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SEVERAL RESULTS ON SET-VALUED POSSIBILISTIC DISTRIBUTIONS

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When proposing and processing uncertainty decision-making algorithms of various kinds and purposes, we more and more often meet probability distributions ascribing non-numerical uncertainty degrees to random events. The reason is that we have to process systems of uncertainties for which the classical conditions like σ-additivity or linear ordering of values are too restrictive to define sufficiently closely the nature of uncertainty we would like to specify and process. In cases of non-numerical uncertainty degrees, at least the following two criteria may be considered. The first criterion should be systems with rather complicated, but sophisticated and nontrivially formally analyzable uncertainty degrees, e. g., uncertainties supported by some algebras or partially ordered structures. Contrarily, we may consider easier relations, which are non-numerical but interpretable on the intuitive level. Well-known examples of such structures are set-valued possibilistic measures. Some specific interesting results in this direction are introduced and analyzed in this contribution.

Keywords: probability measures, possibility measures, non-numerical uncertainty degrees, set-valued uncertainty degrees, possibilistic uncertainty functions, set-valued entropy functions

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1. INTRODUCTION

Since the very origins of modern mathematics, measure theory may be viewed as almost a synonym for mathematical theory of size quantification with the most general and abstract structures in which sizes take their values. On the other hand, the structures over quantity degrees should be rich and flexible enough to enable us to define and process non-trivial deductions with non-trivial results on quantity degrees and their relationships.

In measure theory and, consequently, in probability theory, the sizes of sets and uncertainty (in the sense of randomness as well as of fuzziness and possibility degrees) are quantified by numbers, going from finite natural to rational and then real (or, perhaps, complex-valued) numbers. The development of real-valued probability theory has reached its peak in Kolmogorov axiomatic theory of probability as systematically explained and applied in, e. g., [4, 6].

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From a different angle, the correctness and legality of applying classical probability theory and its consequences (mathematical statistics, Shannon entropy, and information theory...) on problems from real life are based on the assumption that certain non-trivial conditions are satisfied and verified (such as the precise knowledge of a priori probabilities, statistical independence of certain random variables and/or precisely known type and degrees of their dependencies together with the detailed conditional probabilities...). Even though an enormous amount of work has already been completed within the framework of classical probability theory and statistics, and reasonable processing has taken place of input probabilistic data non completely given or known, there are demands for qualitatively different alternative tools for uncertainty (in the sense of randomness as well as fuzziness) processing.

Qualitatively different models of uncertainty quantification and processing, even if still with numerical degrees, are real-valued fuzzy sets, defined by mappings taking the basic space Ω into the unit interval $[0,1]$, hence extending the binary-valued characteristic functions of a standard set, to functions with values in the closed interval $[0,1]$.

Zadeh's pioneering idea of fuzzy sets emerged in 1965 in [5, 12] and, as soon after as in 1967 J. A. Goguen entered the scene with another step of fuzzy sets with non-numerical membership degrees. In particular, J. A. Goguen considered uncertainty in the sense of fuzziness degrees, i. e., as elements of a complete lattice. Let us recall that complete lattice is defined as a p. o. set (partially ordered set) in which supremum and infimum are defined for each nonempty subset.

Up to this point, we have recalled models for uncertainty quantification and processing where the uncertainty degrees take their values in more and more general and less intuitive structures (natural numbers, rationals, real numbers, lattices, semilattices...), so that set-valued possibility degrees occurring in the title of this text seem to be a rather strong step backward, which deserves a rather persuasive explanation. When quantifying sizes by numbers we have to keep in mind that this approach introduces into the model the complete ordering of numbers, which need not correspond to the sizes of pieces of uncertainty in question. Among the structures working with uncertainties, and also keeping in mind the idea to classify incomparable set-quantified degrees of uncertainty with the same values of real-valued measures, set-valued possibilistic measures seem to be sufficiently elastic and resilient to be taken as an intuitively acceptable non-numerical size-quantifying mathematical model.

Let us survey, very briefly, the contents of particular sections of this paper. Our goal will be to minimize the quantity and complexity of preliminaries necessary for a less than fully-oriented reader to understand the text. In Section 2 we introduce the structures for quantifying uncertainty (or uncertainties) by set values. It is perhaps worth mentioning now that probability measure and probability theory are based on the standard combination of set-valued uncertainty quantification (random events are sets) with the standard real-valued quantification of set-valued random events.

In Section 3 we introduce three alternative ways to define mappings, keeping at least some properties of conditional probabilities. This problem seems to be promising for some new and interesting results. In Section 4 we define and analyze set-valued entropy function over set-valued possibilistic function with the goal of solving the problem arising when the possibilistic distribution takes the maximum value $\mathbf{1}_{\mathcal{T}}(=X)$ for at least two different arguments. Analogously to the case of real-valued probability measure the Shannon entropy function [10] takes the maximum value $\mathbf{1}_{\mathcal{T}}(=X)$; hence the qualities of this entropy function cannot be used as a tool for partially ordering different alternatives of possibilistic distribution when choosing the best one for the application in question. Very roughly speaking, the idea is to modify the space of values in which set-valued entropy function takes its values, in such a way that the supremum value of the set-valued entropy function is taken for just one value ω_0 from the basic space Ω of the possibilistic distribution in question. This goal will be achieved by introducing a sophisticated equivalence relation on the basic possibilistic space of our model, for which the supremum is reached in only one value in the resulting factor space. In other words, the unit entropy value for the entropy function does not menace our application of set-valued entropy functions.

Finally, in Section 5 we consider the compositions of set-valued possibilistic distributions.

2. SET-VALUED POSSIBILISTIC DISTRIBUTIONS

Let Ω and X be nonempty sets, let $\mathcal{P}(X)$ be the set of all subsets of X (the power-set over X), let $\pi : \Omega \to \mathcal{P}(X)$ be a mapping ascribing to each $\omega \in \Omega$ a subset $\pi(\omega) \subset X$ (i. e., $\pi(\omega) \in \mathcal{P}(X)$). The mapping π is called a *set-valued* (or precisely $\mathcal{P}(X)$-*valued*) *possibilistic distribution* on Ω, if $\bigcup_{\omega\in\Omega} \pi(\omega) = X$.

For each $A \subset \Omega$, set $\Pi(A) = \bigcup_{\omega\in A} \pi(\omega)$. The mapping $\Pi : \mathcal{P}(\Omega) \to \mathcal{P}(X)$ is called the $\mathcal{P}(X)$-*valued possibilistic measure* induced on $\mathcal{P}(\Omega)$ by the set-valued possibilistic distribution π on Ω. The important characteristic of the $\mathcal{P}(X)$-valued possibilistic distribution π (and of the related $\mathcal{P}(X)$-valued possibilistic measure Π induced by π) is the so-called *possibilistic* (or *Sugeno*) *entropy* defined by the *Sugeno integral* $I(\pi)$. For the particular case of the set-valued possibilistic distribution π on Ω defined above, the definition reads as follows:

$$I(\pi) \;=\; \bigcup_{\omega\in\Omega} [\Pi(\Omega\setminus\{\omega\}) \cap \pi(\omega)] \;\subset\; X. \tag{2.1}$$

E. g., in the simplest case when $\Omega = X$ and $\pi(\omega) = \{\omega\}$, we obtain that $\Pi(A) = \bigcup_{\omega\in A} \pi(\omega) = \bigcup_{\omega\in A}\{\omega\} = A$. For the entropy $I(\pi)$ we obtain that

$$I(\pi) \;=\; \bigcup_{\omega\in\Omega} [\Pi(\Omega\setminus\{\omega\}) \cap \pi(\omega)] \;=\; \bigcup_{\omega\in\Omega} ((\Omega\setminus\{\omega\}) \cap \{\omega\}) \;=\; \emptyset; \tag{2.2}$$

let us recall that the empty subset of X denotes the zero element of the complete lattice (as a matter of fact, complete Boolean algebra) $\langle \mathcal{P}(X), \subseteq \rangle$.

