realize that this change of the strategy has no effect on the values of $(\pi_t, \tilde{\pi}_t, \tilde{\varepsilon}_t)_{t\geq 0}$. □

Remark 3.33. See Lemma 3.23 that there exists a $[(\underline{\pi}, \bar{\pi})]_A$-strategy $(\varphi_t, \psi_t)_{t\geq 0}$ given an initial wealth and given an initial position within $[\underline{\pi}, \bar{\pi}]$ on $A \in \mathcal{F}_0$ with $P(A) = 1$.

Theorem 3.34. Let $(\varphi, \psi), (\hat{\varphi}, \hat{\psi})$ be admissible strategies with the wealth proceses $\mathcal{W}, \hat{\mathcal{W}}$, respectively, such that $\ln \mathcal{W}_0, \ln \hat{\mathcal{W}}_0$ are integrable random variables. If (φ, ψ) is the strategy from Corollary 3.32, then

$$\limsup_{t\to\infty} \tfrac{1}{t}\, E[\ln \hat{\mathcal{W}}_t] \leq \lim_{t\to\infty} \tfrac{1}{t}\, E[\ln \mathcal{W}_t], \tag{112}$$

$$\limsup_{t\to\infty} \tfrac{1}{t}\, \ln \hat{\mathcal{W}}_t \overset{\text{as}}{\leq} \lim_{t\to\infty} \tfrac{1}{t}\, \ln \mathcal{W}_t. \tag{113}$$

Proof. First, without loss of generality, we may assume that $\mathcal{W}_0 = 1$. Further, note that the strategy (φ, ψ) just keeps the position within the interval $[\underline{\pi}, \bar{\pi}]$ up to a null set where $\underline{\pi}, \bar{\pi}$ are the log-optimal policies. Then see [14] and [15] that

$$\lim_{t\to\infty} \tfrac{1}{t}\, \ln \mathcal{W}_t \overset{\text{as}}{=} \lim_{t\to\infty} \tfrac{1}{t}\, E[\ln \mathcal{W}_t] = \nu = \tfrac{1}{2}\, \sigma^2(\theta^2 - \omega^2), \tag{114}$$

where $\omega \in (0, |\theta| \wedge |a - \theta|)$ comes from the definition of the log-optimal policies, i.e. it is the solution of the following equation

$$0 = \Lambda_a(\lambda^\uparrow) - \Lambda_a(-\lambda^\downarrow) + \mathcal{I}(\omega), \qquad \text{where} \qquad \mathcal{I}(\omega) \triangleq \int_{\theta-\omega}^{\theta+\omega} \big(\tfrac{1}{z_a(x)} + \tfrac{1}{q_\omega(x)}\big)\, dx.$$

See Lemma 11.2 in [14] and Lemma 6.3 in [15] that the function $\mathcal{I}$ is a continuous decreasing bijection between $(0, |\theta| \wedge |a - \theta|)$ and $(-\infty, 0)$. Also note that (112−113) hold by Corollary 3.32 if $(\hat{\varphi}, \hat{\psi})$ is an admissible strategy from the statement of Theorem 3.34 with the position attaining values in a compact subset of $\mathcal{A}$.

Let us consider an increasing sequence $0 < \omega_n \uparrow \omega$. Then see the definition of Λ_a and take into account the above mentioned properties of $\mathcal{I}$ in order to see that there exist increasing sequences $0 < \lambda_n^\uparrow \uparrow \lambda^\uparrow$ and $0 < \lambda_n^\downarrow \uparrow \lambda^\downarrow$ such that

$$0 = \Lambda_a(\lambda_n^\uparrow) - \Lambda_a(-\lambda_n^\downarrow) + \mathcal{I}(\omega_n).$$

Let $(\hat{\varphi}, \hat{\psi})$ be an admissible strategy from the statement of Theorem 3.34 and let $(\hat{\varphi}^n, \hat{\psi}^n)$ be a self-financing strategy corresponding to transaction taxes $(\lambda_n^\uparrow, \lambda_n^\downarrow)$ such that $(\hat{\varphi}^n, \hat{\psi}_0^n) = (\hat{\varphi}, \hat{\psi}_0)$. As the transaction taxes $(\lambda_n^\uparrow, \lambda_n^\downarrow)$ are lower than $(\lambda^\uparrow, \lambda^\downarrow)$, we get that $\hat{\mathcal{W}}^n \geq \hat{\mathcal{W}}$ where $\hat{\mathcal{W}}^n$ and $\hat{\mathcal{W}}$ are the wealth processes of $(\hat{\varphi}^n, \hat{\psi}^n)$ and of $(\hat{\varphi}, \hat{\psi})$, respectively. Further, we obtain that $(\hat{\varphi}^n, \hat{\psi}^n)$ is an admissible strategy w.r.t. the transaction taxes $(\lambda_n^\uparrow, \lambda_n^\downarrow)$ as

$$\min\{\hat{\mathcal{W}}_t^n + \lambda_n^\uparrow \hat{\varphi}_t^n S_t^a, \hat{\mathcal{W}}_t^n - \lambda_n^\downarrow \hat{\varphi}_t^n S_t^a\} \geq \hat{\mathcal{W}}_t - (\lambda_n^\uparrow \hat{\varphi}_t^- + \lambda_n^\downarrow \hat{\varphi}_t^+) S_t^a \geq \min\{\hat{\mathcal{W}}_t^\uparrow, \hat{\mathcal{W}}_t^\downarrow\} > 0,$$

