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QUOTIENT ALGEBRAIC STRUCTURES ON THE SET OF FUZZY NUMBERS

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A. M. Bica has constructed in [6] two isomorphic Abelian groups, defined on quotient sets of the set of those unimodal fuzzy numbers which have strictly monotone and continuous sides. In this paper, we extend the results of above mentioned paper, to a larger class of fuzzy numbers, by adding the flat fuzzy numbers. Furthermore, we add the topological structure and we characterize the constructed quotient groups, by using the set of the continuous functions with bounded variation, defined on $[0,1]$.

Keywords: fuzzy number, function with bounded variation, semigroup (monoid) with involution, topological group, metric space

Classification: 08A72, 54H11

1. INTRODUCTION

The study of fuzzy numbers is motivated by their applications, being widely used in engineering and control systems (see [14, 15, 21, 28]). For the convenience of calculus, the fuzzy numbers are usually represented by their level sets, obtaining the parametric representation (see [18, 20, 53]), or by its two sides, considered as a pair of functions x_A^- and x_A^+, defined on the interval $[0,1]$ (see [13, 35]).

In this paper, we provide a completion of the results obtained by A. M. Bica in [6], indicating the nature of the quotient set obtained in this mentioned paper, for the additive and multiplicative structures of the set of fuzzy numbers and extending these results from unimodal fuzzy numbers to flat fuzzy numbers. More precisely, we will characterize the factor groups by using the set of the continuous functions with bounded variation on $[0,1]$. The results will be extended even in the framework of metrizable topological monoids.

The additive quotient group of the set of fuzzy numbers is also studied in [50]. By using some algebraic properties of the equivalence classes, the authors of this mentioned paper have introduced a new concept of convergence in the set of fuzzy numbers.

About the algebraic structure of fuzzy numbers, many results have been obtained. Firstly, using the extension principle, are defined and studied the arithmetic operations with fuzzy numbers and their properties (see [4, 15, 16, 17, 18, 19, 21, 25, 28, 34, 36, 38, 40, 42, 46, 47, 48, 55, 56]). Since in fuzzy arithmetic some of the usual properties

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of operations are missing, such as the nonexistence of the opposite of a (noncrisp) fuzzy number and the absence of the distributivity law of the scalar product for the sum of crisp numbers, several equivalence relations were proposed in order to avoid these defects (see [2, 6, 7, 38, 39, 40, 41, 43, 44, 45, 46, 48, 51]) and obtaining group properties for the quotient set. Since the set of fuzzy numbers is not a group with the addition, the difference of two fuzzy numbers is only a partial operation being defined as a substraction (see [48]) or by using the Hukuhara and generalized Hukuhara difference (see [54]). The same situation can be observed according to the absence of the inverse for fuzzy numbers related to various type of products (see [1, 40, 48, 56]). A recent study of the algebraic properties of the operations with fuzzy numbers, including the partial operations of substraction and division, can be found in [48], where the group properties are obtained on the quotient set up to an equivalence relation (the spread compensation relation).

The study of the algebraic structure for some classes of fuzzy numbers can be found in [2, 5, 8, 9, 10, 31, 29, 30, 33, 37, 51, 53, 56]. A general framework has been recently proposed in [11, 12, 22, 24] and [23].

In [6, Remark 25], it is mentioned that the quotient set $FV/\sim_{\oplus}$ of the fuzzy numbers set FV, has the properties $\mathbb{R} \subset FV/\sim_{\oplus}$ and $FV/\sim_{\oplus} \neq \mathbb{R}$, but the nature of $FV/\sim_{\oplus}$ is not specified. As a main contribution of this paper we determine this set $FV/\sim_{\oplus}$ (denoted here by $\widehat{\mathfrak{F}}$), showing that it is topologically isomorphic with the set $\mathrm{BVC}\,[0,1]$ of all continuous functions with bounded variation on $[0,1]$. In this context, $\mathbb{R}$ can be identified with the subset of all constant functions in $\mathrm{BVC}\,[0,1]$. A similar result is also obtained for the multiplicative structure. The results obtained in [6] are concerning to unimodal continuous fuzzy numbers, and here we extend all these results for flat fuzzy numbers.