Fact 2.1. Let Ω and X be nonempty sets, and $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω such that, for each ω_1, ω_2, $\omega_1 \neq \omega_2$, $\pi(\omega_1) \cap \pi(\omega_2) = \emptyset$ holds. Then we obtain that $\Pi(A) \cap \Pi(B) = \emptyset$ holds for each $A, B \subset X, A \cap B = \emptyset$.

P r o o f. An easy calculation yields that

$$\Pi(A) \cap \Pi(B) = \left(\bigcup_{\omega \in A} \pi(\omega) \right) \cap \left(\bigcup_{\omega \in B} \pi(\omega) \right)$$

$$= \bigcup_{\omega_1 \in B} \left[\left(\bigcup_{\omega \in A} \pi(\omega) \right) \cap \pi(\omega_1) \right] = \bigcup_{\omega_1 \in B} \bigcup_{\omega \in A} (\pi(\omega_1) \cap \pi(\omega)) = \emptyset, \tag{2.3}$$

as the sets A and B are disjoint. The assertion is proven. □

Lemma 2.2. Let Ω and X be nonempty sets, and $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω. Then $I(\pi) = \emptyset$ iff $\pi(\omega_1) \cap \pi(\omega_2) = \emptyset$ for each $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$.

P r o o f. If $\pi(\omega_1) \cap \pi(\omega_2) = \emptyset$ for each $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$, then

$$I(\pi) \ = \bigcup_{\omega \in \Omega} [\pi(\Omega \setminus \{\omega\}) \cap \pi(\omega)] \ = \bigcup_{\omega \in \Omega} [\pi(\Omega \setminus \{\omega\}) \cap \Pi(\{\omega\})] \ = \ \emptyset \tag{2.4}$$

holds, due to Fact 2.1.

On the other hand, let $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$, be such that $\pi(\omega_1) \cap \pi(\omega_2) \neq \emptyset$. Then $\omega_2 \in \Omega \setminus \{\omega_1\}$ holds, so that

$$\Pi(\Omega \setminus \{\omega_1\}) \cap \pi(\omega_1) \ \supset \ \pi(\omega_2) \cap \pi(\omega_1) \ \neq \ \emptyset; \tag{2.5}$$

consequently,

$$I(\pi) \ \supset \ \Pi(\Omega \setminus \{\omega_1\}) \cap \pi(\omega_1) \ \supset \ \pi(\omega_2) \cap \pi(\omega_1) \ \neq \ \emptyset, \tag{2.6}$$

follows. The assertion is proven. □

Theorem 2.3. Let Ω and X be nonempty sets, and π_1, π_2 be $\mathcal{P}(X)$-valued possibilistic distributions such that, for each $\omega \in \Omega$, $\pi_1(\omega) \subset \pi_2(\omega)$ holds. Then $I(\pi_1) \subset I(\pi_2)$ holds.

P r o o f. By definition,

$$I(\pi_1) \ = \bigcup_{\omega \in \Omega} [\pi_1(\Omega \setminus \{\omega\}) \cap \pi_2(\omega)]. \tag{2.7}$$

For each $\omega \in \Omega$, the inclusion

$$\Pi_1(\Omega \setminus \{\omega\}) \quad = \bigcup_{\omega^* \in \Omega \setminus \{\omega\}} \pi_1(\omega^*) \quad \subset \bigcup_{\omega^* \in \Omega \setminus \{\omega\}} \pi_2(\omega^*) \ = \ \Pi_2(\Omega \setminus \{\omega\}) \tag{2.8}$$

is valid, as $\pi_1(\omega^*) \subseteq \pi_2(\omega^*)$ holds for each $\omega^* \in \Omega$. Consequently, the inclusion

$$\Pi_1(\Omega \setminus \{\omega\}) \cap \pi_1(\omega) \ \subset \ \Pi_2(\Omega \setminus \{\omega\}) \cap \pi_2(\omega) \tag{2.9}$$

holds for each $\omega \in \Omega$, so that the inclusion $I(\pi_1) \subset I(\pi_2)$ immediately follows. The assertion is proven. □

The following fact is almost trivial, but it seems worth being explicitly recalled. In the space of set-valued possibilistic distributions it may easily happen that $\pi_1(\omega) \subset \pi_2(\omega)$ holds for each $\omega \in \Omega$, at least for some $\omega \in \Omega$ this inclusion is strict (i. e., $\pi_1(\omega) \neq \pi_2(\omega)$, but the identity $\bigcup_{\omega\in\Omega} \pi_1(\omega) = \bigcup_{\omega\in\Omega} \pi_2(\omega) = X$ is valid.

This property qualitatively differentiates a possibilistic case from a finite probability distribution, where the inequality $p_1(\omega_i) \leq p_2(\omega_i)$ for each $i = 1, 2, \dots,$ together with $\sum_{i=1}^n p_1(\omega_i) = \sum_{i=1}^n p_2(\omega_i) = 1$ implies that the probability distributions p_1 and p_2 are indentical on $\{\omega_1, \omega_2, \dots, \omega_n\}$.

Lemma 2.4. Let Ω, X be nonempty sets, let $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution. If there are $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$, such that $\pi(\omega_1) = \pi(\omega_2) = X$, then $I(\pi) = X = \mathbf{1}_{\mathcal{P}(X)}$.

Proof. Let $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$, be such that $\pi(\omega_1) = \pi(\omega_2) = X$, consider the set $\Pi(\Omega \setminus \{\omega_1\}) \cap \pi(\omega_1)$. Then $\omega_2 \in \Omega \setminus \{\omega_1\}$ holds, hence,

$$\Pi(\Omega \setminus \{\omega_1\}) \;=\; \bigvee^{\omega^* \in \Omega\setminus\{\omega_1\}} \pi(\omega^*) \supset \pi(\omega_2) \;=\; X \tag{2.10}$$

holds and $\Pi(\Omega \setminus \{\omega_1\}) = X$ follows. Replacing mutually ω_1 and ω_2 we obtain that $\Pi(\Omega \setminus \{\omega_2\}) = X$ holds as well, hence,

$$X \;=\; \Pi(\Omega \setminus \{\omega_j\}) \cap \pi(\omega_j) \;=\; \bigcup_{\omega\in\Omega} [\Pi(\Omega \setminus \{\omega\}) \cap \pi(\omega)] \;=\; I(\pi) \tag{2.11}$$

holds for any j and the assertion is proven. □

Denote by Q the set of all $\mathcal{P}(X)$-valued possibilistic distributions on Ω. If π_1, π_2 are $\mathcal{P}(X)$-possibilistic distributions on Ω such that $\pi_1(\omega) \subset \pi_2(\omega)$ holds for each $\omega \in \Omega$, we write $\pi_1 \leq \pi_2$ and say that π_1 is *majorized* by π_2 or that π_2 is an upper bound for π_1. As proved in Theorem 2.3 if $\pi_1 \leq \pi_2$ holds, then $I(\pi_1) \subseteq I(\pi_2)$ holds as well.