where $\hat{\mathcal{W}}_t^\uparrow \triangleq \hat{\mathcal{W}}_t + \lambda^\uparrow \hat{\varphi}_t S_t^a$ and $\hat{\mathcal{W}}_t^\downarrow \triangleq \hat{\mathcal{W}}_t - \lambda^\downarrow \hat{\varphi}_t S_t^a$. Obviously, the position $(\hat{\pi}_t)_{t\geq 0}$ of the competing admissible strategy $(\hat{\varphi}, \hat{\psi})$ attains values in $\mathcal{A}$. Then also

$$\hat{\pi}_t^n \triangleq \hat{\varphi}_t^n S_t^a / \hat{\mathcal{W}}_t^n = \hat{\varphi}_t S_t^a / \hat{\mathcal{W}}_t^n = \hat{\pi}_t \hat{\mathcal{W}}_t / \hat{\mathcal{W}}_t^n \in \mathcal{A} = (-1/\lambda^\uparrow, 1/\lambda^\downarrow),$$

as $\mathcal{A}$ is a convex set containing zero and as $\operatorname{sign}(\hat{\pi}_t^n) = \operatorname{sign}(\hat{\pi}_t)$ and $|\hat{\pi}_t^n| \leq |\hat{\pi}_t|$. In particular, $(\hat{\pi}_t^n)_{t\geq 0}$ attains values in a compact subset $[-1/\lambda^\uparrow, 1/\lambda^\downarrow]$ of $\mathcal{A}_n \triangleq (-1/\lambda_n^\uparrow, 1/\lambda_n^\downarrow)$. By the first part of the proof applied with transaction taxes $(\lambda_n^\uparrow, \lambda_n^\downarrow)$, we get that

$$\limsup_{t\to\infty} \tfrac{1}{t}\, E[\ln \hat{\mathcal{W}}_t] \leq \limsup_{t\to\infty} \tfrac{1}{t}\, E[\ln \hat{\mathcal{W}}_t^n] \leq \nu_n \triangleq \tfrac{1}{2}\, \sigma^2(\theta^2 - \omega_n^2) \to \nu,$$

as $n \to \infty$, which together with (114) ensures that (112) holds. The relation (113) can be obtained in a similar manner. □

4. COMPARISON OF TECHNICAL TOOLS

The aim of this section is to provide links between the technical tools considered in this paper and the tools considered in some other papers solving the same problem, i. e. the problem of maximization of growth rate of the wealth process in the long run.

4.1. Comparison to [17]

See [17] for a solution of maximization of the long-run growth rate of the wealth process corresponding to the case $a = 1$ based on shadow prices. They seek for a shadow price $(\tilde{S}_t)_{t\geq 0}$ in the form

$$\tilde{S}_t = m_t\, g(\tfrac{S_t}{m_t}), \tag{115}$$

where g is a smooth function and $(m_t)_{t\geq 0}$ is a certain process of locally finite variation, which is in the end of the form

$$m_t = \tfrac{1}{c}\, \tilde{S}_t(\tfrac{1}{\tilde{\pi}_t} - 1), \qquad \text{with} \qquad c = \tfrac{1-\theta+\omega_\lambda}{\theta-\omega_\lambda},$$

where $(\tilde{S}_t, \tilde{\pi}_t)_{t\geq 0}$ are the obtained shadow price and the corresponding shadow position.

They assume without loss of generality that $\lambda^\uparrow = 0$ and their results really correspond to the ones presented in this paper when $a = 1$. Note that we assume that $\lambda^\uparrow > 0$ in this paper but admitting the case $\lambda^\uparrow = 0$ would require only minor changes. The function g considered in (115) and function f satisfying assumption (A1) of Theorem 3.21 are connected as follows

$$f(x) = -\ln \tfrac{1-x}{x} - \ln g^{-1}(\tfrac{cx}{1-x}), \qquad g^{-1}(y) = \tfrac{y}{c}\, e^{-f\left(\frac{y}{y+c}\right)}.$$

Note that our ODE for f is simpler than the ODE for g considered in [17].

4.2. Comparison to martingale approach

A martingale approach considered in [14] and [15] provides a solution to the problem

$$\max\ \liminf_{t\to\infty} \tfrac{1}{t}\, \mathcal{U}_\gamma^{-1} E\mathcal{U}_\gamma(\mathcal{W}_t), \qquad \text{where} \qquad \mathcal{U}_\gamma(y) = \begin{cases} \ln y & \text{if } \gamma = 0 \\ \frac{1}{\gamma}\, y^\gamma & \text{if } \gamma < 0 \end{cases} \tag{116}$$

of maximization of the long-run growth rate of the certainty equivalent from the wealth process among a certain family of strategies. The main task there is to find a smooth function g and $\nu \in \mathbb{R}$ such that

$$\mathcal{U}_\gamma(\mathcal{W}_t e^{-g(\pi_t)-\nu t}), \quad t \geq 0, \tag{117}$$

is a martingale in the optimal case and such that it is a supermartingale, in general. If $\gamma = 0$, we have the following link between the process $\tilde{\varepsilon} = (\tilde{\varepsilon}_t)_{t\geq 0}$ determining the obtained shadow price in this paper and the function g from (117) as follows

$$0 = \tilde{\varepsilon}_t + G(\pi_t), \qquad \text{where} \qquad G(x) \triangleq \tfrac{g'(x)}{1+xg'(x)}, \tag{118}$$

provided that $(\pi_t)_{t\geq 0}$ is the position of the optimal strategy.