The paper is organized as follows: in Section 2 we remember some preliminary notions and results about fuzzy numbers and functions with bounded variation. Section 3 is devoted to present the algebraic framework of cancelative monoids with involution, and metrizable topological monoids and groups, adequate to obtain the algebraic properties of the set of fuzzy numbers. In the final part of this section, an interesting isomorphism theorem is obtained. The main results concerning to the quotient algebraic and topological structures on the set of fuzzy numbers are presented in Section 4.

2. PRELIMINARIES

Recall that a fuzzy number (see, for example [3]) is a function $A : \mathbb{R} \to [0,1]$ which is normal (i.e., there exists $x_0 \in \mathbb{R}$, such that $A(x_0) = 1$), convex (i.e., $A(\lambda x + (1-\lambda)y) \geq \min\{A(x), A(y)\}$, for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$), upper semicontinuous on $\mathbb{R}$ and has compact support (i.e., $\operatorname{supp} A$ being the closure of the set $\{x \in \mathbb{R} : A(x) > 0\}$ is a compact interval of $\mathbb{R}$). For the concept of fuzzy number and operations with fuzzy numbers we can mention [16] and [15].

For a fuzzy number $A : \mathbb{R} \to [0,1]$, the set $\operatorname{core} A = \{x \in \mathbb{R} : A(x) = 1\}$ is called the core of A. Obviously, by the definition of the fuzzy numbers, $\operatorname{supp} A$ and $\operatorname{core} A$ are compact intervals. In the case that $\operatorname{core} A$ is a singleton (one point set) we say that A is unimodal, respectively, if $\operatorname{core} A$ is a nontrivial compact interval, then we say that the fuzzy number A is flat.

If $A : \mathbb{R} \to [0,1]$ is a fuzzy number, the $t-$level sets $[A]_t$ of A, defined by $[A]_0 =$

$\overline{\{x \in \mathbb{R} : A(x) > 0\}}$ and $[A]_t = \{x \in \mathbb{R} : A(x) \geq t\}$ if $t \in (0,1]$, are compact intervals for each $t \in [0,1]$ and we see that $\operatorname{supp} A = [A]_0$, respectively $\operatorname{core} A = [A]_1$. Goetschel and Voxman in [18], proves that if $[A]_t = \left[x_A^-(t), x_A^+(t)\right]$, for each $t \in [0,1]$, then the functions $x_A^-, x_A^+ : [0,1] \to \mathbb{R}$ (defining the endpoints of the $t-$level sets) are bounded, left-continuous in $(0,1]$ and continuous in 0, x_A^- is increasing, x_A^+ is decreasing and $x_A^-(t) \leq x_A^+(t)$, for all $t \in [0,1]$. Moreover, a fuzzy number A is completely determined by a pair $x_A = \left(x_A^-, x_A^+\right)$ of functions $x_A^-, x_A^+ : [0,1] \to \mathbb{R}$ satisfying these conditions.

In the following, in the purpose to extend the results in the framework of topological monoids and groups, we consider only those fuzzy numbers for which the functions x_A^- and x_A^+ are continuous and denote by $\mathfrak{F}$ the set of all these fuzzy numbers. The purely algebraic results can be obtained without the hypothesis of continuity, but for the extension in the framework of topological monoids this hypothesis is necessary (see the proof of Theorem 4.5). Although, this is not too restrictive because it is known that the set of points of discontinuity of a function with bounded variation is at most countable (see [52]). Thus, the set $\mathfrak{F}$ can be represented as the set of elements of the type $A = \left[x_A^-, x_A^+\right]$, where $x_A^-, x_A^+ \in \mathrm{C}[0,1]$, x_A^- is increasing, x_A^+ is decreasing and $x_A^-(t) \leq x_A^+(t)$, for all $t \in [0,1]$. A characterization of the fuzzy numbers belonging to $\mathfrak{F}$ can be found in [35]. We denote by $\mathfrak{F}_+$ the set of all positive fuzzy numbers $A \in \mathfrak{F}$ $\left(\text{i. e., } x_A^-(t) > 0, \text{ for all } t \in [0,1]\right)$.