The inverse implication does not hold in general, i. e., if $I(\pi_1) \subseteq I(\pi_2)$ is valid, then $\pi_1 \leq \pi_2$ need not hold. Indeed, let $\Omega = \{\omega_1, \omega_2\}$, let $\pi_1(\omega_1) = \pi_2(\omega_2) = X$, $\pi_1(\omega_2) = \pi_2(\omega_1) = \emptyset$, so that neither $\pi_1 \leq \pi_2$ nor $\pi_2 \leq \pi_1$ holds. For entropy $I(\pi_1)$ we obtain that

$$\begin{aligned} I(\pi_1) &= \bigcup_{\omega\in\Omega} [\Pi_1(\Omega \setminus \{\omega\}) \cap \pi_2(\omega)] \\ &= [\Pi_1(\Omega \setminus \{\omega_1\}) \cap \pi_1(\omega_1)] \;\cup\; [\Pi_1(\Omega \setminus \{\omega_2\}) \cap \pi_1(\omega_2)] \\ &= (\pi_1(\omega_2) \cap \pi_1(\omega_1)) \;\cup\; (\pi_1(\omega_1) \cap \pi_1(\omega_2)) \\ &= (\emptyset \cap X) \cup (X \cap \emptyset) \;=\; \emptyset. \end{aligned} \tag{2.12}$$

For $I(\pi_2)$ the calculations and the results are the same, so that $I(\pi_1) = I(\pi_2)$, but neither $\pi_1 \leq \pi_2$ nor $\pi_2 \leq \pi_1$ holds.

Lemma 2.5. Let π be a $\mathcal{P}(X)$-valued distribution on Ω. Then for each $\mathcal{S} \subset \mathcal{P}(\Omega)$ the relation

$$\begin{aligned} \Pi\left(\bigcup \mathcal{S}\right) &= \Pi\left(\bigcup \{A : A \in \mathcal{S}\}\right) = \bigvee^{\mathcal{T}} \left\{\pi(\omega) : \omega \in \bigcup \mathcal{S}\right\} \\ &= \bigcup \{\{\pi(\omega) : \omega \in A\} : A \in \mathcal{S}\} \\ &= \bigvee^{\mathcal{T}} \{\Pi(A) : A \in \mathcal{S}\} = \bigcup \{\Pi(A) : A \in \mathcal{S}\} \end{aligned} \tag{2.13}$$

holds.

Proof. Obvious. □

Let us denote by $Q(\Omega, X)$ the space of all $\mathcal{P}(A)$-valued possibilistic distributions over the space Ω, in symbols,

$$Q(\Omega, X) = \{\ \pi\ :\ \pi : \Omega \to \mathcal{P}(X),\ \bigcup \{\pi(\omega) : \omega \in \Omega\} = \mathbf{1}_{\mathcal{T}} = X\ \}. \tag{2.14}$$

Let $\leq^*$ be the binary relation, on $Q(\Omega, X)$, i.e., the subset of the Cartesian product $Q(\Omega, X) \times Q(\Omega, X)$ defined in this way: for each $\pi_1, \pi_2 \in Q(\Omega, X)$, $\pi_1 <^* \pi_2$ holds iff $\pi_1(\omega) \subseteq \pi_2(\omega)$ holds for each $\omega \in \Omega$. It is possible that $\pi_1 <^* \pi_2$ holds for two $\mathcal{P}(X)$-distribution π_1, π_2 such that $\pi_1(\omega) \subset \pi_2(\omega)$ is the case for some $\omega \in \Omega$ and, of course, $\pi_1(\omega^*) \subseteq \pi_2(\omega^*)$ holds for two $\mathcal{P}(X)$-distributions π_1, π_2 such that $\pi_1(\omega) \subset \pi_2(\omega)$ is the case for some $\omega \in \Omega$ and, of course, $\pi_1(\omega^*) \subseteq \pi_2(\omega^*)$ holds for each $\omega^* \in \Omega$.

Lemma 2.6. The ordered pair $\mathcal{D} = \langle Q(\Omega, X), \leq^* \rangle$ is a p. o. set which defines a complete upper semilattice, so that for each nonempty subset $E \subset D$ the supremum $\pi(E) = \bigvee^{\mathcal{D}} \{\pi : \pi \in E\}$ is defined. Given explicitly, π^E is the mapping which takes Ω into $\mathcal{P}(X)$ in such a way that for each $\omega \in \Omega$

$$\pi^E(\omega)\ =\ \bigcup \{\pi \in E : \pi(\omega)\}. \tag{2.15}$$

This mapping obviously defines a π-valued possibilistic distribution on Ω.

Proof. Obvious. □

However, the situation with the infimum of a set E of $\mathcal{P}(X)$-distributions is not dual to $\bigvee^{\mathcal{D}} E$. We may define the mapping $M(E) : \Omega \to \mathcal{P}(X)$ in such a way that, for each $\omega \in \Omega$, $M(E)(\omega) = \bigcap \{\pi(\omega) : \pi \in E\}$, but this mapping does not meet the condition $\bigvee^{\mathcal{D}} \{M(E)(\omega) : \omega \in \Omega\} = \mathbf{1}_{\mathcal{P}(X)} = X$. Indeed, let $E = \{\pi_1, \pi_2\}$ be such that $\pi_1(\omega) = \mathbf{1}_{\mathcal{P}(X)}$ for $\omega \in \Omega_0$, $\emptyset \neq \Omega_0 \neq \Omega$, and $\pi_2(\omega) = \mathbf{1}_{\mathcal{P}(X)}$ for $\omega \in \Omega \setminus \Omega_0$, $\pi_2(\omega) = \emptyset_{\mathcal{P}(X)}$ for $\omega \in \Omega$. Obviously, $M(E)(\omega) = \emptyset_{\mathcal{P}(X)}$ for each $\omega \in \Omega$, so that $M(E)$ is not a $\mathcal{P}(X)$-distribution. Neither the operation of $\mathcal{P}(X)$-valued complements, defined by $\pi^C(\omega) = X \setminus \pi(\omega)$ yields the results meeting the conditions imposed on $\mathcal{P}(X)$-distributions.

Lemma 2.7. Let $E \subset Q$ be a nonempty set of $\mathcal{P}(X)$-distributions, for each $\pi \in Q$ let $\Pi_\pi : \mathcal{P}(\Omega) \to \mathcal{P}(X)$ denote the corresponding induced $\mathcal{P}(X)$-possibilistic measure on $\mathcal{P}(\Omega)$. Then, for each $A \subset \Omega$, the relation $\Pi_{\pi^E}(A) = \bigvee^{\mathcal{T}}\{\Pi_\pi(A) : \pi \in E\}$ holds.

Proof. For each $A \subset \Omega$ we obtain that

$$\begin{aligned} \bigvee^{\mathcal{T}}\{\Pi_\pi(A) : \pi \in E\} &= \bigvee^{\mathcal{T}}\{\,\{\bigvee^{\mathcal{T}}\pi(\omega) : \omega \in A\} \;:\; \pi \in E\,\} \\ = \bigvee^{\mathcal{T}}\{\pi(\omega) : \omega \in \Omega, \pi \in E\} &= \bigvee^{\mathcal{T}}\{\,\{\bigvee^{\mathcal{T}}\pi(\omega) : \pi \in E\} \;:\; \omega \in A\,\} \\ = \bigvee^{\mathcal{T}}\{\pi^E(\omega) : \omega \in A\} &= \Pi_{\pi^E}(A). \end{aligned} \tag{2.16}$$

The assertion is proven. □

According to the way in which $\mathcal{P}(X)$-valued possibilistic measure Π on $\mathcal{P}(\Omega)$ induced by a $\mathcal{P}(X)$-valued possibilistic distribution π on Ω is defined, the set function Π is extensional with respect to the supremum operation $\bigvee^{\mathcal{T}}$ on $\mathcal{T} = \langle \mathcal{P}(X), \subseteq \rangle$ in the sense that for each nonempty system $\mathcal{A}$ of subsets of Ω, the identity

$$\Pi\left(\bigcup \mathcal{A}\right) = \bigvee^{\mathcal{T}}\{\Pi(A) : A \in \mathcal{A}\} \tag{2.17}$$

holds. In particular, for $\mathcal{A} = \{A_1, A_2\}$, $\Pi(A_1) \cup \Pi(A_2) = \Pi(A_1 \cup A_2)$. For the operation of infimum the relation dual to (2.17) is not the case, in general, only the inclusion $\Pi(A \cap B) \subseteq \Pi(A) \cap \Pi(B)$ is valid, as $\Pi(A \cap B) \subset \Pi(A)$ and $\Pi(A \cap B) \subset \Pi(B)$ holds trivially. The difference between the values $\Pi(A_1 \cap A_2)$ and $\Pi(A_1) \cap \Pi(A_2)$ may range over all the Boolean interval $\langle \emptyset, X \rangle$ of $\mathcal{T} = \langle \mathcal{P}(X), \subseteq \rangle$. Indeed, let $X = \{0, 1\}$, let $\pi(\omega_1) = \pi(\omega_2) = 1$, let $A_1 = \{\omega_1\}$, $A_2 = \{\omega_2\}$. Then $\Pi(A_1) = \Pi(A_2) = 1$, so that $\Pi(A_1) \wedge \Pi(A_2) = 1$, but $\Pi(A_1 \cap A_2) = \Pi(\emptyset) = 0$.