Remark 4.1. The interested reader may appreciate a very brief and rough presentation of the solution to the problem (116) solved in [14] and [15]. The optimal policies $\underline{\pi}, \bar{\pi}$ for the position process $(\pi_t)_{t\geq 0}$ and the maximal long-run growth rate ν of the certainty equivalent from the wealth process are as follows

$$\underline{\pi} = \xi^{-1}_{a,\Lambda_a(\lambda^\uparrow)}(\Theta - \omega), \quad \bar{\pi} = \xi^{-1}_{a,\Lambda_a(-\lambda^\downarrow)}(\Theta + \omega), \quad \nu = (1-\gamma)\,\tfrac{\sigma^2}{2}\,(\Theta^2 - \omega^2),$$

where $\Theta \triangleq \theta/(1-\gamma)$ is the corresponding Merton proportion and where $\omega = \omega_\lambda > 0$ is the unique root of the following equation

$$0 = \lambda + \textstyle\int_{\Theta-\omega}^{\Theta+\omega} \big(\tfrac{1}{z_a(x)} + \tfrac{1}{\mathbb{D}_\omega(x)}\big)\,\mathrm{d}x, \;\text{ where }\; \mathbb{D}_\omega(x) \triangleq \gamma x^2 + (2\theta - a)x + (1-\gamma)(\Theta^2 - \omega^2).$$

The function g from (117) is of the form $g(x) = \int \frac{x - y(u(x))}{z_a(x)}\,\mathrm{d}x$, where y solves

$$y'(u) + \mathbb{D}_\omega(y(u)) = 0, \quad y(u(\underline{\pi})) = \Theta - \omega, \quad y(u(\bar{\pi})) = \Theta + \omega,$$

where $u(x) = \frac{1}{x}$ if $a = 0$ and $u(x) = \ln|\frac{x}{1-x}|$ if $a = 1$. Finally note that the function $x \mapsto y_u(x)$ from (97−98) is nothing else but $x \mapsto y(u(x))$ if $\gamma = 0$.

5. APPENDIX

In this section, namely in Lemma 5.5, we show that a self-financing strategy with a positive wealth process is, up to a null set, equal to an admissible strategy.

Notation 5.1. If B is a subset of the extended real line $\mathbb{R} \cup \{\infty, -\infty\}$, we write simply $B_{\mathbb{R}}$ instead of $B \cap \mathbb{R}$, and if τ is a random time and $(X_t)_{t\geq 0}$ a real-valued process, we simply write $[X_\tau \in B]$ instead of $[\tau < \infty, X_\tau \in B]$ as X_∞ is not defined in this section. Let α be a fixed stopping time. Put

$$\tau^\uparrow_\alpha \triangleq \inf\{t \geq \alpha; S_t = S^\uparrow_\alpha\}, \quad \tau^\downarrow_\alpha \triangleq \inf\{t \geq \alpha; S_t = S^\downarrow_\alpha\}, \tag{119}$$

if $a = 1$ and in the same way with S replaced by F if $a = 0$. Let us consider a fixed self-financing strategy $(\varphi_t, \psi_t)_{t\geq 0}$ with the ask and bid wealth processes $(\mathcal{W}^\uparrow_t, \mathcal{W}^\downarrow_t)_{t\geq 0}$ defined in (9, 15). Then we put

$$\beta^\uparrow_\alpha \triangleq \inf\{t \geq \alpha; \mathcal{W}^\uparrow_t = 0\}, \quad \beta^\downarrow_\alpha \triangleq \inf\{t \geq \alpha; \mathcal{W}^\downarrow_t = 0\}. \tag{120}$$

Further, we consider a self-financing strategy $(\hat{\varphi}_t, \hat{\psi}_t)_{t\geq 0} \triangleq (\varphi_{t\wedge\alpha}, \psi_{t\wedge\alpha})_{t\geq 0}$ with the ask and bid wealth processes $(\hat{\mathcal{W}}_t^{\uparrow}, \hat{\mathcal{W}}_t^{\downarrow})_{t\geq 0}$ defined as follows

$$(\hat{\mathcal{W}}_t^{\uparrow}, \hat{\mathcal{W}}_t^{\downarrow}) \triangleq \hat{\psi}_t + \hat{\varphi}_t(S_t^{\uparrow}, S_t^{\downarrow}), \tag{121}$$

if $a = 1$ and in the same way with S replaced by F if $a = 0$, cf. $(9, 15)$. Then we consider $\hat{\beta}_\alpha^{\uparrow}, \hat{\beta}_\alpha^{\downarrow}$ defined similarly as in (120) but with $\mathcal{W}$ replaced by $\hat{\mathcal{W}}$.

If α is a current time, we can interpret $\tau_\alpha^{\uparrow}, \tau_\alpha^{\downarrow}$ as the times when the nominal price reaches the current ask and bid price, respectively, and $\beta_\alpha^{\uparrow}, \beta_\alpha^{\downarrow}$ as the times when the ask and bid wealth process reaches zero, respectively. The β's can be understood as the bankruptcy times if the current wealth is positive but they will also be used differently.

Lemma 5.2. Let $(\varphi_t, \psi_t)_{t\geq 0}$ be a self-financing strategy such that the corresponding wealth process $(\mathcal{W}_t)_{t\geq 0}$ attains only positive values. Then

$$[\mathcal{W}_t^{\uparrow} \leq 0] \subseteq [\varphi_t < 0], \quad [\mathcal{W}_t^{\downarrow} \leq 0] \subseteq [\varphi_t > 0] \tag{122}$$

hold whenever $t \in [0, \infty)$, where $(\mathcal{W}_t^{\uparrow}, \mathcal{W}_t^{\downarrow})_{t\geq 0}$ are the processes from (9,15). Moreover, let $(\alpha, \beta_\alpha^{\uparrow}, \beta_\alpha^{\downarrow})$ be as in Notation 5.1. Then

$$\beta_\alpha^{\uparrow} \overset{\text{as}}{=} \inf\{t \geq \beta_\alpha^{\uparrow}; \mathcal{W}_t^{\uparrow} < 0\}, \qquad \beta_\alpha^{\downarrow} \overset{\text{as}}{=} \inf\{t \geq \beta_\alpha^{\downarrow}; \mathcal{W}_t^{\downarrow} < 0\}, \tag{123}$$

i. e. the processes $(\mathcal{W}_t^{\uparrow}, \mathcal{W}_t^{\downarrow})_{t\geq\alpha}$ reach negative values immediately after reaching the value zero up to a null set.

