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ON BEST APPROXIMATION IN FUZZY METRIC SPACES

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In this paper we introduce the notation of t-best approximatively compact sets, t-best approximation points, t-proximinal sets, t-boundedly compact sets and t-best proximity pair in fuzzy metric spaces. The results derived in this paper are more general than the corresponding results of metric spaces, fuzzy metric spaces, fuzzy normed spaces and probabilistic metric spaces.

Keywords: best approximation, topology, fuzzy metric spaces

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1. INTRODUCTION AND PRELIMINARIES

The concept of probabilistic metric spaces and probabilistic normed spaces has been investigated by numerous authors. It is also of fundamental importance in probabilistic functional analysis and nonlinear analysis and applications, e. g. see [2]. These notions have been introduced and studied by many authors from different points of view. Following Menger [13], Kramosil and Michálek [12] introduced the fuzzy metric space by generalizing the concept of probabilistic metric space to the fuzzy situation with the help of continuous t-norm which was proved to be equivalent [12, Theorem 1] in a certain sense, to a probabilistic metric space. Later on, in order to construct a Hausdorff topology, George and Veeramani [4] modified the concept of fuzzy metric space, introduced by Kramosil and Michálek and obtained several classical theorems on this new structure. Actually, this topology is first countable and metrizable [8].

Best approximation has important applications in diverse disciplines of mathematics, engineering and economics in dealing with problems arising in: Fixed point theory, Approximation theory, game theory, mathematical economics, best proximity pairs, Equilibrium pairs, etc. Many authors have studied best approximation and best proximity pair in the both metric and fuzzy metric spaces (e. g. see [3,11,18-22]). Best proximity pair theorems in the metric space (X, d) are consider to expound the sufficient conditions that ensure the existence of $x \in A$ such that $d(x, Tx) = d(A, B) := \inf\{d(a, b); a \in A, b \in B\}$, where $T : A \to 2^B$ is a multifunction defined on suitable subsets A, B of X. Also, a best proximity pair theorem evolves as a generalization of the problem, considered by Beer and Pai [1], Sahney and Singh [16], Singer [19] and Xu [22], of exploring the

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sufficient conditions for the non-emptiness of the set

$$\mathrm{Prox}(A,B) = \{(a,b) \in A \times B : d(a,b) = d(A,B)\},$$

where A, B are suitable subsets of metric or linear normed space X. In this paper, we generalize some notions, definitions and results in [10, 1821] such as set of best approximation points, proximinal sets and approximatively compact sets for the fuzzy metric space in the sense of Kramosil and Michálek [12]. In addition, some examples and applications are presented.

Recall [17] that a continuous t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], \leq, *)$ is an ordered Abelian topological monoid with unit 1.

Definition 1.1. (Kramosil and Michálek [12]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set of $X \times X \times [0, \infty)$ satisfying the following properties, for all $x, y, z \in X, s, t > 0$:

(KM1) $M(x, y, 0) = 0$;

(KM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(KM3) $M(x, y, t) = M(y, x, t)$;

(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(KM5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

The following is given in [17].

Definition 1.2. Let x be a point of the fuzzy metric space $(X, M, *)$. The set of all points in X

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\},$$

where $r \in (0, 1), t > 0$ is called a neighborhood of x.

According to this definition, a sequence $\{x_n\}$ in X converges to x, denoted by $\lim_{n\to\infty} x_n = x$, if for every $r \in (0, 1), t > 0$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B_M(x, r, t)$ whenever $n > n_0$. Notice that $\lim_{n\to\infty} x_n = x$ if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$.

If $(X, M, *)$ is a fuzzy metric space then the family of $\{B_M(x, r, t) : x \in X, r \in (0, 1), t > 0\}$ is a base for a topology τ_M on X, called topology induced by M. Obviously the family $\{B_M(x, r, t) : r \in (0, 1), t > 0\}$ is a local base of each $x \in X$ in the topology τ_M, and this topology is Hausdorff. A fuzzy metric space $(X, M, *)$ is called compact if (X, τ_M) is a compact topological space. A subset A of X is said to be bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$. Some results related to the theory of fuzzy metric spaces can be found in references [4-9,12-15,17,18,20,21].