Let us consider the simplest $\mathcal{P}(X)$-valued possibilistic distribution π for which the induced $\mathcal{P}(X)$-measure Π on $\mathcal{P}(\Omega)$ is also extensional w.r.t. the operation of infimum $\bigwedge$: the identity mapping on $\mathcal{P}(\Omega)$. Take $\Omega = X$, and $\pi(\omega) = \{\omega\}$ for every $\omega \in \Omega$, so that, for each $A \subset \Omega$, $\Pi(A) = \bigcap_{A \in \mathcal{A}} \Pi(A)$ follows, in particular, $\Pi(A \cap B) = \Pi(A) \cap \Pi(B)$ holds.

Definition 2.8. $\mathcal{P}(X)$-valued possibilistic distribution π taking a nonempty set Ω into the power-set $\mathcal{P}(X)$ over a nonempty set X is called *completely extensional* if, for each nonempty system $\mathcal{A}$ of subsets of Ω, the relation

$$\Pi\left(\bigcap \mathcal{A}\right) = \Pi\left(\bigcap_{A \in \mathcal{A}} A\right) = \bigcap_{A \in \mathcal{A}} \Pi(A) \tag{2.18}$$

holds. The $\mathcal{P}(X)$-distribution π is called *extensional* if

$$\Pi(A \cap B) = \Pi(A) \cap \Pi(B) \tag{2.19}$$

holds for each pair $A, B \subset \Omega$.

Lemma 2.9. Let π be a $\mathcal{P}(X)$-valued possibilistic distribution defined on a nonempty space Ω, taking its values in the power-set $\mathcal{P}(X)$ over a nonempty space X and such that $\pi(\omega_1) \cap \pi(\omega_2) = \emptyset$ holds for each $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$. Then the induced $\mathcal{P}(X)$-possibilistic measure on $\mathcal{P}(\Omega)$ is extensional in the sense that $\Pi(A) \cap \Pi(B) = \Pi(A \cap B)$ is valid for each $A, B \subset \Omega$.

Proof. First of all, let us consider the case when the sets A, B are disjoint. Then

$$\begin{aligned} \Pi(A) \cap \Pi(B) &= \left(\bigvee_{\omega_1 \in A} \pi(\omega_1) \right) \cap \left(\bigvee_{\omega_2 \in B} \pi(\omega_2) \right) \\ &= \bigcup_{\langle \omega_1, \omega_2 \rangle,\, \omega_1 \in A,\, \omega_2 \in B} (\pi(\omega_1) \cap \pi(\omega_2)) \;=\; \emptyset \\ &= \Pi(\emptyset) \;=\; A \cap B \;=\; \Pi(A \cap B), \end{aligned} \tag{2.20}$$

as for each $\omega_1 \in A$, $\omega_2 \in B$, $\omega_1 \neq \omega_2$ and $\pi(\omega_1) \cap \pi(\omega_2) = \emptyset$ holds.

For each $A, B \subset \Omega$, $A = (A \setminus B) \cup (A \cap B)$, $B = (B \setminus A) \cup (A \cap B)$ holds, so that

$$\begin{aligned} &\Pi(A) \cap \Pi(B) \;=\; [\Pi((A \setminus B) \cup (A \cap B))] \;\cap\; [\Pi((B \setminus A) \cup (A \cap B))] \\ =\;& [\Pi(A \setminus B) \cup \Pi(A \cap B)] \;\cap\; [\Pi(B \setminus A) \cup \Pi(A \cap B)] \\ =\;& [\Pi(A \setminus B) \cap \Pi(B \setminus A)] \;\cup\; [\Pi(A \cap B) \cap \Pi(B \setminus A)] \\ &\cup [\Pi(A \cap B) \cap \Pi(A \setminus B)] \;\cup\; \Pi(A \cap B) \;=\; \Pi(A \cap B), \end{aligned} \tag{2.21}$$

as

$$(A \setminus B) \cap (B \setminus A) \;=\; (A \cap B) \cap (B \setminus A) \;=\; (A \cap B) \cap (A \setminus B) \;=\; \emptyset, \tag{2.22}$$

so that, due to (2.20)

$$\Pi(A \setminus B) \cap \Pi(B \setminus A) \;=\; (A \cap B) \cap (B \setminus A) \;=\; (A \cap B) \cap (A \setminus B) \;=\; \emptyset. \tag{2.23}$$

The assertion is proven. □

3. CONDITIONED SET-VALUED POSSIBILISTIC DISTRIBUTIONS AND MEASURES

Conditioned (or conditional) probability distributions are very important tools in probability theory. Roughly speaking, conditioned probabilities enable us, on the grounds of newly obtained evidence, to transform the probability values in such a way that the random events incompatible with the new pieces of evidence are eliminated from evidence – they obtain the zero-valued conditioned probability. Within the framework of the standard Kolmogorov axiomatic probability theory mathematical formalization of this transformation is very simple and well-known. Let $\langle \Omega, \mathcal{A}, P \rangle$ be a probability space. Hence, Ω is a nonempty space (no relationship to the support set of the possibilistic distribution introduced above is being assumed at this moment), $\mathcal{A}$ is a nonempty σ-field of subsets of Ω, and $P : \mathcal{A} \to [0, 1]$ is a σ-additive real-valued set function. Subsets of Ω belonging to $\mathcal{A}$ are called *random events*; hence, to each $A \in \mathcal{A}$ the real number $P(A) \in [0,1]$ is ascribed and called the *probability of (the random event) A*. Given another random event $B \in \mathcal{A}$ such that $P(B) > 0$ holds, the conditioned probability of (the random event) A under the condition that (the random event) B holds is denoted by $P(A|B)$ and defined by the well-known formula

$$P(A|B) = P(A \cap B) / P(B). \tag{3.24}$$

This definition cannot be immediately translated into the model and language of $\mathcal{T}$-valued possibilistic distributions because of the fact that operation of division between the values $P(A \cap B)$ and $P(B)$ cannot be defined in $\mathcal{T}$. Let us proceed in this way: we introduce three alternative approaches and we will examine the role of each of them when considering conditioned possibilistic measure.