Consider now the set $\mathrm{C}[a,b]$ of real-valued continuous functions on $[a,b]$, $\mathrm{C}_+[a,b]$ the subset of $\mathrm{C}[a,b]$ of strictly positive-valued functions and $\mathrm{BV}[a,b]$ the set of real-valued functions with bounded variation on $[a,b]$. Denote

$$\mathrm{BVC}[a,b] = \mathrm{C}[a,b] \cap \mathrm{BV}[a,b],$$

respectively,

$$\mathrm{BVC}_+[a,b] = \mathrm{C}_+[a,b] \cap \mathrm{BV}[a,b].$$

In the theory of the functions with bounded variation, it is well known that if $f, g \in \mathrm{BVC}[a,b]$ and $\lambda \in \mathbb{R}$, then $f \pm g$, λf, $f \cdot g \in \mathrm{BVC}[a,b]$ and if $\frac{1}{g}$ is bounded, then $\frac{f}{g} \in \mathrm{BVC}[a,b]$. Consequently, $(\mathrm{BVC}[a,b], +)$ and $(\mathrm{BVC}_+[a,b], \cdot)$ are Abelian groups. Also, by the Jordan's decomposition theorem, a function f is with bounded variation on $[a,b]$ if and only if there exist two increasing functions f_1 and f_2, such that $f = f_1 - f_2$ (see, for example [49] and [52]). Moreover, if $f \in \mathrm{BVC}[a,b]$, then f_1 and f_2 can be chosen to be continuous and conversely, if f_1 and f_2 are two continuous and increasing functions, such that $f = f_1 - f_2$, then $f \in \mathrm{BVC}[a,b]$.

Theorem 2.1. (Josephy [27]) If $[a,b] \xrightarrow{f} [c,d] \xrightarrow{g} \mathbb{R}$ where $f \in \mathrm{BV}[a,b]$, then $g \circ f \in \mathrm{BV}[a,b]$ if and only if g satisfies the Lipschitz condition on $[c,d]$.

Proposition 2.2. A continuous function $f \in \mathrm{C}_+[a,b]$ is of bounded variation on $[a,b]$ if and only if there exist two increasing functions $\alpha, \beta \in \mathrm{C}_+[a,b]$, such that $f = \frac{\alpha}{\beta}$.

Proof. Since $\operatorname{Im} f$ is a compact subinterval of $(0, +\infty)$ and the function ln satisfies the Lipschitz condition on every compact interval, there exist two increasing functions $f_1, f_2 \in \mathrm{C}[a,b]$, such that $\ln \circ f = f_1 - f_2$ and so, $f = e^{f_1 - f_2} = \frac{e^{f_1}}{e^{f_2}}$.

Conversely, if α and β are increasing, then they are of bounded variation and so $f = \frac{\alpha}{\beta}$ is of bounded variation. □

Remark 2.3. If $f \in \mathrm{BVC}[a,b]$ then, according to the Jordan's decomposition theorem, we can choose an increasing function $u \in \mathrm{C}[a,b]$ and a decreasing function $v \in \mathrm{C}[a,b]$ such that $f = \frac{u+v}{2}$. Moreover, the functions u and v can be chosen such that $u(t) < v(t)$, for all $t \in [a,b]$ (if the functions u and v from Jordan's decomposition theorem do not satisfy this condition, put $\widetilde{u}(t) = u(t) - \alpha$ and $\widetilde{v}(t) = v(t) + \alpha$, where $\alpha > 0$; obviously, $\widetilde{u}, \widetilde{v} \in \mathrm{C}[a,b]$, $\widetilde{u}$ is increasing, $\widetilde{v}$ is decreasing, $f = \frac{\widetilde{u}+\widetilde{v}}{2}$ and $\widetilde{u}(t) < \widetilde{v}(t)$, for all $t \in [a,b]$, if α is large enough.)

Similarly, according to Proposition 2.2, if $f \in \mathrm{BVC}_+[a,b]$, then we can choose an increasing function $u \in \mathrm{C}_+[a,b]$ and a decreasing function $v \in \mathrm{C}_+[a,b]$ such that $f = \sqrt{u \cdot v}$ and $u(t) < v(t)$, for all $t \in [a,b]$.