Example 1.3. (Schweizer and Sklar [17]) Let (X, d) be a metric space and $G : [0, \infty) \to [0, 1]$ a non-decreasing, left-continuous function with $G(0) = 0$ and $\lim_{r\to\infty} G(r) = 1$, we define the map $M : X \times X \times [0, \infty) \to [0, 1]$ as

$$M(x, y, t) = \begin{cases} G\left(\dfrac{t}{d(x, y)}\right), & x \neq y; \\ 0, & x = y, t = 0; \\ 1, & x = y, t > 0. \end{cases}$$

then for any choice of t-norm, $(X, M, *)$ is a fuzzy metric space. If the function G is defined by $G(r) = \frac{r}{r+1}$, then fuzzy metric space $(X, M_d, *)$ is called standard fuzzy metric space [4].

In the same way that a classical metric does not take the value ∞ and in order to obtain a Hausdorff topology, George and Veeramani [4, 5] defined the fuzzy set M on $X \times X \times (0, \infty)$ that satisfies axioms (KM3) and (KM4), and the axioms (KM1), (KM2) and (KM5) are replaced by the following respectively:

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ if and only if $x = y$ (see also [7, Remark 1]);

(GV5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

for all $x, y \in X, s, t > 0$.

Example 1.4. Every fuzzy normed space, probabilistic normed space, probabilistic metric space in a certain sense (see [12, Theorem 1]) and fuzzy metric space in the senses of George and Veeramani (see [20], [18], [12] and [4], respectively) are fuzzy metric spaces.

Remark 1.5. (Grabiec [6]) In a fuzzy metric space $(X, M, *)$, $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.

Remark 1.6. (George and Veeramani [4]) In a GV-fuzzy metric space $(X, M, *)$ for each x in X, $0 < r < 1$ and $t > 0$, the set $B_M[x, r, t]$ defined as

$$B_M[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\},$$

is a closed set.

Remark 1.7. (Rodríguez-López and Romaguera [14], see also Corollary 7 of Grabiec [6]) Let $(X, M, *)$ be a fuzzy metric space such that the axiom (GV5) holds then M is a continuous function on $X \times X \times (0, \infty)$.

Example 1.8. Let $X = \mathbb{R}$. For every $x, y \in X, t > 0$ define the metric d_t on $X \times X$ by $d_t(x, y) = \min\{| x - y |, t\}$, and the map $M : \mathbb{R}^2 \times [0, \infty) \to [0, 1]$ by $M(x, y, 0) = 0$ and

$$M(x, y, t) = \frac{t}{t + d_t(x, y)},$$

then $(X, M, \cdot)$ is a fuzzy metric space, wherein $\cdot$ is the product t-norm. We only show the axiom (KM4) of Definition 1.1. For any $x, y, z \in X, t, s > 0$ we have

$$\left(1+\frac{1}{t}\right) d_t(x,y) + \left(1+\frac{1}{s}\right) d_s(y,z) \geq d_{t+s}(x,z),$$

so

$$\frac{d_t(x,y)}{t} + \frac{d_s(y,z)}{s} \geq \frac{d_{t+s}(x,z)}{t+s},$$

consequently we have

$$\begin{aligned}\frac{1}{M(x,y,t)M(y,z,s)} &= \left(1+\frac{d_t(x,y)}{t}\right)\left(1+\frac{d_s(y,z)}{s}\right)\\ &= 1+\frac{d_t(x,y)}{t}+\frac{d_s(y,z)}{s}+\frac{1}{st}d_t(x,y)d_s(y,z)\\ &\geq 1+\frac{d_{t+s}(x,z)}{t+s}\\ &= \frac{1}{M(x,z,t+s)}.\end{aligned}$$

2. BEST APPROXIMATION

We begin this section with the concept of t-best approximation points in fuzzy metric spaces introduced by Veeramani [21] and we some known results in this spaces. Our reference for best approximation in metric spaces is [19].

Definition 2.1. Let A be a non-empty subset of fuzzy metric space $(X, M, *)$. For each $x \in X$ and $t > 0$, define

$$M(A, x, t) = \sup\{M(x, y, t) : y \in A\}.$$

An element $y_0 \in A$ is said to be a t-best approximation point to x from A if

$$M(y_0, x, t) = M(A, x, t).$$

We denote by $P_A^M(x,t)$ the set of t-best approximation points to x. For $t > 0$ a subset A of a fuzzy metric space $(X, M, *)$ is called t-proximinal if for every point $x \in X$, $P_A^M(x,t) \neq \emptyset$.