So, let $\mathcal{T} = \langle \mathcal{P}(X), \subseteq \rangle$, Ω, $\pi : \Omega \to \mathcal{P}(X)$ be such that $\bigcup_{\omega \in \Omega} \pi(\omega) = X = \mathbf{1}_{\mathcal{T}}$ and $\Pi : \mathcal{P}(\Omega) \to \mathcal{P}(X)$ defined by $\Pi(A) = \bigcup_{\omega \in A} \pi(\omega)$ for each $A \subset X$ be as above. Given $B \subset \Omega$, let us define three mappings $\pi^i(\omega|B) : \Omega \to \mathcal{P}(X)$ in the following way:

$$\begin{aligned}
(i) \quad & \pi^1(\omega|B) \;=\; \pi(\omega) \cap \Pi(B), && (3.25\text{a})\\
(ii) \quad & \pi^2(\omega|B) \;=\; \pi(\omega), \qquad \text{if } \omega \in B, &&\\
& \pi^2(\omega|B) \;=\; \emptyset \; (= \emptyset_{\mathcal{T}}), \quad \text{if } \omega \in \Omega \setminus B, && (3.25\text{b})\\
(iii) \quad & \pi^3(\omega|B) \;=\; \Pi(\Omega \setminus B) \cup \pi(\omega) \;=\; \Pi((\Omega \setminus B) \cup \{\omega\}). && (3.25\text{c})
\end{aligned}$$

Let us investigate simple properties of these three mappings. Define, for each $i = 1, 2, 3$ and each $B \subset \Omega$, the mapping $\Pi^i(\,\cdot\,|B) : \mathcal{P}(\Omega) \to \mathcal{P}(X)$ for each $A \subset \Omega$,

$$\Pi^i(A|B) \;=\; \bigvee_{\omega \in A}^{\mathcal{T}} \pi^i(\omega|B) \;=\; \bigcup_{\omega \in A} \pi^i(\omega|B). \tag{3.26}$$

Hence, for each $i = 1, 2, 3$, we explicitly obtain that

$$\begin{aligned}
\Pi^1(A|B) \;&=\; \bigcup_{\omega \in A} \pi^1(\omega|B) \;=\; \bigcup_{\omega \in A} (\pi(\omega) \cap \Pi(B)) \;=\; \left(\bigcup_{\omega \in A} \pi(\omega) \right) \cap \Pi(B)\\
&=\; \Pi(A) \cap \Pi(B),
\end{aligned} \tag{3.27}$$

$$\Pi^2(A|B) \;=\; \bigcup_{\omega \in A} \pi^2(\omega|B) \;=\; \bigcup_{\omega \in A \cap B} \pi(\omega) \;=\; \Pi(A \cap B), \tag{3.28}$$

$$\begin{aligned}
\Pi^3(A|B) \;&=\; \bigcup_{\omega \in A} \pi^3(\omega|B) \;=\; \bigcup_{\omega \in A} (\Pi(\Omega \setminus B) \cup \pi(\omega))\\
&=\; \Pi(\Omega \setminus B) \cup \bigcup_{\omega \in A} \pi(\omega) \;=\; \Pi(\Omega \setminus B) \cup \Pi(A)\\
&=\; \Pi((\Omega \setminus B) \cup A).
\end{aligned} \tag{3.29}$$

For the extremum values $A = \Omega$ or $B = \Omega$ we obtain that

$$\begin{aligned}
\Pi^1(\Omega|B) &= \Pi(\Omega) \cap \Pi(B) = \Pi(B),\\
\Pi^2(\Omega|B) &= \Pi(\Omega \cap B) = \Pi(B),\\
\Pi^3(\Omega|B) &= \Pi((\Omega \setminus B) \cup \Omega) = \Pi(\Omega) = \mathbf{1}_{\mathcal{T}},\\
\Pi^1(A|\Omega) &= \Pi(A) \cap \Pi(\Omega) = \Pi(A),\\
\Pi^2(A|\Omega) &= \Pi(A \cap \Omega) = \Pi(A),\\
\Pi^3(A|\Omega) &= \Pi((\Omega \setminus \Omega) \cup A) = \Pi(A). \qquad (3.30)
\end{aligned}$$

So, $\pi^1(\,\cdot\,|B)$ and $\pi^2(\,\cdot\,|B)$ define $\mathcal{T}$-possibilistic distributions on B (supposing that $B \neq \emptyset$), and $\pi^3(\,\cdot\,|B)$ defines a $\mathcal{T}$-possibilistic distribution on Ω. Moreover, if $B = \Omega$, then $\Pi^i(\,\cdot\,|B)$ is identical with the a priori possibilistic distribution π on Ω for each $i = 1, 2, 3$. Let us recall that, in standard probability theory, if $B \subset \Omega$ is such that $P(B) = 1$, then for each $A \subset \Omega$ the identity $P(A|B) = P(A \cap B)/P(B) = P(A)$ holds. The intuition behind this fact is quite simple – the occurrence of a certain random event (i. e., for which the probability equals one) does not bring any new information, so that no modification of the a priori probability measure occurs. All of the three set functions $\Pi^i(\,\cdot\,|B)$, $i = 1, 2, 3$, also possess this important property.

More generally, the result $\Pi^i(A|B) = \Pi(B)$ (for $i = 1, 2$) or $\Pi^3(A|B) = \mathbf{1}_{\mathcal{T}}$ holds not only for $A = \Omega$, but also for each $A \supseteq B$, as may be easily checked by inspecting the formulas (3.27), (3.28), and (3.29).

When approaching a more detailed analysis of the three $\mathcal{P}(X)$-valued mappings $\pi^i(\omega|B), i = 1, 2, 3$, let us begin with the mapping $\pi^3(\omega|B)$ defined by (3.25c) and related conditional possibility $\Pi^3(A|B)$ given by (3.29).
The reason for our giving preference to $\pi^3(\,\cdot\,|B)$ is the fact that $\pi^3(\omega|B)$ is, for each B, the only one of the three mappings in question which meets the condition of normalization, i. e., for which

$$\begin{aligned}
\bigcup_{\omega\in\Omega} \pi^3(\omega|B) &= \bigcup_{\omega\in\Omega} (\Pi(\Omega \setminus B) \cup \pi(\omega))\\
= \Pi(\Omega \setminus B) \cup \bigcup_{\omega\in\Omega} \pi(\omega) &= \Pi(\Omega \setminus B) \cup X = X = \mathbf{1}_{\mathcal{T}}. \qquad (3.31)
\end{aligned}$$

So, the $\mathcal{P}(X)$-valued entropy $I(\pi^3(\,\cdot\,|B))$ is defined and, by writing $\hat{\pi}(\omega)$ instead of $\pi^3(\omega|B)$ in order to simplify the notation, we can write

$$I(\pi^3(\,\cdot\,|B)) = I(\hat{\pi}) = \bigcup_{\omega\in\Omega} (\Pi^3(\Omega \setminus \{\omega\}) \cap \hat{\pi}(\omega)). \qquad (3.32)$$

Let $\omega_0 \in \Omega$ be such that $\hat{\pi}(\omega_0) = X$. Then

$$\begin{aligned}
I(\pi^3(\,\cdot\,|B)) &= I(\hat{\pi}) = \bigcup_{\omega\in\Omega,\,\omega\neq\omega_0} \hat{\Pi}((\Omega \setminus \{\omega\}) \cap \hat{\pi}(\omega)) \cup \hat{\Pi}(\Omega \setminus \{\omega_0\}) \cap \hat{\pi}(\omega_0)\\
&= \bigcup_{\omega\in\Omega,\,\omega\neq\omega_0} (X \cap \hat{\pi}(\omega)) \cup (\hat{\Pi}(\Omega \setminus \{\omega_0\}) \cap X
\end{aligned}$$

$$= \bigcup_{\omega\in\Omega,\,\omega\neq\omega_0} \hat{\pi}(\omega) \cup \hat{\Pi}(\Omega\setminus\{\omega_0\}) = \hat{\Pi}(\Omega\setminus\{\omega_0\})$$
$$= \Pi^3((\Omega\setminus\{\omega_0\})|B). \tag{3.33}$$

4. REFINED SET-VALUED ENTROPY FUNCTIONS

Let us reconsider and analyze Lemma 2.4 in more detail. According to that result, if there are $\omega_1, \omega_2 \in \Omega$, $\omega_1 \neq \omega_2$, such that $\pi(\omega_1) = \pi(\omega_2) = X$, then $I(\pi) = X = \mathbf{1}_{\mathcal{P}(X)}$.