P r o o f. See (10,15) that $\operatorname{sign}(\varphi_t) = \operatorname{sign}(\mathcal{W}_t^{\uparrow} - \mathcal{W}_t) = \operatorname{sign}(\mathcal{W}_t - \mathcal{W}_t^{\downarrow})$ hold whenever $t \in [0, \infty)$. As $(\mathcal{W}_t)_{t\geq 0}$ is assumed to be a positive process, we get that

$$\begin{array}{l}[\mathcal{W}_t^{\uparrow} \leq 0] \subseteq [\mathcal{W}_t^{\uparrow} < \mathcal{W}_t] \subseteq [\varphi_t < 0], \\ [\mathcal{W}_t^{\downarrow} \leq 0] \subseteq [\mathcal{W}_t^{\downarrow} < \mathcal{W}_t] \subseteq [\varphi_t > 0],\end{array} \qquad t \in [0, \infty),$$

i. e. (122) is verified and it remains to show (123). Put

$$N_t \triangleq \textstyle\int_0^t \varphi_s S_s^a \,\mathrm{d}F_s \overset{\text{as}}{=} \mu \int_0^t \varphi_s S_s^a \,\mathrm{d}s + \sigma \int_0^t \varphi_s S_s^a \,\mathrm{d}B_s. \tag{124}$$

First, we will show that the following implication holds

$$[\beta < \infty] \subseteq [\varphi_\beta \neq 0] \quad \Rightarrow \quad \beta \overset{\text{as}}{=} \inf\{t \geq \beta; N_t < N_\beta\}, \tag{125}$$

whenever β is a stopping time. Let $[\beta < \infty] \subseteq [\varphi_\beta \neq 0]$. As $(\varphi_t)_{t\geq 0}$ is a right-continuous process, we get that $\int_\beta^t \varphi_s^2 S_s^{2a} \,\mathrm{d}s > 0$ holds on $[\beta < t]$. Then we obtain by Schwartz inequality that

$$\limsup_{t\to\beta-} |\textstyle\int_\beta^t \varphi_s S_s^a \,\mathrm{d}s| (\int_\beta^t \varphi_s^2 S_s^{2a} \,\mathrm{d}s)^{-1/2} \leq \lim_{t\to\beta-} \sqrt{t-\beta} = 0 \tag{126}$$

holds on $[\beta < \infty]$. Further, as $\langle N \rangle_t \overset{\text{as}}{=} \sigma^2 \int_0^t \varphi_s^2 S_s^{2a} \,\mathrm{d}s$, we get that

$$\beta \overset{\text{as}}{=} \inf\{t \geq \beta; \langle N \rangle_t > \langle N \rangle_\beta\}.$$

Then by the law of iterated logarithm and Dambis–Dubins–Schwartz Theorem, see Theorems 11.18 and 16.4 in [23], we get by (124,126) that the following implication

$$\beta < \infty \quad \Rightarrow \quad \liminf_{t\to\beta-} \frac{N_t - N_\beta}{\sqrt{\langle N\rangle_t - \langle N\rangle_\beta}} = \liminf_{t\to\beta-} \frac{\int_\beta^t \varphi_s S_s^a \,\mathrm{d}B_s}{\sqrt{\int_\beta^t \varphi_s^2 S_s^{2a}\,\mathrm{d}s}} = -\infty \tag{127}$$

holds up to a null set, which verifies the validity of the implication (125).

Further, as $(\mathcal{W}_t^\uparrow, \mathcal{W}_t^\downarrow)_{t\geq 0}$ are rcll-process, we get by the definition of $(\beta_\alpha^\uparrow, \beta_\alpha^\downarrow)$ in (120) the left-hand inclusions in

$$\begin{aligned} [\beta_\alpha^\uparrow < \infty] &\subseteq [\mathcal{W}^\uparrow_{\beta_\alpha^\uparrow} = 0] \subseteq [\varphi_{\beta_\alpha^\uparrow} < 0], \\ [\beta_\alpha^\downarrow < \infty] &\subseteq [\mathcal{W}^\downarrow_{\beta_\alpha^\downarrow} = 0] \subseteq [\varphi_{\beta_\alpha^\downarrow} > 0], \end{aligned} \tag{128}$$

while the right-hand inclusions follow from the already proved properties written in (122). Thus, $(\beta_\alpha^\uparrow, \beta_\alpha^\downarrow)$ satisfy the assumption of the implication (125) and hence, we get that also the conclusion in (125) holds with β replaced by $\beta_\alpha^\uparrow$ and $\beta_\alpha^\downarrow$, respectively.