It is elementary to prove that $(\mathrm{BVC}[a,b], +)$ and $(\mathrm{BVC}_+[a,b], \cdot)$ are Abelian topological groups with the topology induced by the distance defined as

$$\mathrm{D}(f,g) = \sup_{t\in[a,b]} |f(t) - g(t)|. \tag{1}$$

Moreover, the correspondence $f \mapsto e^f$ establishes a topological isomorphism between the topological groups $(\mathrm{BVC}[a,b], +)$ and $(\mathrm{BVC}_+[a,b], \cdot)$.

3. THE ALGEBRAIC FRAMEWORK

Recall that, if $(M, \cdot)$ is a semigroup, an involution in M is a unary operation $x \mapsto x^*$ on M, satisfying the following conditions: $(x \cdot y)^* = y^* \cdot x^*$ and $x^{**} = x$, for all $x, y \in M$. An element $x \in M$ is called Hermitian if and only if $x^* = x$.

Since the following results are true only for the commutative case, in what follows we will consider that all monoids and all groups are commutative.

Consider now the class $\mathfrak{M}$ of all systems $(M, \cdot, e, ^*)$, where $(M, \cdot, e)$ is a cancelative and commutative monoid and * is an involution in M.

If $(M_1, \cdot, e_1, ^*)$ and $(M_2, \bullet, e_2, ^\star)$ are in $\mathfrak{M}$, a function $f : M_1 \to M_2$ is called a $\mathfrak{M}$-homomorphism, if f is a monoid homomorphism and $f(x^*) = (f(x))^\star$, for all $x \in M_1$.

Remark 3.1. If $(G, \cdot)$ is an Abelian group, then $\left(G, \cdot, 1, \cdot^{-1}\right) \in \mathfrak{M}$ and every group homomorphism between two Abelian groups is a $\mathfrak{M}$-homomorphism.

The algebraic results obtained in [6] can be presented now, by considering cancelative and commutative monoids with involution.

Remark 3.2. If $(M, \cdot, e, ^*) \in \mathfrak{M}$, since M is commutative, the set

$$S(M) = \{x \in M : x^* = x\}$$

of all Hermitian elements of M, is a submonoid of M and its elements have the following properties:

1. $x \in S(M) \Leftrightarrow x^* \in S(M)$;
2. $x \cdot x^* \in S(M)$, $\forall\, x \in M$;

3. if $x, x \cdot y \in S(M)$, then $y \in S(M)$;
4. if $x, y \in M$, then $x \cdot y^* \in S(M) \Leftrightarrow x \cdot y^* = x^* \cdot y$.

Definition 3.3. If $(M, \cdot, e, ^*) \in \mathfrak{M}$, we define a relation on M, denoted by " $\sim_*$ ", as follows:

$$x \sim_* y \Longleftrightarrow x \cdot y^* \in S(M) \tag{2}$$

Proposition 3.4. If $(M, \cdot, e, ^*) \in \mathfrak{M}$, then:

1. the relation " $\sim_*$ " on M, defined by (2), is a congruence relation on $(M, \cdot, e, ^*)$;
2. the corresponding quotient set $\widehat{M} = \{[x] : x \in M\}$, where

$$[x] = \{y \in M : x \cdot y^* = x^* \cdot y\}$$

is the equivalence class of $x \in M$, is an Abelian group with the induced operation

$$[x] \odot [y] = [x \cdot y].$$

The neutral element of the group $\left(\widehat{M}, \odot\right)$ is $[e] = S(M)$ and the inverse of $[x] \in \widehat{M}$ is $[x^*] \in \widehat{M}$.

Proof. Elementary. □

As above, the canonical homomorphism $p : M \to \widehat{M}$, is defined by $x \mapsto [x]$.