Example 2.2. (Veeramani [21]) Let $X = \mathbb{N}$, define $a * b = ab$ for all $a, b \in [0,1]$, let M be a fuzzy set on $X \times X \times (0,\infty)$ as follows

$$M(x,y,t) = \begin{cases} \dfrac{x+t}{y+t} & x \leq y \\[2ex] \dfrac{y+t}{x+t} & y \leq x \end{cases}$$

for all $x, y \in X$ and $t > 0$, then $(X, M, *)$ is a fuzzy metric space. Let $A = \{2, 4, 6, \ldots\}$, we conclude

$$M(A, 3, t) = \max\left\{\frac{2+t}{3+t}, \frac{3+t}{4+t}\right\} = \frac{3+t}{4+t} = M(3, 4, t),$$

Hence, for each $t > 0$, 4 is t-best approximation point to 3 from A. As $M(3, 4, t) > M(2, 3, t)$, 2 is not a t-best approximation point to 3, so $P_A^M(3, t) = \{4\}$.

Definition 2.3. (Veeramani [21]) For $t > 0$, a non-empty subset A of a fuzzy metric space $(X, M, *)$ is said to be t-approximatively compact if for each x in X and each sequence y_n in A with $M(y_n, x, t) \to M(A, x, t)$, there exists a subsequence y_{n_k} of y_n converging to an element y_0 in A.

Definition 2.4. (Veeramani [21]) For $t > 0$, a non-empty closed subset A of a fuzzy metric space $(X, M, *)$ is said to be t-boundedly compact if for each x in X and $0 < r < 1$, the set $B[x, r, t] \cap A$ is a compact subset of X.

Remark 2.5. (Veeramani [21]) Let (X, d) be a metric space and $A \subseteq X$, then A is a approximatively compact set in the metric space (X, d) if and only if for any $t > 0$, A is a t-approximatively compact set in the induced fuzzy metric space $(X, M_d, *)$.

Veeramani proved that every non-empty t-approximatively compact subset of a fuzzy metric space is t-proximinal and every t-boundedly compact subset is t-approximatively compact in GV-fuzzy metric space [21, Theorem 2.10 and Theorem 2.16, respectively].

Remark 2.6. Let (X, d) be a metric space and A a non-empty subset of M, then the following are equivalent.

(a) $y_0 \in A$ is a best approximation point to $x \in X$ in the metric space (X, d).

(b) $y_0 \in A$ is a t-best approximation point to $x \in X$ in the induced fuzzy metric space $(X, M_d, *)$.

By a slight modification in the definitions and the results in [20,18,21] we can extend those results to the fuzzy metric spaces, e. g., the following is given for fuzzy normed spaces in [20].

Definition 2.7. Let A be a non-empty subset of a fuzzy metric space $(X, M, *)$. An element $y_0 \in A$ is said to be an F-best approximation of $x \in X$ from A if it is a t-best approximation of x from A, for every $t > 0$. The set of all elements of F-best approximations of X from A is denoted by

$$FP_A^M(x) = \bigcap_{t \in (0,\infty)} P_A^M(x, t).$$

If each $x \in X$ has at least one F-best approximation in A, then A is called a F-proximinal set.

Remark 2.8. Let $(X, M_d, *)$ be a standard fuzzy metric space in Example 1.3 and $A \subseteq X$ and $x \in X$, then for every $t_1, t_2 > 0$, $P_A^M(x, t_1) = P_A^M(x, t_2)$, thus, $FP_A^M(x) = P_A^M(x, t_1) = P_A^M(x, 1)$. Also this property holds for Example 2.2 and Example 2.15 of [20] and other known examples in the literature, the following shows that the above property is not true in general and the definition of best approximation point in fuzzy metric spaces is related to parameter t in its definition, so it is different from the classical theory of metric spaces.

Example 2.9. Consider Example 1.8, take $A = [0, 1]$ and $y_0 = 2$ then one can easily shows that if $t \geq 1$ then $P_A^M(y_0, t) = \{1\}$ and if $0 < t < 1$ then $P_A^M(y_0, t) = A$.

3. GENERALIZATION

Following the approach of Kainen [10] we introduce a new definition to generalize t-approximatively compact set, then, we introduce t-best approximation point, t-proximinal set and t-boundedly compact set relative to set in fuzzy metric spaces.