By Lemma 2.15 with $\tilde{\varepsilon}_t = \lambda^\uparrow$, we get that

$$\mathcal{W}_t^\uparrow \overset{\text{as}}{=} \mathcal{W}_0^\uparrow + \int_0^t \varphi_s (S_s^\uparrow)^a \,\mathrm{d}F_s - C_t^\uparrow, \qquad \text{where} \qquad C_t^\uparrow \triangleq (\lambda^\uparrow + \lambda^\downarrow)\int_0^t S_s^a \,\mathrm{d}\varphi_s^\downarrow.$$

As $(C_t^\uparrow)_{t\geq 0}$ is a non-decreasing process, we obtain that

$$\mathcal{W}_t^\uparrow - \mathcal{W}_s^\uparrow \overset{\text{as}}{\leq} \int_s^t \varphi_u (S_u^\uparrow)^a \,\mathrm{d}F_u \overset{\text{as}}{=} (1+\lambda^\uparrow)^a [N_t - N_s], \qquad 0 \leq s \leq t < \infty. \tag{129}$$

Then we obtain the left-hand equality in (123). The second equality in (123) can be obtained similarly simply by replacing $\uparrow$ by $\downarrow$, and vice versa, in the end of this proof beginning with using Lemma 2.15. □

Lemma 5.3. Let $\mu \in \mathbb{R}, \sigma \in (0,\infty)$ and $(B_t)_{t\geq 0}$ be a standard Brownian motion. Put

$$\mathbb{B}_t \triangleq \mu t + \sigma B_t, \quad t \geq 0 \qquad \text{and} \qquad \tau_c \triangleq \inf\{t \geq 0; \mathbb{B}_t = c\}, \quad c \in \mathbb{R}.$$

If $u, -v \in (0,\infty)$, then $P(\tau_u < \tau_v) > 0$.

Proof. 1. Let $\mu = 0$. By Proposition 3.8 in [31, part II], $P(\tau_u < \tau_v) = \frac{v}{v-u} > 0$. Obviously, there exists $T \in [0,\infty)$ such that $0 < p_T(u,v,\sigma) \triangleq P(\tau_u < \tau_v \wedge T)$ holds as

$$0 < P(\tau_u < \tau_v) = \lim_{T\to\infty} p_T(u,v,\sigma).$$

2. Generally, let $\mu \in \mathbb{R}$. By the first part of the proof, there exists $T \in [0,\infty)$ such that $0 < p_T(u,v,\sigma)$. By Girsanov Theorem, see Corollary 16.25 in [23], there exists a probability measure $Q_T \sim P$ such that

$$P \circ (B_t)^{-1}_{t\leq T} = Q_T \circ (\bar{B}_t)^{-1}_{t\leq T}, \qquad \text{where} \qquad \bar{B}_t \triangleq B_t + \tfrac{\mu}{\sigma}\, t, \quad t \geq 0.$$

Then $0 < p_T(u,v,\sigma) = Q_T(\tau_u < \tau_v \wedge T) \leq Q_T(\tau_u < \tau_v)$, and also $0 < P(\tau_u < \tau_v)$ as $P \sim Q_T$. □

In the proof of Lemma 5.5, we will need strong Markov property of a standard Brownian motion formulated as follows.

Lemma 5.4. Let $(B_t)_{t\geq 0}$ be a standard $\mathcal{F}_t$-Brownian motion and τ be an $\mathcal{F}_t$-stopping time. Assume that there exists a standard Brownian motion $(\tilde{B}_t)_{t\geq 0}$ independent of $\mathcal{F}_\infty$. Then there exists a standard Brownian motion $(B^\tau_t)_{t\geq 0}$ such that

$$[\tau < \infty] \subseteq \bigcap_{t\geq 0} [B_{t+\tau} - B_\tau = B^\tau_t] \tag{130}$$

and that

$$1_{[\tau<\infty]} P(B^\tau \in A|\mathcal{F}_\tau) \overset{\text{as}}{=} 1_{[\tau<\infty]} P(B \in A), \tag{131}$$

whenever $A \subseteq \mathbb{R}^{[0,\infty)}$ is a Borel subset of the set of all continuous functions on $[0,\infty)$ endowed with compact open topology.

P r o o f. Put

$$B^\tau_t \triangleq 1_{[\tau<\infty]}(B_{t+\tau} - B_\tau) + 1_{[\tau=\infty]}\tilde{B}_t, \quad t \geq 0.$$

Then (130) obviously holds. Further, we will show that B, B^τ have the same distribution, which will ensure that B^τ is also a standard Brownian motion. Let $A \subseteq \mathbb{R}^{[0,\infty)}$ be a Borel subset of the set of all continuous functions on $[0,\infty)$ endowed with compact open topology. Obviously,

$$1_{[\tau=\infty]} P(B^\tau \in A|\mathcal{F}_\infty) \overset{\text{as}}{=} 1_{[\tau=\infty]} P(\tilde{B} \in A|\mathcal{F}_\infty) \overset{\text{as}}{=} 1_{[\tau=\infty]} P(\tilde{B} \in A),$$

as $B^\tau = \tilde{B}$ on $[\tau = \infty] \in \mathcal{F}_\infty$ and $\tilde{B}$ is independent of $\mathcal{F}_\infty$. Put

$$\mathbb{A}_\tau \triangleq [B^\tau \in A] \cap [\tau < \infty], \quad \mathbb{A}_{\tau\wedge n} \triangleq [B^{\tau\wedge n} \in A]. \tag{132}$$