This fact may be understood in the sense that the above-defined set-valued entropy function I is a very weak, poor and rough quantitative tool when seeking for an element $\omega_0 \in \Omega$ which could be preferred as the most expectable state of the universe Ω on the ground of the criteria that may be formalized within the framework of possibilistic distributions and measures taking their values in the power-set $\mathcal{P}(X)$.

Hence, each decision rule picking up just one $\omega_0 \in \Omega$ must be based on more input parameters than those expressible by the values of the entropy function $I(\pi)$. However, the situation is the same in the simplest probability space $\langle\Omega, \mathcal{A}, P\rangle$, where $\Omega = \{\omega_1, \omega_2\}$ and $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$. When we have to pick up just one of the states ω_1, ω_2 as the better solution of the problem in question, we have to do so on the grounds of utilizing more data and criteria than from the two values $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$. The following lemma may be taken as a complementary formulation of the conditions when $I(\pi) \neq 1_{\mathcal{P}(X)} = X$ is the case.

Lemma 4.1. Let Ω, X be nonempty sets, $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-possibilistic distribution on Ω, and $\omega_0 \in \Omega$ be such that $\pi(\omega_0) = X$. Then

$$I(\pi) = \Pi(\Omega\setminus\{\omega_0\}) \tag{4.34}$$

holds. Consequently, if $\Pi(\Omega\setminus\{\omega_0\}) \subsetneq X$ holds, then $I(\pi) \subsetneq X$ follows.

P r o o f. For $I(\pi)$ we have

$$I(\pi) = \bigcup_{\omega\in\Omega} (\Pi(\Omega\setminus\{\omega\}) \cap \pi(\omega)]$$
$$= \bigcup_{\omega\in\Omega,\,\omega\neq\omega_0} [\Pi(\Omega\setminus\{\omega\}) \cap \pi(\omega)] \cup \Pi(\Omega\setminus\{\omega_0\}) \cap \pi(\omega_0). \tag{4.35}$$

If $\omega \neq \omega_0$, then $\omega_0 \in (\Omega\setminus\{\omega\})$ and $\Pi(\Omega\setminus\{\omega\}) = X = \pi(\omega_0)$ holds, so that

$$I(\pi) = \left(\bigcup_{\omega\in\Omega,\,\omega\neq\omega_0} \pi(\omega)\right) \cup \Pi(\Omega\setminus\{\omega_0\})$$
$$= \Pi(\Omega\setminus\{\omega_0\}) \tag{4.36}$$

is valid and the assertion is proven. □

An easy corollary of Lemma 4.1 reads as follows. Let Ω, X and π be the same as in Lemma 4.1, and let there exist $x_0 \in X$ such that there is only one $\omega_0 \in \Omega$ with the properties $x_0 \in \pi(\omega_0)$ and $\Pi(\Omega\setminus\{\omega_0\}) \subsetneq X$. Then $I(\pi) = \Pi(\Omega\setminus\{\omega_0\}) \subsetneq X$ follows.

Inspired by Lemma 2.4 and Lemma 4.1, we propose in [7, 8, 9] some modifications of the space of values in which the mapping $\pi : \Omega \to T$ takes its values in such a way that $\pi(\omega_0) = \mathbf{1}_{\mathcal{T}}$ is valid for only one $\omega_0 \in \Omega$. In [7], the mapping π, defined on Ω, takes its values in a *complete chained lattice*; let us recall, for the reader's convenience, the way leading to this notion.

A p. o. set (partially ordered set) $\mathcal{T} = \langle T, \leq \rangle$ is called a *lattice* if, for each $t_1, t_2 \in T$, the elements $t_1 \vee t_2$ and $t_1 \wedge t_2$ are defined; and $\mathcal{T}$ is called a *complete lattice* if $\bigvee S$ and $\bigwedge S$ are defined for each $S \subset T$, applying a convention according to which $\bigwedge \emptyset = \bigvee T$ and $\bigvee \emptyset = \bigwedge T$ for the empty subset of T ($\vee$ and $\wedge$, as well as $\bigvee$ and $\bigwedge$ are supremum and infimum operations related to the partial ordering relation $\leq$ on T). The element $\bigvee T$ (or $\bigwedge T$) is called the *unit (element)* of $\mathcal{T}$ (or the *zero (element)* of $\mathcal{T}$) and is denoted by $\mathbf{1}_{\mathcal{T}}$ (or $\emptyset_{\mathcal{T}}$), respectively.

A complete lattice $\mathcal{T} = \langle T, \leq \rangle$ is called *distributive* if, for each $s \in T$ and $S \subset T$, the relations

$$s \wedge \left(\bigvee S\right) = \bigvee (s \wedge t), s \vee \left(\bigwedge S\right) = \bigwedge (s \vee t) \tag{4.37}$$

are valid. A complete lattice $\mathcal{T}$ is called *chained*, if the partial ordering $\leq$ on T is linear: that is, either $t_1 \leq t_2$ or $t_2 \leq t_1$ holds for each $t_1, t_2 \in T$. Consequently, for each mutually different $t_1, t_2 \in T$ either $t_1 < t_2$ or $t_2 < t_1$ holds.

Obviously, each complete chained lattice $\mathcal{T} = \langle T, \leq \rangle$ is distributive.

For more detail on binary relations, partial orderings and chains (linear orderings), semilattices and lattices, Boolean algebras, and the related structures and notions cf. [1, 3, 11] or any more recent textbook or monograph on this.

The values of possibilistic distributions are taken from complete lattices in [7], but they are bound by the condition of chained structure, so that each two possibility degrees are comparable by the partial ordering relation $\leq$ defined on $\mathcal{T} = \langle T, \leq \rangle$. In what follows, we use a more intuitive space of values, namely, that of the power-set over the space X. However, the conditions imposed on chained lattices need not be valid in general; that is, the structures from [7] and that introduced in this text cannot be classified as being a particular case or, in contrary, a generalization of each other.

Theorem 4.2. Let Ω, X be nonempty sets, and $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω. Let $\omega_0 \in \Omega$ be such that $\pi(\omega_0) = X$, and $\Omega_0 = \{\omega \in \Omega : \omega \neq \omega_0, \pi(\omega) = X\}$. Define $\pi^0 : \Omega \to \mathcal{P}(X)$ in this way: if $\omega \in \Omega \setminus \Omega_0$, then $\pi^0(\omega) = \pi(\omega)$, if $\omega \in \Omega_0$, then $\pi^0(\omega) = \emptyset$ $(= 0_{\mathcal{P}(X)})$. Then

$$I(\pi^0) \;=\; \Pi(\Omega \setminus \{\omega_0\}). \tag{4.38}$$

P r o o f. By definition,

$$\begin{aligned} I(\pi^0) &= \bigcup_{\omega \in \Omega} [\Pi^0(\Omega \setminus \{\omega\}) \cap \pi^0(\omega)] \\ &= \bigcup_{\omega \in \Omega \setminus \Omega_0} [\Pi^0(\Omega \setminus \{\omega\}) \cap \pi^0(\omega) \\ &\quad \cup \bigcup_{\omega \in \Omega_0} [\Pi^0(\Omega \setminus \{\omega\}) \cap \pi^0(\omega)]. \end{aligned} \tag{4.39}$$

The last line in (4.39) is identical with $\emptyset$, as $\pi^0(\omega) = \emptyset$ for each $\omega \in \Omega_0$, so that, as $\pi^0(\omega)$ and $\pi(\omega)$ are identical for each $\omega \in \Omega \setminus \Omega_0$, we obtain that

$$I(\pi^0) = \Pi(\Omega \setminus \{\omega_0\}) \cap \pi(\omega_0) \cup \bigcup_{\omega \in (\Omega \setminus \Omega_0) \setminus \{\omega_0\}} [\Pi(\Omega \setminus \{\omega\}) \cap \pi(\omega)]. \tag{4.40}$$

If $\omega \in (\Omega \setminus \Omega_0) \setminus \{\omega_0\}$ is the case, then $\omega_0 \in \Omega \setminus \{\omega\}$ thus $\Pi(\Omega \setminus \{\omega\}) = X$ holds, moreover, $\pi(\omega_0) = X$ holds as well. Consequently, we can reformulate (4.40) as

$$\begin{aligned} I(\pi^0) &= \left(\bigcup_{\omega \in (\Omega \setminus \Omega_0) \setminus \{\omega_0\}} (\pi(\omega) \cap X) \cup (\Pi(\Omega \setminus \{\omega_0\}) \cap X \right) \\ &= \Pi(\Omega \setminus \{\omega_0\}). \end{aligned} \tag{4.41}$$

The assertion is proven. □

Let us consider another example of restricted set-valued possibilistic distributions, which may be viewed as a more severe application of the reduction principle leading from π to π^0 in Theorem 4.2.