Remark 3.5. Let $(M, \cdot, e, ^*) \in \mathfrak{M}$ and $\left(G, \bullet, 1, \cdot^{-1}\right)$ be an Abelian group. If there exists a surjective $\mathfrak{M}$–homomorphism $f : (M, \cdot, e, ^*) \to \left(G, \bullet, 1, \cdot^{-1}\right)$, such that

$$x \sim_* y \Leftrightarrow f(x) = f(y) \tag{3}$$

for all $x, y \in M$, then (by the first isomorphism theorem), the function $\overline{f} : \widehat{M} \to G$, $[x] \mapsto f(x)$, is a group isomorphism and $\overline{f} \circ p = f$.

Moreover, in these conditions, if

$$\ker f = \{(x, y) \in M \times M : f(x) = f(y)\}$$

is the kernel of the function f, the condition (3) is equivalent with $\sim_* = \ker f$, respectively, if

$$\operatorname{Ker} f = \{x \in M : f(x) = 1\}$$

is the kernel of f as a monoid homomorphism, the condition (3) is equivalent with $\operatorname{Ker} f = S(M)$, too.

We extend now the above remark, by adding the topological structure:

Theorem 3.6. If (M, d_1) and (G, d_2) are metric spaces such that:

1. $(M, \cdot, e, ^*, \tau_{\mathrm{d}_1})$ is a topological commutative monoid with continuous involution (where τ_{d_1} is the topology induced by the metric d_1);
2. $(G, \bullet, \tau_{\mathrm{d}_2})$ is a topological Abelian group (where τ_{d_2} is the topology induced by the metric d_2);
3. there exists a continuous and surjective $\mathfrak{M}$ - homomorphism $f : M \to G$ which satisfies the condition (3), for all $x, y \in M$;

then $\left(\widehat{M}, \widehat{\mathrm{d}}\right)$ is a metric space, where $\widehat{\mathrm{d}} : \widehat{M} \times \widehat{M} \to \mathbb{R}$ is defined by

$$\widehat{\mathrm{d}}\left([x], [y]\right) = \mathrm{d}_2\left(f(x), f(y)\right), \quad \text{for all } [x], [y] \in \widehat{M}. \tag{4}$$

Moreover, the canonical homomorphism $p : M \to \widehat{M}$ is continuous and $\left(\widehat{M}, \odot, \tau_{\widehat{\mathrm{d}}}\right)$ is a topological Abelian group (with the induced topology) which is topologically isomorphic with $(G, \bullet, \tau_{\mathrm{d}_2})$.

P r o o f. Obviously, $\widehat{\mathrm{d}}$ is a metric on $\widehat{M}$ and the continuity of p follows by the continuity of f. The above equality means that the function $\overline{f} : \widehat{M} \to G$, $[x] \mapsto f(x)$ is an isometry, and so, $\overline{f}$ and $\overline{f}^{-1}$ are continuous.

If $[a], [b] \in \widehat{M}$, since $(G, \bullet)$ is a topological group, for each $\varepsilon > 0$ there exists $\delta > 0$, such that for all $u, v \in G$ with $\mathrm{d}_2(u, f(a)) < \delta$ and $\mathrm{d}_2(v, f(b)) < \delta$, we have that $\mathrm{d}_2(u \bullet v, f(a) \bullet f(b)) < \varepsilon$. Then, if $[x], [y] \in \widehat{M}$ such that $\widehat{\mathrm{d}}([x], [a]) < \delta$ and $\widehat{\mathrm{d}}([y], [b]) < \delta$, then $\mathrm{d}_2(f(x), f(a)) < \delta$ and $\mathrm{d}_2(f(y), f(b)) < \delta$ and so

$$\begin{aligned}\varepsilon &> \mathrm{d}_2\left(f(x) \bullet f(y), f(a) \bullet f(b)\right) = \mathrm{d}_2\left(f(x \cdot y), f(a \cdot b)\right) = \widehat{\mathrm{d}}\left([x \cdot y], [a \cdot b]\right) \\ &= \widehat{\mathrm{d}}\left([x] \odot [y], [a] \odot [b]\right),\end{aligned}$$

which proves the continuity of $\odot$.