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space and A, B are non-empty subsets of X and $t > 0$, let

$$M(A, B, t) = \sup\{M(a, b, t); a \in A, b \in B\}.$$

We say a sequence $x_n \in A$, t-converges in distance to B if

$$M(x_n, B, t) \to M(A, B, t).$$

If $B = \{b\}$ is singleton then we use b instead of $\{b\}$. Let $\mathfrak{B}$ denote the family of non-empty subsets of X, we say the subset A is t-approximatively compact relative to $\mathfrak{B}$ if for every $B \in \mathfrak{B}$ and every sequence $x_n \in A$ which converges in distance to B, then there exists a subsequence y_{n_k} of y_n and $y_0 \in A$ such that $y_{n_k} \to y_0$. If $\mathfrak{B} = \{B\}$ is singleton then we use B instead of $\{B\}$.

Definition 3.2. For $t > 0$, an element $y_0 \in A$ is said to be a t-best approximation point to B from A if

$$M(y_0, B, t) = M(A, B, t).$$

We denote by $P_A^M(B, t)$ the set of t-best approximation points to B. A subset A is called t-proximinal relative to $\mathfrak{B}$ if for every $B \in \mathfrak{B}$, $P_A^M(B, t) \neq \emptyset$ and A is called t-quasi Chebyshev relative to $\mathfrak{B}$ if for every $B \in \mathfrak{B}$, $P_A^M(B, t)$ be a compact set.

Next examples illustrate the last definition.

Example 3.3. Consider Example 2.2.

(1) If $A = \{2, 4, \ldots\}$ and $B = \{1, 3, \ldots\}$ then

$$M(A, B, t) = \sup\left\{\frac{\min\{m, n\} + t}{\max\{m, n\} + t}; m \in A, n \in B\right\} = 1,$$

also for any $m \in A$, we have

$$M(m, B, t) = \sup\left\{\frac{\min\{m,n\}+t}{\max\{m,n\}+t}; n \in B\right\} = \frac{m+t}{m+1+t},$$

thus, $P_A^M(B,t) = \emptyset$.

(2) If $A = \{2, 4, \ldots\}$ and $B = \{1, 3\}$ then

$$M(A, B, t) = \sup\left\{\frac{\min\{m,n\}+t}{\max\{m,n\}+t}; m \in A, n \in B\right\} = \frac{3+t}{4+t}$$

and $y_0 = 4$ is the only element in A such that $M(y_0, B, t) = (3+t)/(4+t)$, thus, $P_A^M(B,t) = \{4\}$.

Let $(X, M, *)$ be a fuzzy metric spaces. In the sequel for arbitrary $t > 0$, let $\mathcal{C}(X), \mathcal{A}(X)$ and $\mathcal{B}(X)$ denote the set of compact, t-approximatively compact and t-boundedly compact subsets of X respectively. Also we denote by $(\mathcal{A}(X), \mathfrak{B})$ the set of t-approximatively compact subsets of X relative to $\mathfrak{B}$ and for non-empty subsets A, B of X, denote by $\operatorname{Prox}(A, B, t)$ the set of t-best proximity pairs, i. e. $(a, b) \in A \times B$ such that $M(a, b, t) = M(A, B, t)$.

Remark 3.4. If A be a subset of fuzzy metric space $(X, M, *)$, then A is t-approximatively compact if and only if A is t-approximatively compact relative to x, for all $x \in X$.

The following main result shows that the notion of t-approximatively compact set can be applied to compact sets.

Theorem 3.5. Let $t > 0$. A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$. If $A \in \mathcal{A}(X)$ and $B \in \mathcal{C}(X)$ then $A \in (\mathcal{A}(X), B)$.

Proof. The case $M(A, B, t) = 0$ is trivial. Suppose $M(A, B, t) > 0$. Let $x_n \in A$ be any sequence converges in distance to B, so for any $t > 0$,

$$M(x_n, B, t) \to M(A, B, t).$$

Let sequence $y_n \in B$ satisfy

$$M(x_n, y_n, t) \to M(A, B, t).$$

Since B is compact, there exists a subsequence y_{n_k} of y_n such that converges to an element y_0 in B. For every $0 < \epsilon < M(A, B, t)$, there exists $K \in \mathbb{N}$ such that for every $k > K$