Then we easily obtain, from strong Markov property presented by Theorem 11.11 in [23], that $P(\mathbb{A}_{\tau\wedge n}|\mathcal{F}_{\tau\wedge n}) \overset{\text{as}}{=} P(B \in A)$. As $\mathbb{A}_\tau, \mathbb{A}_{\tau\wedge n}$ differ only on set $[\tau > n]$, we get that

$$\begin{aligned} 1_{[\tau\leq n]} P(\mathbb{A}_\tau|\mathcal{F}_\tau) &\overset{\text{as}}{=} P([\tau \leq n] \cap \mathbb{A}_\tau|\mathcal{F}_\tau) \overset{\text{as}}{=} P([\tau \leq n] \cap \mathbb{A}_{\tau\wedge n}|\mathcal{F}_\tau) \\ &\overset{\text{as}}{=} 1_{[\tau\leq n]} P(\mathbb{A}_{\tau\wedge n}|\mathcal{F}_\tau) \overset{\text{as}}{=} 1_{[\tau\leq n]} P(\mathbb{A}_{\tau\wedge n}|\mathcal{F}_{\tau\wedge n}) \overset{\text{as}}{=} 1_{[\tau\leq n]} P(B \in A). \end{aligned}$$

If we pass to the limit $n \to \infty$, we obtain that $P(\mathbb{A}_\tau|\mathcal{F}_\tau) \overset{\text{as}}{=} 1_{[\tau<\infty]} P(B \in A)$. Finally, we get that

$$P(B^\tau \in A) = P(\tau = \infty)P(\tilde{B} \in A) + P(\tau < \infty)P(B \in A) = P(B \in A). \tag{133}$$

Note that (131) follows from the equality written just above (133). □

Lemma 5.5. Let $(\varphi_t, \psi_t)_{t\geq 0}$ be a self-financing strategy with positive wealth process $(\mathcal{W}_t)_{t\geq 0}$, then $\mathcal{W}^\uparrow_t \wedge \mathcal{W}^\downarrow_t > 0, t \geq 0$, up to a null set. In particular, there exists an admissible strategy $(\varphi^*_t, \psi^*_t)_{t\geq 0} \overset{\text{as}}{=} (\varphi_t, \psi_t)_{t\geq 0}$.

P r o o f. First, we show that $\alpha \overset{\text{as}}{=} \infty$ holds whenever α is a stopping time such that

$$[\alpha < \infty] \subseteq [\mathcal{W}^\updownarrow_\alpha < 0]. \tag{134}$$

As in Notation 5.1, we consider a self-financing strategy $(\hat{\varphi}_t, \hat{\psi}_t)_{t\geq 0} \triangleq (\varphi_{t\wedge\alpha}, \psi_{t\wedge\alpha})_{t\geq 0}$ that stops trading at α with the wealth process $(\hat{\mathcal{W}}_t)_{t\geq 0}$ and with the ask and bid wealth processes $(\hat{\mathcal{W}}^{\uparrow}_t, \hat{\mathcal{W}}^{\downarrow}_t)_{t\geq 0}$ introduced in (121). As $0 < \mathcal{W}_t \leq \mathcal{W}^{\uparrow}_t \vee \mathcal{W}^{\downarrow}_t, t \geq 0$, we get from (134) that also

$$[\alpha < \infty] \subseteq [\hat{\mathcal{W}}^{\uparrow}_\alpha = \mathcal{W}^{\uparrow}_\alpha < 0 < \mathcal{W}^{\downarrow}_\alpha = \hat{\mathcal{W}}^{\downarrow}_\alpha]. \tag{135}$$

As $(\hat{\mathcal{W}}^{\uparrow}_t, \hat{\mathcal{W}}^{\downarrow}_t)_{t\geq 0}$ do not jump at any $t \geq \alpha$, we get from (135) the left-hand equalities in

$$\begin{aligned} &\hat{\beta}^{\uparrow}_\alpha = \inf\{t \geq \alpha; \hat{\mathcal{W}}^{\uparrow}_t \geq 0\}, \qquad && \beta^{\uparrow}_\alpha = \inf\{t \geq \alpha; \mathcal{W}^{\uparrow}_t \geq 0\}, \\ &\hat{\beta}^{\downarrow}_\alpha = \inf\{t \geq \alpha; \hat{\mathcal{W}}^{\downarrow}_t \leq 0\}, \end{aligned} \tag{136}$$

cf. (120). In order to verify the equality on the right-hand side, see (134) and notice that self-financing condition gives that $\Delta\mathcal{W}^{\uparrow}_t \leq 0, t \geq 0$. These conditions ensure that $(\mathcal{W}^{\uparrow}_t)_{t\geq\alpha}$ has to reach zero before reaching any positive value and we get that also the last equality in (136) holds.

First, we have by (122) in Lemma 5.2 that $[\mathcal{W}^{\uparrow}_t \leq 0] \subseteq [\varphi_t < 0] \subseteq [\mathcal{W}^{\downarrow}_t > 0]$ hold if $t \geq 0$. Further, as $(\mathcal{W}^{\uparrow}_t, \hat{\mathcal{W}}^{\uparrow}_t, -\hat{\mathcal{W}}^{\downarrow}_t)_{t\geq\alpha}$ do not jump upwards, we get from (136) that

$$\begin{aligned} &\hat{\mathcal{W}}^{\uparrow}_t \leq 0, \quad t \in [\alpha, \hat{\beta}^{\uparrow}_\alpha]_{\mathbb{R}}, \qquad && \left.\begin{aligned}\mathcal{W}^{\uparrow}_t \leq 0 < \mathcal{W}^{\downarrow}_t \\ \varphi_t < 0\end{aligned}\right\} \; t \in [\alpha, \beta^{\uparrow}_\alpha]_{\mathbb{R}}. \\ &\hat{\mathcal{W}}^{\downarrow}_t \geq 0, \quad t \in [\alpha, \hat{\beta}^{\downarrow}_\alpha]_{\mathbb{R}}, \end{aligned} \tag{137}$$