It is easy to prove that for each $\varepsilon > 0$ there exists $\delta > 0$, such that for all $u \in G$ with $\mathrm{d}_2(u, f(a)) < \delta$, we have that $\mathrm{d}_2\left(u^{-1}, f(a)^{-1}\right) < \varepsilon$. So, if $[x] \in \widehat{M}$ such that $\widehat{\mathrm{d}}([x], [a]) < \delta$, then $\mathrm{d}_2(f(x), f(a)) < \delta$ and

$$\widehat{\mathrm{d}}\left([x^*], [a^*]\right) = \mathrm{d}_2\left(f(x^*), f(a^*)\right) = \mathrm{d}_2\left(f(x)^{-1}, f(a)^{-1}\right) < \varepsilon.$$

Thus, we have proved that $\left(\widehat{M}, \odot\right)$ is a topological group. □

4. THE MAIN RESULTS

If $A = \left[x_A^-, x_A^+\right] \in \mathfrak{F}$ and $B = \left[x_B^-, x_B^+\right] \in \mathfrak{F}$, then their (usual) sum is defined by

$$A + B = \left[x_A^- + x_B^-, x_A^+ + x_B^+\right]$$

and $-A$ is defined by

$$-A = \left[-x_A^+, -x_A^-\right].$$

Also, if A and B are positive fuzzy numbers (i. e., $A, B \in \mathfrak{F}_+$), their (usual) product is defined by

$$A \cdot B = \left[x_A^- \cdot x_B^-, x_A^+ \cdot x_B^+\right]$$

and $A^{-1} = \frac{1}{A}$ is defined by

$$\frac{1}{A} = \left[\frac{1}{x_A^+}, \frac{1}{x_A^-}\right].$$

Obviously, $\left(\mathfrak{F}, +, \overline{0}, -\right) \in \mathfrak{M}$ and $\left(\mathfrak{F}_+, \cdot, \overline{1}, ^{-1}\right) \in \mathfrak{M}$, where $\overline{0} = [0, 0]$ and $\overline{1} = [1, 1]$.

Proposition 4.1. $\left(\mathfrak{F}, +, \overline{0}, -, \tau_{\mathrm{d}}\right)$ and $\left(\mathfrak{F}_+, \cdot, \overline{1}, ^{-1}, \tau_{\mathrm{d}}\right)$ are topological monoids with continuous involutions, where τ_{d} is the topology induced by the Hausdorff metric $\mathrm{d} : \mathfrak{F} \times \mathfrak{F} \to [0, +\infty)$, defined by

$$\mathrm{d}\left(A, B\right) = \sup_{t \in [0,1]} \left(\left|x_A^-\left(t\right) - x_B^-\left(t\right)\right| + \left|x_A^+\left(t\right) - x_B^+\left(t\right)\right|\right). \qquad (5)$$

Proof. Elementary. □

Denoting $S_0 = \mathrm{S}\left(\mathfrak{F}, +, \overline{0}, -\right)$ and $S_1 = \mathrm{S}\left(\mathfrak{F}_+, \cdot, \overline{1}, ^{-1}\right)$, we see that,

$$S_0 = \{A \in \mathfrak{F} : A = -A\} = \left\{A \in \mathfrak{F} : x_A^- + x_A^+ = 0\right\}$$
$$S_1 = \left\{A \in \mathfrak{F}_+ : A = A^{-1}\right\} = \left\{A \in \mathfrak{F}_+ : x_A^- \cdot x_A^+ = 1\right\}.$$

The induced congruence relations on $\left(\mathfrak{F}, +, \overline{0}, -\right)$ and $\left(\mathfrak{F}_+, \cdot, \overline{1}, ^{-1}\right)$ are defined by

$$A \sim B \Leftrightarrow A + (-B) \in S_0 \Leftrightarrow x_A^- + x_A^+ = x_B^- + x_B^+$$

if $A, B \in \mathfrak{F}$, respectively

$$A \approx B \Leftrightarrow A \cdot B^{-1} \in S_1 \Leftrightarrow x_A^- \cdot x_A^+ = x_B^- \cdot x_B^+$$

if $A, B \in \mathfrak{F}_+$ and the corresponding equivalence classes are

$$[A] = \{B \in \mathfrak{F} : A \sim B\}, \quad \text{if } A \in \mathfrak{F}$$
$$\langle A\rangle = \{B \in \mathfrak{F}_+ : A \approx B\}, \quad \text{if } A \in \mathfrak{F}_+.$$