$$M(y_{n_k}, y_0, (t-\delta)/2) > 1 - \epsilon$$

and

$$M(x_{n_k}, y_{n_k}, t) > M(A, B, t) - \epsilon$$

where $0 < \delta < t$, so by triangular inequality we have

$$\begin{aligned} M(A, B, t) \geq M(x_{n_k}, y_0, t) &\geq M(x_{n_k}, y_{n_k}, (t+\delta)/2) * M(y_{n_k}, y_0, (t-\delta)/2) \\ &\geq M(x_{n_k}, y_{n_k}, (t+\delta)/2) * (1-\epsilon), \end{aligned}$$

by taking $\delta \to t^-$ for each $k > K$ we have

$$M(A,B,t) \geq M(x_{n_k}, y_0, t) \geq (M(A,B,t) - \epsilon) * (1 - \epsilon),$$

thus

$$M(x_{n_k}, y_0, t) \to M(A,B,t).$$

Consequently for every $\epsilon > 0$, there exists $k_0 > 0$ such that for every $k > k_0$,

$$M(A, y_0, t) - \epsilon \leq M(A,B,t) - \epsilon \leq M(x_{n_k}, y_0, t) \leq M(A, y_0, t),$$

this shows

$$M(x_{n_k}, y_0, t) \to M(A, y_0, t).$$

Since A is t-approximatively compact, so there exists a subsequence of x_{n_k} such that converges to an element $x_0 \in A$, thus x_n converges subsequently to an element of A, that is, A is t-approximatively compact relative to B. □

Corollary 3.6. If A be a subset of fuzzy metric space $(X, M, *)$ then $A \in \mathcal{A}(X)$ if and only if $A \in (\mathcal{A}(X), \mathcal{C}(X))$.

Since every t-boundedly compact set is t-approximatively compact set we have the following result.

Corollary 3.7. Let A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$. If $A \in \mathcal{B}(X)$ and $B \in \mathcal{C}(X)$, then $A \in (\mathcal{A}(X), B)$.

Theorem 3.8. Let A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$. If $A \in \mathcal{A}(X)$ and bounded and $B \in \mathcal{B}(X)$ then $A \in (\mathcal{A}(X), B)$.

Proof. Let $t > 0$ and $x_n \in A$ be any sequence converges in distance to B and let the sequence $y_n \in B$ satisfy

$$M(x_n, y_n, t) \to M(A,B,t).$$

As x_n is bounded, so y_n is bounded. Since B is boundedly compact, there exists subsequence y_{n_k} of y_n converges to an element y_0 in B, continue as in the proof of Theorem 3.5. □

Remark 3.9. In Theorems 3.5 and 3.8 we need only A is t-approximatively compact relative to all x in the closure of B and closure of B is compact.

Theorem 3.10. Let A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$. If A is closed and t-boundedly compact and B is bounded, then $A \in (\mathcal{A}(X), B)$.

Proof. Let $t > 0$ and x_n converges in distance to B and choose y_n in B such that

$$M(x_n, y_n, t) \to M(A,B,t).$$

As x_n is bounded, so is y_n and hence there is subsequence y_{n_k} of y_n converges to an element y_0 in B, continue as in the proof of Theorem 3.5. □

Theorem 3.11. Let A and B be non-empty subsets of a metric space $(X, M, *)$. If $A \in \mathcal{B}(X)$ and $B \in \mathcal{C}(X)$, then $\text{Prox}(A, B, t) \neq \emptyset$, for every $t > 0$.

P r o o f. Suppose $t > 0$ and y_n be a sequence that satisfies $M(A, y_n, t) \to M(A, B, t)$. By compactness of B, there is a subsequence y_{n_k} of y_n such that y_{n_k} converges to an element y_0 in B. So $M(A, y_0, t) = M(A, B, t)$. Now since A is t-boundedly compact then A is t-proximinal, so we can choose $x_0 \in A$ such that $M(x_0, y_0, t) = M(A, y_0, t)$. □

Theorem 3.12. Let A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$. If $A, B \in \mathcal{B}(X)$, A is bounded and B is closed, then $\text{Prox}(A, B, t) \neq \emptyset$, for every $t > 0$.

P r o o f. Suppose $t > 0$ and y_n be a sequence that satisfies $M(A, y_n, t) \to M(A, B, t)$. Since A is bounded, y_n must also be bounded, so there is a subsequence y_{n_k} of y_n such that y_{n_k} converges to an element y_0 in $\bar{B} = B$, continue as in the proof of Theorem 3.11. □

4. OPERATIONS PRESERVING COMPACTNESS

In the following, using the results of the previous section we introduce some actions that preserve compactness. First, we define product of two fuzzy metric spaces.