By the self-financing condition (6,16) and the definition of $(\mathcal{W}^{\uparrow}_t, \mathcal{W}^{\downarrow}_t)_{t\geq 0}$ in (9,15), we obtain the equality on the left-hand side in the following

$$\psi_t\varphi_\alpha - \psi_\alpha\varphi_t = \int_\alpha^t \hat{\mathcal{W}}^{\downarrow}_s \,\mathrm{d}\varphi^{\downarrow}_s - \int_\alpha^t \hat{\mathcal{W}}^{\uparrow}_s \,\mathrm{d}\varphi^{\uparrow}_s \geq 0, \quad t \in [\alpha, \hat{\beta}^{\uparrow}_\alpha \wedge \hat{\beta}^{\downarrow}_\alpha]_{\mathbb{R}}, \tag{138}$$

while the inequality on the right in (138) follows from the inequalities in (137) on the left. Further, we put $\beta^{\Uparrow}_\alpha \triangleq \hat{\beta}^{\uparrow}_\alpha \wedge \hat{\beta}^{\downarrow}_\alpha \wedge \beta^{\uparrow}_\alpha$. Then we get from (137,138) that

$$\psi_t/\varphi_t \geq \psi_\alpha/\varphi_\alpha, \qquad t \in [\alpha, \beta^{\Uparrow}_\alpha]_{\mathbb{R}}. \tag{139}$$

Now, we are going to show that $\tau^{\uparrow}_\alpha$ introduced in (119) is never lower than $\beta^{\Uparrow}_\alpha$, i.e. that $A_t \triangleq [t = \tau^{\uparrow}_\alpha < \beta^{\Uparrow}_\alpha] = \emptyset$ holds for every $t \geq 0$. As $(S_t, F_t)_{t\geq 0}$ are continuous processes by assumption, we obtain from (119) that

$$\frac{\mathcal{W}_t - \psi_t}{\varphi_t} = \left\{\begin{matrix} S_t = S^{\uparrow}_\alpha & \text{if} & a = 1 \\ F_t = F^{\uparrow}_\alpha & \text{if} & a = 0 \end{matrix}\right\} = \frac{\mathcal{W}^{\uparrow}_\alpha - \psi_\alpha}{\varphi_\alpha} \quad \text{hold on } A_t. \tag{140}$$

Further, we get by the assumption (134) and the inequalities in (137) on the right that

$$A_t \subseteq [\varphi_\alpha, \varphi_t < 0] \subseteq [\mathcal{W}^{\uparrow}_\alpha \leq 0 < \mathcal{W}_t] \tag{141}$$

as $\mathcal{W}_t > 0$ holds by assumption. Then we get from (139,140,141) that

$$A_t \subseteq \left[\tfrac{\mathcal{W}_t}{\varphi_t} < 0 \leq \tfrac{\mathcal{W}^{\uparrow}_\alpha}{\varphi_\alpha}\right] \cap \left[\tfrac{\mathcal{W}_t}{\varphi_t} - \tfrac{\mathcal{W}^{\uparrow}_\alpha}{\varphi_\alpha} = \tfrac{\psi_t}{\varphi_t} - \tfrac{\psi_\alpha}{\varphi_\alpha} \geq 0\right] = \emptyset$$

holds whenever $t \geq 0$. Thus, we have indeed verified that $\beta^{\Uparrow}_\alpha \leq \tau^{\uparrow}_\alpha$ holds.

If we add a process from $(S_t^\uparrow, S_t^\downarrow, F_t^\uparrow, F_t^\downarrow)_{t\geq\alpha}$ to both sides of (139) and use the following equality $(\hat{\varphi}_t, \hat{\psi}_t) = (\varphi_\alpha, \psi_\alpha)$ holding on $[\alpha \leq t < \infty]$, we get that

$$\left.\begin{array}{l} \mathcal{W}_t^\uparrow/\varphi_t \geq \hat{\mathcal{W}}_t^\uparrow/\hat{\varphi}_t \\ \mathcal{W}_t^\downarrow/\varphi_t \geq \hat{\mathcal{W}}_t^\downarrow/\hat{\varphi}_t \end{array}\right\} \quad t \in [\alpha, \beta_\alpha^\Uparrow]_{\mathbb{R}}. \tag{142}$$

Next, we show that $\hat{\beta}_\alpha^\downarrow \geq \hat{\beta}_\alpha^\uparrow \wedge \beta_\alpha^\uparrow$. Namely, we show that $B_t \triangleq [t = \hat{\beta}_\alpha^\downarrow < \hat{\beta}_\alpha^\uparrow \wedge \beta_\alpha^\uparrow] = \emptyset$ holds whenever $t \in [0, \infty)$. As $(\mathcal{W}_t^\downarrow)_{t\geq 0}$ is a right-continuous process, we obtain by the definition of $\hat{\beta}_\alpha^\downarrow \triangleq \inf\{t \geq \alpha; \hat{\mathcal{W}}_t^\downarrow = 0\}$, by the second row in (142) and by the right-hand inequalities in (137) that

$$B_t \subseteq \left[0 = \tfrac{\hat{\mathcal{W}}_t^\downarrow}{\hat{\varphi}_t} \leq \tfrac{\mathcal{W}_t^\downarrow}{\varphi_t} < 0\right] = \emptyset.$$

Hence, $\hat{\beta}_\alpha^\downarrow \geq \hat{\beta}_\alpha^\uparrow \wedge \beta_\alpha^\uparrow = \beta_\alpha^\Uparrow$. Further, as $(\mathcal{W}_t^\uparrow)_{t\geq 0}$ is a right-continuous process, we similarly obtain that

$$[\alpha \leq t = \beta_\alpha^\uparrow < \hat{\beta}_\alpha^\uparrow] \subseteq \left[0 = \tfrac{\mathcal{W}_t^\uparrow}{\varphi_t} \geq \tfrac{\hat{\mathcal{W}}_t^\uparrow}{\hat{\varphi}_t} = \tfrac{\hat{\mathcal{W}}_t^\uparrow}{\varphi_\alpha} > 0\right] = \emptyset.$$

from the definition of $\beta_\alpha^\uparrow$ in (120), from the first row in (142) and by the equality for $\hat{\beta}_\alpha^\uparrow$ in (136) together with the inequalities on the second row of the right-hand side in (137). This shows that $\beta_\alpha^\uparrow \geq \hat{\beta}_\alpha^\uparrow$, and hence we have that $\hat{\beta}_\alpha^\uparrow = \beta_\alpha^\Uparrow \leq \tau_\alpha^\uparrow$.