Denote by $\widehat{\mathfrak{F}}$ and $\widetilde{\mathfrak{F}}_+$ the corresponding quotient sets $\mathfrak{F}/_\sim$ and $\mathfrak{F}_+/_\approx$, respectively, and so, $\widehat{\mathfrak{F}} = \{[A] : A \in \mathfrak{F}\}$ and $\widetilde{\mathfrak{F}}_+ = \{\langle A\rangle : A \in \mathfrak{F}_+\}$.

By Proposition 3.4, $\left(\widehat{\mathfrak{F}}, \oplus\right)$ is an Abelian group with the operation defined by $[A] \oplus [B] = [A + B]$. The neutral element is $\left[\overline{0}\right] = S_0$ and the additive inverse of $[A] \in \widehat{\mathfrak{F}}$ is $-[A] = [-A]$. Also, $\left(\widetilde{\mathfrak{F}}_+, \odot\right)$ is an Abelian group with the operation defined by $\langle A\rangle \odot \langle B\rangle = \langle A \cdot B\rangle$. The neutral element is $\left\langle \overline{1}\right\rangle = S_1$ and the multiplicative inverse of $\langle A\rangle \in \widetilde{\mathfrak{F}}_+$ is $\langle A\rangle^{-1} = \left\langle A^{-1}\right\rangle$.

Remark 4.2. In [6] it is shown that the quotient set $FV/\sim_{\oplus}$ of the fuzzy numbers set FV, has the property

$$\mathbb{R} \subset FV/\sim_{\oplus} \quad \text{and} \quad FV/\sim_{\oplus} \neq \mathbb{R},$$

but the structure of $FV/\sim_{\oplus}$ is not specified. A similar observation can be done for the quotient set $FV_{+}^{*}/\sim_{\odot}$. Now, we want to complete this result by obtaining the structure of the quotient sets $\widehat{\mathfrak{F}}$ and $\widetilde{\mathfrak{F}}_{+}$ (in our notations) and presenting the main results of this paper. These two results are obtained as applications of Theorem 3.6.

Theorem 4.3. $\left(\widehat{\mathfrak{F}}, \oplus\right)$ is a metrizable topological group which is topologically isomorphic with $(\mathrm{BVC}\,[0,1]\,,+)$.

P r o o f. The function $\mathrm{m}_a : \mathfrak{F} \to \mathrm{BVC}\,[0,1]\,,$ defined by $\mathrm{m}_a\,(A) = \frac{x_A^- + x_A^+}{2}$, is a surjective $\mathfrak{M}$-homomorphism and if $A, B \in \mathfrak{F}$, then $[A] = [B]$ if and only if $\mathrm{m}_a\,(A) = \mathrm{m}_a\,(B)\,.$

Moreover, m_a is continuous. Indeed, if $A \in \mathfrak{F}$ and $\varepsilon > 0$, we choose $\delta > 0$, such that $\delta < \varepsilon$; if $B = \left[x_B^-, x_B^+\right] \in \mathfrak{F}$ and $\mathrm{d}\,(A, B) < \delta$, then for all $t \in [0,1]\,,$ we have that $\left|x_A^-\,(t) - x_B^-\,(t)\right| < \delta$ and $\left|x_A^+\,(t) - x_B^+\,(t)\right| < \delta$, and so

$$\left|\frac{x_A^-\,(t) + x_A^+\,(t)}{2} - \frac{x_B^-\,(t) + x_B^+\,(t)}{2}\right| \leq \frac{\left|x_A^-\,(t) - x_B^-\,(t)\right| + \left|x_A^+\,(t) - x_B^+\,(t)\right|}{2} < \delta < \varepsilon.$$

That is $\mathrm{D}\,(\mathrm{m}_a\,(A)\,, \mathrm{m}_a\,(B)) < \varepsilon$, which proves the continuity of m_a in $A \in \mathfrak{F}_{+}$.