Theorem 4.3. Let Ω, X be nonempty spaces, and $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω. Further let $\pi(\omega_0) = X$ holds for a certain element $\omega_0 \in \Omega$, and $\pi(\omega) \subset X_0 \subsetneq X$ for a certain proper subset X_0 of X and for each $\omega \in \Omega$, $\omega \neq \omega_0$. Then $I(\pi) \subseteq X_0$ holds, and the equality take place when there is $\omega_1 \in \Omega$ such that $\pi(\omega_1) = X_0$.

P r o o f. By definition,

$$\begin{aligned} I(\pi) &= \bigcup_{\omega \in \Omega} [\Pi(\Omega \setminus \{\omega\}) \cap \pi(\omega)] \\ &= \bigcup_{\omega \in \Omega, \omega \neq \omega_0} [\Pi(\Omega \setminus \{\omega\}) \cap \pi(\omega)] \cup [\Pi(\Omega \setminus \{\omega_0\}) \cap \pi(\omega_0)] \\ &= \bigcup_{\omega \in \Omega, \omega \neq \omega_0} (X \cap \pi(\omega)) \cup [\Pi(\Omega \setminus \{\omega_0\}) \cap X] \\ &= \bigcup_{\omega \in \Omega, \omega \neq \omega_0} \pi(\omega) \cup \Pi(\Omega \setminus \{\omega_0\}) \\ &= \Pi(\Omega \setminus \{\omega_0\}) \cup \Pi(\Omega \setminus \{\omega_0\}) = \Pi(\Omega \setminus \{\omega_0\}) \subset X_0 \end{aligned} \tag{4.42}$$

as the relation $\Pi(\Omega \setminus \{\omega\}) = X$ holds for each $\omega \neq \omega_0$. The inclusion $I(\pi) \subset X_0$ easily follows, with the equality in the particular case of $\pi(\omega_1) = X_0$ for some $\omega_1 \in \Omega$. □

It is worth explicitly mentioning, that for each $\mathcal{P}(X)$-valued possibilistic distribution $\pi : \Omega \to \mathcal{P}(X)$ we may obtain reduced possibilistic distribution π^0, setting $\pi^0(\omega_0) = \pi(\omega_0) = X$, and $\pi^0(\omega) = \pi(\omega) \cap X_0$ for a fixed proper subset $X_0 \subset X$ and for each $\omega \in \Omega$, $\omega \neq \omega_0$.

5. COMPOSITIONS OF SET-VALUED POSSIBILISTIC DISTRIBUTIONS

Let Ω and X be nonempty sets, and H be a nonempty parameter set. For each $i \in H$, let $\pi_i : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω; that is, for each $\omega \in \Omega$, $\pi_i(\omega) \subset X$ and $\bigcup_{\omega\in\Omega} \pi_i(\omega) = X$ holds. Let π^H be the following $\mathcal{P}(X)$-valued mapping defined on Ω: for each $\omega \in \Omega$,

$$\pi^H(\omega) \;=\; \bigcup_{i\in H} \pi_i(\omega). \tag{5.43}$$

This mapping is called the supremum of the $\mathcal{P}(X)$-valued possibilistic distribution over the parameter set H. Instead of $\pi^H = \bigcup_{i\in H} \pi_i$ we also write $\bigvee^H \pi_i$ or $\bigvee_{i\in H} \pi_i$ (the symbol for supremum being used in order to save the symbol $\bigcup$ of set union just for subsets of the spaces Ω and X). In order to apply (2.1) we obtain for the $\mathcal{P}(X)$-valued entropy of π_i, $i \in H$, the value

$$I(\pi_i) \;=\; \bigcup_{\omega\in\Omega} [\Pi^i(\Omega \setminus \{\omega\}) \cap \pi_i(\omega)]. \tag{5.44}$$

The mapping π^H obviously meets the conditions imposed on $\mathcal{P}(X)$-valued possibilistic distribution, so that the related entropy value $I(\pi^H)$ is defined by

$$I(\pi^H) \;=\; \bigcup_{\omega\in\Omega} [\Pi^H(\Omega \setminus \{\omega\}) \cap \pi^H(\omega)]. \tag{5.45}$$

Here Π^i is the $\mathcal{P}(X)$-valued possiblistic measure on $\mathcal{P}(\Omega)$ defined by the distribution π_i on Ω, and Π^H is the $\mathcal{P}(X)$-valued possibilistic measure defined by the distribution π^H on Ω. Applying Theorem 2.3 on π_i and π^H, the inclusion $I(\pi_i) \subseteq I(\pi^H)$ holds for each $i \in H$; so the inclusion $\bigcup_{i\in H} I(\pi_i) \subseteq I(\pi^H)$ is also valid. The equality need not hold, as the following very simple example demonstrates.

Let there be $\Omega = \{\omega_1, \omega_2\}$, let $X \neq \emptyset$, $\pi_1 : \Omega \to \mathcal{P}(X)$ be defined by $\pi_1(\omega_1) = X$, $\pi_1(\omega_2) = \emptyset$, and $\pi_2 : \Omega \to \mathcal{P}(X)$ be defined by $\pi_2(\omega_1) = \emptyset$, $\pi_2(\omega_2) = X$. For both $i = 1, 2$, the identity $\bigcup_{\omega\in\Omega} \pi_i(\omega) = X$ obviously holds. Moreover

$$\begin{aligned} I(\pi_1) &= \bigcup_{\omega\in\Omega} (\Pi_1(\Omega \setminus \{\omega\}) \cap \pi_1(\omega)) \\ &= (\Pi_1(\Omega \setminus \{\omega_1\}) \cap \pi_1(\omega)) \;\cup\; (\Pi_1(\Omega \setminus \{\omega_2\}) \cap \pi_1(\omega_2)) \\ &= (\pi_1(\omega_2) \cap \pi_1(\omega_1)) \;\cup\; (\pi_1(\omega_1) \cap \pi_1(\omega_2)) \\ &= (\emptyset \cap X) \cup (X \cap \emptyset) \;=\; \emptyset. \end{aligned} \tag{5.46}$$

Analogously, we obtain that $I(\pi_2) = \emptyset$, hence, $I(\pi_1) \cup I(\pi_2) = \emptyset$. For $\pi_1 \vee \pi_2$ we obtain that

$$(\pi_1 \vee \pi_2)(\omega_1) \;=\; \pi_1(\omega_1) \cup \pi_2(\omega_1) \;=\; X \cup \emptyset \;=\; X, \tag{5.47}$$

$$(\pi_1 \vee \pi_2)(\omega_2) \;=\; \pi_1(\omega_2) \cup \pi_2(\omega_2) \;=\; \emptyset \cup X \;=\; X, \tag{5.48}$$

and $I(\pi_1 \vee \pi_2) = X \neq \emptyset = I(\pi_1) \cup I(\pi_2)$.