Definition 4.1. Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy metric spaces. We define the product fuzzy metric space $(X_1 \times X_2, M_1 \times M_2, *)$, where the fuzzy set $M_1 \times M_2$ on $X_1 \times X_2 \times [0, \infty)$ is given by

$$M_1 \times M_2((x, y), (x', y'), t) = M_1(x, x', t) * M_2(y, y', t),$$

for all $(x, y), (x', y') \in X_1 \times X_2, t \geq 0$. Also, in this space we define

$$M_1 \times M_2(A \times B, (x, y), t) = \sup\{M_1 \times M_2((a, b), (x, y), t); (a, b) \in A \times B\}.$$

One can easily see

$$M_1 \times M_2(A \times B, (x, y), t) = M_1(A, x, t) * M_2(B, y, t),$$

for all $(x, y) \in X_1 \times X_2, t \geq 0$.

For obtain the next theorem we need replace the axiom (KM1) in Definition 1.1 by the following

(KM1)' $M(x, y, t) = 0$ iff $t = 0$.

Note that the fuzzy metric space that induced by a GV-fuzzy metric space is satisfied the above condition.

The following investigates the above notions for product of fuzzy metric spaces.

Theorem 4.2. Let A and B be non-empty subsets of a fuzzy metric space $(X_1, M_1, *)$ and $(X_2, M_2, *)$, respectively. Suppose $B \in \mathcal{C}(X_2)$, if $A \in \mathcal{B}(X_1)$ or $A \in \mathcal{A}(X_1)$ then $A \times B \in \mathcal{B}(X_1 \times X_2)$ or $A \times B \in \mathcal{A}(X_1 \times X_2)$, respectively.

P r o o f. Let $t > 0$ and $A \in \mathcal{B}(X_1)$. We will show every bounded sequence (a_n, b_n) in $A \times B$ has a convergent subsequence. By definition of product fuzzy metric space, a_n is bounded and since A is boundedly compact, there exist subsequence a_{n_k} of a_n and $a_0 \in A$ such that $a_{n_k} \to a_0$. By compactness of B there exist subsequence $b_{n_{k_l}}$ of b_{n_k} and $b_0 \in A$ such that $b_{n_{k_l}} \to b_0$, hence, $(a_{n_{k_l}}, b_{n_{k_l}}) \to (a_0, b_0) \in A \times B$. That is, $A \times B$ is t-boundedly compact. Suppose $A \in \mathcal{A}(X_1)$, let $(a, b) \in X_1 \times X_2$ and (a_n, b_n) is a sequence in $A \times B$ which converges in distance to (a, b), that is, $M_1 \times M_2((a_n, b_n), (a, b), t) \to M_1 \times M_2(A \times B, (a, b), t)$, by compactness of B, there exist subsequence b_{n_k} of b_n and $b_0 \in B$ such that $b_{n_k} \to b_0$, hence,

$$M_1(a_{n_k}, a, t) * M_2(b_0, b, t) \to M_1(A, a, t) * M_2(B, b, t),$$

so $L = \lim_{k\to\infty} M_1(a_{n_k}, a, t)$ exists and

$$\begin{aligned} L = & \lim_{k\to\infty} M_1(a_{n_k}, a, t) \geq \lim_{k\to\infty} M_1(a_{n_k}, a, t) * \frac{M_2(b_0, b, t)}{M_2(B, b, t)} \\ & = M_1(A, a, t) * \frac{M_2(B, b, t)}{M_2(B, b, t)} = M_1(A, a, t). \end{aligned}$$

Since $M_1(y, a, t) \leq M_1(A, a, t)$ for all $y \in A$, we have

$$L = \lim_{k\to\infty} M_1(a_{n_k}, a, t) = M_1(A, a, t),$$

hence, a_{n_k} converges in distance to a and since A is t-approximatively compact, there exists subsequence $a_{n_{k_l}}$ of a_{n_k} and $a_0 \in A$ such that $a_{n_{k_l}} \to a_0$, thus, $(a_{n_{k_l}}, b_{n_{k_l}}) \to (a_0, b_0) \in A \times B$. That is, $A \times B$ is t-approximatively compact set. □

The following generalizes [18, Theorem 2.19] and shows that the metric projection $P_A^M(x, t)$ also preserves compactness.