By the strong Markov property of a standard Brownian motion stated in Lemma 5.4, there exists (on an enlargement of the original probability space) an arithmetic Brownian motion $(\mathbb{B}_s)_{s\geq 0}$ starting from $\mathbb{B}_0 = 0$ with the same distribution as $\ln(S_t/S_0)_{t\geq 0}$ if $a = 1$ and as $(F_t)_{t\geq 0}$ if $a = 0$ such that $1_{[\alpha<\infty]}\, P_{\mathbb{B}|\mathcal{F}_\alpha} \overset{\text{as}}{=} 1_{[\alpha<\infty]}\, P_{\mathbb{B}}$ and that

$$\alpha < \infty \quad \Rightarrow \quad \mathbb{B}_s = \begin{cases} \ln(S_{s+\alpha}/S_\alpha) & \text{if} \quad a = 1 \\ F_{s+\alpha} - F_\alpha & \text{if} \quad a = 0 \end{cases}$$

holds. Here, $P_{\mathbb{B}}$ stands for the distribution of $\mathbb{B}$ and $P_{\mathbb{B}|\mathcal{F}_\alpha}$ stands for the conditional distribution of $\mathbb{B}$ given $\mathcal{F}_\alpha$. See the definition of $\tau_\alpha^\uparrow$ in (119) and of $\hat{\beta}_\alpha^\uparrow$ in (120) with $\mathcal{W}$ replaced by $\hat{\mathcal{W}}$ in order to agree that

$$\tau_\alpha^\uparrow = \inf\{t \geq \alpha; \mathbb{B}_{t-\alpha} = u\} \quad \text{and} \quad \hat{\beta}_\alpha^\uparrow = \inf\{t \geq \alpha; \mathbb{B}_{t-\alpha} = V_\alpha^\uparrow\},$$

where $u \triangleq \Lambda_a(\lambda^\uparrow) > 0$ and where

$$V_t^\uparrow \triangleq \left\{ \begin{array}{ll} \ln\big(-\psi_t/\varphi_t\big) - \ln S_t - \ln(1+\lambda^\uparrow) & \text{if} \quad a = 1 \\ -\psi_t/\varphi_t - F_t - \lambda^\uparrow & \text{if} \quad a = 0 \end{array} \right\} = \Lambda_a\left(\frac{\mathcal{W}_t^\uparrow}{-\varphi_t (S_t^\uparrow)^a}\right)$$

is such that $[\alpha < \infty] \subseteq [V_\alpha^\uparrow < 0]$ by (134,137). By Lemma 5.3, we get that

$$\alpha < \infty \quad \Rightarrow \quad P(\tau_\alpha^\uparrow < \hat{\beta}_\alpha^\uparrow | \mathcal{F}_\alpha) > 0 \tag{143}$$

holds almost surely. By the previous part of the proof we have that $\hat{\beta}_\alpha^\uparrow \leq \tau_\alpha^\uparrow$ and therefore, we get that $0 = P(\tau_\alpha^\uparrow < \hat{\beta}_\alpha^\uparrow) = EP(\tau_\alpha^\uparrow < \hat{\beta}_\alpha^\uparrow | \mathcal{F}_\alpha)$. This together with (143) gives that $\alpha \overset{\text{as}}{=} \infty$.

Now, we put $\alpha \triangleq 0$. We will show that $\beta_0^{\uparrow} \overset{\text{as}}{=} \infty$. Whenever $n \in \mathbb{N} \cup \{\infty\}$, we consider

$$\alpha_n \triangleq \inf\{t \geq \beta_0^{\uparrow}; \mathcal{W}_t^{\uparrow} < -1/n\}.$$

As $(\mathcal{W}_t^{\uparrow})_{t\geq 0}$ is a right-continuous process, we have that $[\alpha_n < \infty] \subseteq [\mathcal{W}_{\alpha_n}^{\uparrow} \leq -1/n]$. Hence, (134) holds with α replaced by α_n, and we get by the first part of the proof that $\alpha_n \overset{\text{as}}{=} \infty$ holds whenever $n \in \mathbb{N}$, which gives that also $\alpha_\infty = \inf_n \alpha_n \overset{\text{as}}{=} \infty$. Then we get by Lemma 5.2 that also $\beta_0^{\uparrow} \overset{\text{as}}{=} \alpha_\infty \overset{\text{as}}{=} \infty$. Hence, $\beta_0^{\uparrow} \overset{\text{as}}{=} \infty$ and, similarly, we would show that also $\beta_0^{\downarrow} \overset{\text{as}}{=} \infty$. Note that showing that $\beta_0^{\downarrow} \overset{\text{as}}{=} \infty$ needs to modify also the first part of the proof but the modification is straightforward. Then we get that $(\mathcal{W}_t^{\uparrow} \wedge \mathcal{W}_t^{\downarrow})_{t\geq 0}$ attains positive values up to a null set. □

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