Let Ω, X be nonempty spaces, and $\pi_1, \pi_2 : \Omega \to \mathcal{P}(X)$ be $\mathcal{P}(X)$-valued possibilistic distributions on Ω such that $\Pi_1 \subseteq \Pi_2$ holds; hence, the inclusion $\pi_1(\omega) \subseteq \pi_2(\omega)$ is valid for each $\omega \in \Omega$. As proved in Section 2, in this case $I(\pi_1) \subseteq I(\pi_2)$ follows and $\pi_1(\omega) \subseteq \pi_2(\omega)$ is valid for each $\omega \in \Omega$. Consequently, $I(\pi_1 \vee \pi_2) = I(\pi_2) = I(\pi_1) \cup I(\pi_2)$ follows.

Let us mention two almost immediate consequences of the relations proven above.

Lemma 5.1. Let Ω, X be nonempty sets, and $\pi : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-valued possibilistic distribution on Ω such that there exist different elements $\omega_1, \omega_2 \in \Omega$ for which the intersection $\pi(\omega_1) \cap \pi(\omega_2)$ defines a nonempty subset of X. Then the $\mathcal{P}(X)$-valued entropy function I ascribes to π the nonempty (i. e., nonzero in the sense of the structure $\mathcal{T}$ on $\mathcal{P}(\Omega)$) value

$$I(\pi) \;=\; \bigcup_{\omega \in \Omega} (\Pi(\Omega \setminus \{\omega\}) \cap \pi(\omega)) \;\neq\; \emptyset. \tag{5.49}$$

P r o o f. As $\omega_1 \neq \omega_2$, the membership relations $\omega_2 \in \Omega \setminus \{\omega_1\}$ and $\omega_1 \in \Omega \setminus \{\omega_2\}$ are valid. Hence, the relation

$$I(\pi) \;\supseteq\; \pi(\omega_1) \cap \pi(\omega_2) \;\neq\; \emptyset \tag{5.50}$$

holds and the assertion is proven. □

Lemma 5.2. Let Ω and X be nonempty sets, and $\pi_1, \pi_2 : \Omega \to \mathcal{P}(X)$ be $\mathcal{P}(X)$-valued possibilistic distributions on Ω such that $\pi_1(\omega_1) = X$, $\pi_1(\omega) = \emptyset$, if $\omega \neq \omega_1$, $\pi_2(\omega_2) = X$, $\pi_2(\omega) = \emptyset$, if $\omega \neq \omega_2$. Then

$$\emptyset \;=\; I(\pi_1) \cup I(\pi_2) \;\subsetneq\; I(\pi_1 \vee \pi_2) \;\supseteq\; X. \tag{5.51}$$

P r o o f. $I(\pi_1) = \emptyset$, as $\pi_1(\omega_1) \cap \pi_1(\omega) = \emptyset$ for any $\omega \in \Omega$ different from ω_1 (cf. Lemma 2.2); similarly also holds $I(\pi_2) = \emptyset$. Thus we have $\emptyset = I(\pi_1) = I(\pi_2) = I(\pi_1) \cup I(\pi_2)$. Analogously to (5.47) and (5.48) we have two different elements $\omega_1, \omega_2 \in \Omega$ for which $(\pi_1 \vee \pi_2)(\omega_1) \cap (\pi_1 \vee \pi_2)(\omega_2) = X \neq \emptyset$, thus the right-hand side of (5.51) follows from (5.49): $I(\pi_1 \vee \pi_2) = \bigcup_{\omega \in \Omega} (\Pi(\Omega \setminus \{\omega\}) \cap \pi(\omega)) \supseteq \pi(\omega_1) \cap \pi(\omega_2) = X$. □

Let Ω and X be nonempty sets, and G be a nonempty parameter set. For each $i \in G$, let $\pi_i : \Omega \to \mathcal{P}(X)$ be a $\mathcal{P}(X)$-possibilistic distribution on Ω; hence, $\pi_i(\omega) \subset X$ and $\bigcup_{\omega \in \Omega} \pi_i(\omega) = X$ hold for each $i \in G$. Let $\pi^G : \Omega \to \mathcal{P}(X)$ be the following $\mathcal{P}(X)$-valued mapping defined on Ω: for each $\omega \in \Omega$,

$$\pi^G(\omega) \;=\; \bigcap_{i \in G} \pi_i(\omega). \tag{5.52}$$

The following lemma is obvious, but worth being introduced explicitly.

Lemma 5.3. Let there exist $\omega_0 \in \Omega$ such that for each $i \in G$, $\pi_i(\omega_0) = X$. Then the mapping $\pi^G : \Omega \to \mathcal{P}(X)$ defined by (5.52), meets the conditions imposed on $\mathcal{P}(X)$-valued possibilistic distributions.

6. CONCLUSIONS

According to what we said in the introductory section, our aim was to introduce and analyze some possibilistic distributions and related possibilistic measures with non-numerical, but intuitive enough uncertainty (in the sense of fuzziness and vagueness) degrees – as the simplest structure for these purposes we have taken the classical Boolean algebra over the power-set of all subsets of a basic set Ω, together with sizes of elements of Ω and their collections quantified by subsets of another space X. The contents of particular sections as scheduled in the introduction have been more or less tightly kept; that is why we do not feel it is necessary to repeat them now, rather focusing our attention on some inspirations for further developments.

First, worthy of interest are set-valued distributions taking values in power-sets of particular sets X; they are both interesting and important. E. g., take a map of a region with different subregions colored in different colors, yielding some information on different regions due to the system of colors known to the user.

More theoretical, but still interesting enough are the problems of incomplete set-valued possibilistic distributions and measures. In [7], we proposed possibilistic distributions $\pi : \Omega \to \mathcal{P}(X)$ and possibilistic measures $\Pi : \mathcal{P}(\Omega) \to \mathcal{P}(X)$ as complete mappings, so that for each $\omega \in \Omega$ and each $A \subset \Omega$ the values $\pi(\omega) \in \mathcal{P}(X)$ and $\Pi(A) = \bigcup_{\omega \in A} \pi(\omega)$ are defined. However, only the value $P(\bigcup_{A \in \mathcal{S}} A)$ for a finite system $\mathcal{S}$ of a disjoint subsystem of $\mathcal{A}$ may be defined and computed from values of P on $\mathcal{A}$ in probability spaces $\langle \Omega, \mathcal{A}, P \rangle$ with finitely additive probability measure P on a finite field $\mathcal{A}$. Hence, only probability spaces which can be fully described by relative frequencies of their results can be fully defined by probability spaces and, if this is the case, finitely additive probability measures suffice. For infinite spaces $\langle \Omega, \mathcal{A}, P \rangle$ and for the Borel or Lebesque subsets of real line, the Borel measure defined for semi-open interval by their length may be in a consistent and conservative way extended to Borel or Lebesque sets, but there are subsets of the real line which are measurable neither in the Borel nor in the Lebesque sense, so that the system of all Borel and Lebesque subsets of the real line measurable in the Borel or Lebesque sense remains incomplete.

As it is well-known, in the competition of set-functions in general and measures, including the probabilistic ones, in particular much more applications in various practical computational and technical problems have been based on set-functions based on Borel and Lebesque real-valued measures. They are measures not defined on all subsets of the basic space, but on those sets where their values are defined they provide an intuitive and easy way to compute and process values. Consequently, even when set-valued distributions and measures introduced and analyzed above lead to complete measures, it should be useful and interesting to admit the incompleteness of the resulting set-valued possibilistic distributions and measures from the very primary and axiomatic approach to set-valued possibilistic distributions and measures. Let us hope to have an opportunity to analyze this problem in more detail in our future work.

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