Theorem 4.3. Let A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$. Suppose $B \in \mathcal{C}(X)$, if $A \in \mathcal{B}(X)$ or $A \in \mathcal{A}(X)$ then A is t-quasi Chebyshev relative to B.

P r o o f. Since $\mathcal{B}(X) \subseteq \mathcal{A}(X)$, we only need prove assertion for $A \in \mathcal{A}(X)$. For $t > 0$, we show $C = \{x \in A; \exists b \in B, M(a, b, t) = M(A, b, t)\}$ is compact. Suppose y_n be a sequence in C, for every $n \in \mathbb{N}$ define φ_n on B by $\varphi_n(b) = M(y_n, b, t)$, since B is compact, there exists a b_n in B such that maximizes φ_n, so we have

$$M(y_n, b_n, t) = M(A, b_n, t).$$

Since B is compact, there exists a subsequence b_{n_k} of b_n such that converges to an element b_0 in B. Suppose $M(A, b_0, t) > 0$. For every $0 < \epsilon < M(A, b_0, t)$, there exists $K \in \mathbb{N}$ such that for every $k > K$,

$$M(b_{n_k}, b_0, (t - \delta)/2) \geq 1 - \epsilon$$

and

$$M(A, b_{n_k}, t) \geq M(A, b_0, t) - \epsilon,$$

where $0 < \delta < t$, therefore

$$\begin{aligned} M(A, b_0, t) \geq M(y_{n_k}, b_0, t) &\geq M(y_{n_k}, b_{n_k}, (t+\delta)/2) * M(b_{n_k}, b_0, (t-\delta)/2) \\ &\geq M(y_{n_k}, b_{n_k}, (t+\delta)/2) * (1-\epsilon), \end{aligned}$$

by taking $\delta \to t^-$ we conclude

$$\begin{aligned} M(A, b_0, t) \geq M(y_{n_k}, b_0, t) &\geq M(y_{n_k}, b_{n_k}, t) * (1-\epsilon) \\ &= M(A, b_{n_k}, t) * (1-\epsilon) \\ &\geq (M(A, b_0, t) - \epsilon) * (1-\epsilon), \end{aligned}$$

so

$$M(y_{n_k}, b_0, t) \to M(A, b_0, t).$$

Since A is in $\mathcal{A}(X)$, y_{n_k} converges subsequently. So when $B \in \mathcal{C}(X)$ and $A \in \mathcal{A}(X)$, it follows $P_A^M(B, t) = \{x \in A; M(x, B, t) = M(A, B, t)\}$ is compact. In case $M(A, b_0, t) = 0$, the latter limit holds for every sequence y_n in A, so the same result is derived. □

Remark 4.4. Another actions preserve compactness are adding a compact set in probabilistic normed spaces and acting a compact set in fuzzy metric groups (see [18] and [15] for definitions, respectively) i. e. if A is t-approximatively compact and B is compact then $A + B$ and AB are t-approximatively compact in probabilistic normed spaces and fuzzy metric groups, respectively.

5. CONCLUSION AND PERSPECTIVE

Best approximation and its related subjects is widely used in mathematical fields (see e. g. [3,10,11,18-22]). We have tried to unify the matter which is scarcely discussed in the literature. Then we have examined its use in fuzzy metric spaces, this can be of much benefit in a variety of fields. Also the concept of probabilistic metric spaces and probabilistic normed spaces and fuzzy metric spaces has been investigated by numerous authors (see e. g. [2, 9]). Best approximation and its related subjects can be applied in probabilistic metric and normed spaces, fuzzy metric spaces, fixed point theory, etc. Based on its importance, in this paper we generalized the theory of best approximation point and some related notions in metric spaces and unified some results in [10, 18, 20, 21, 19] to fuzzy metric spaces in the sense of Kramosil and Michálek [12]. Example 2.9 shows the notion of best approximation is related to parameter the t and is different from the classical theory of metric spaces. In view of Remarks 2.5 and 2.6, the results obtained in this paper are applicable to in the both metric spaces and fuzzy metric spaces. We concluded that each of the following properties in fuzzy metric spaces implies the next one: compact, t-boundedly compact, t-approximatively compact relative to $\mathcal{C}(X)$, t-proximinal relative to $\mathcal{C}(X)$ and closed.

Finally, these notions in fuzzy metric spaces can constitute the start of a theory in which the best approximation and its related topics must play an important role in best proximity pair, neural network approximation.

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REFERENCES

[1] G. Beer and D. Pai: Proximal maps, prox maps and coincidence points. Numer. Funct. Anal. Optim. *11* (1990), 5–6, 429–448. DOI:10.1080/01630569008816382

[2] S. S. Chang, Y. J. Cho, and S. M. Kang: Nonlinear Operator Theory in Probabilistic Metric Spaces. Nova Science Publishers Inc, New York 2001.

[3] A. A. Eldred and P. Veeramani: Existence and convergence of best proximity points. J. Math. Anal. Appl. *323* (2006), 2, 1001–1006. DOI:10.1016/j.jmaa.2005.10.081

[4] A. George and P. Veeramani: On some results in fuzzy metric spaces. Fuzzy Sets and Systems *64* (1994), 3, 395–399. DOI:10.1016/0165-0114(94)90162-7

[5] A. George and P. Veeramani: On some results of analysis for fuzzy metric spaces. Fuzzy Sets and Systems *90* (1997), 3, 365–368. DOI:10.1016/s0165-0114(96)00207-2

[6] M. Grabiec: Fixed points in fuzzy metric spaces. Fuzzy Sets and Systems *27* (1998), 3, 385–389. DOI:10.1016/0165-0114(88)90064-4

[7] V. Gregori and S. Romaguera: Characterizing completable fuzzy metric spaces. Fuzzy Sets and Systems *144* (2004), 3, 411–420. DOI:10.1016/s0165-0114(03)00161-1

[8] V. Gregori and S. Romaguera: Some properties of fuzzy metric spaces. Fuzzy Sets and Systems *115* (2000), 3, 485–489. DOI:10.1016/s0165-0114(98)00281-4

[9] O. Hadžić and E. Pap: Fixed Point Theory in Probabilistic Metric Spaces. Kluwer Academic Publishers, Dordrecht 2001. DOI:10.1007/978-94-017-1560-7

[10] P. C. Kainen: Replacing points by compacta in neural network approximation. J. Franklin Inst. *341* (2004), 4, 391–399. DOI:10.1016/j.jfranklin.2004.03.001

[11] W. A. Kirk, P. S. Srinivasan, and P. Veeramani: Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory *4* (2003), 1, 79–89.

[12] I. Kramosil and J. Michálek: Fuzzy metrics and statistical metric spaces. Kybernetika *11* (1975), 5, 336–344.

[13] K. Menger: Statistical metrics. Proc. Nat. Acad. Sci. U. S., A. *28* (1942), 535–537. DOI:10.1073/pnas.28.12.535

[14] J. Rodríguez-López and S. Romaguera: The Hausdorff fuzzy metric on compact sets. Fuzzy Sets and Systems *147* (2004), 2, 273–283. DOI:10.1016/j.fss.2003.09.007

[15] S. Romaguera and M. Sanchis: On fuzzy metric groups. Fuzzy Sets and Systems *124* (2001), 1, 109–115. DOI:10.1016/s0165-0114(00)00085-3

[16] B. E. Sahney and S. P. Singh: On best simultaneous approximation. In: Approximation Theory III (E. W. Cheney, ed.), 1980, pp. 783–789.

[17] B. Schweizer and A. Sklar: Statistical metric spaces. Pacific J. Math. *10* (1960), 314–334. DOI:10.2140/pjm.1960.10.313

[18] M. Shams and S. M. Vaezpour: Best approximation on probabilistic normed spaces. Chaos Solitons Fractals *41* (2009), 4, 1661–1667. DOI:10.1016/j.chaos.2008.07.009

[19] I. Singer: Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces. Springer-Verlag 1970. DOI:10.1007/978-3-662-41583-2

[20] S. M. Vaezpour and F. Karimi: t-best approximation in fuzzy normed spaces. Iran. J. Fuzzy Syst. *5* (2008), 2, 93–99.

[21] P. Veeramani: Best approximation in fuzzy metric spaces. J. Fuzzy Math. *9* (2001), 1, 75–80.

[22] X. Xu: A result on best proximity pair of two sets. J. Approx. Theory *54* (1988), 3, 322–325. DOI:10.1016/0021-9045(88)90008-1

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