Therefore $\mathrm{m}_a : \left(\mathfrak{F}, +, \overline{0}, -, \tau_{\mathrm{d}}\right) \to (\mathrm{BVC}\,[0,1]\,, +, 0, -, \tau_{\mathrm{D}})$ is continuous and surjective $\mathfrak{M}$-homomorphism and by Theorem 3.6,

$$\widehat{\mathrm{d}}\,([A]\,, [B]) = \sup_{t \in [0,1]} \left|\frac{x_A^-\,(t) + x_A^+\,(t)}{2} - \frac{x_B^-\,(t) + x_B^+\,(t)}{2}\right| \tag{6}$$

is a metric on $\widehat{\mathfrak{F}}$, $\left(\widehat{\mathfrak{F}}, \oplus\right)$ is a topological group and $\left(\widehat{\mathfrak{F}}, \oplus\right) \cong_{\mathrm{top}} (\mathrm{BVC}\,[0,1]\,,+)\,.$ □

Remark 4.4. It is easy to prove that the distance $\widehat{\mathrm{d}}$ has the following properties:

1. $\widehat{\mathrm{d}}\,([A] + [C]\,, [B] + [D]) \leq \widehat{\mathrm{d}}\,([A]\,, [B]) + \widehat{\mathrm{d}}\,([C]\,, [D])\,;$
2. $\widehat{\mathrm{d}}\,([-A]\,, [-B]) = \widehat{\mathrm{d}}\,([A]\,, [B])\,;$

for all $[A]\,, [B]\,, [C]\,, [D] \in \widehat{\mathfrak{F}}$.

Theorem 4.5. $\left(\widetilde{\mathfrak{F}}_{+}, \odot\right)$ is a metrizable topological group which is topologically isomorphic with $(\mathrm{BVC}_{+}\,[0,1]\,, \cdot)$.

Proof. The function $\mathrm{m}_g : \mathfrak{F}_+ \to \mathrm{BVC}_+[0,1]$ defined by $\mathrm{m}_g(A) = \sqrt{x_A^- \cdot x_A^+}$ is a surjective $\mathfrak{M}$-homomorphism and $\langle A \rangle = \langle B \rangle$ if and only if $\mathrm{m}_g(A) = \mathrm{m}_g(B)$.

Moreover, m_g is continuous. Indeed, let $A = [x_A^-, x_A^+] \in \mathfrak{F}_+$ and $\varepsilon > 0$. Since x_A^-, x_A^+ are continuous, there exists $m, M > 0$ such that $m \le x_A^-(t) \le M$ and $m \le x_A^+(t) \le M$, for all $t \in [0,1]$. Also, there exists $\delta > 0$ such that $\frac{\delta^2 + 2M\delta}{m} < \varepsilon$. If $B = [x_B^-, x_B^+] \in \mathfrak{F}_+$ such that $\mathrm{d}(A,B) < \delta$, then for all $t \in [0,1]$, $|x_A^-(t) - x_B^-(t)| < \delta$ and $|x_A^+(t) - x_B^+(t)| < \delta$, and so

$$\left|\sqrt{x_A^-(t) \cdot x_A^+(t)} - \sqrt{x_B^-(t) \cdot x_B^+(t)}\right| = \frac{\left|x_A^-(t) \cdot x_A^+(t) - x_B^-(t) \cdot x_B^+(t)\right|}{\sqrt{x_A^-(t) \cdot x_A^+(t)} + \sqrt{x_B^-(t) \cdot x_B^+(t)}}$$

$$\le \frac{x_A^-(t) \cdot \left|x_A^+(t) - x_B^+(t)\right|}{\sqrt{x_A^-(t) \cdot x_A^+(t)}} + \frac{\left(\left|x_A^+(t) - x_B^+(t)\right| + x_A^+(t)\right) \cdot \left|x_A^-(t) - x_B^-(t)\right|}{\sqrt{x_A^-(t) \cdot x_A^+(t)}}$$
$$< \frac{\delta^2 + 2M\delta}{m} < \varepsilon.$$

That is $\mathrm{D}(\mathrm{m}_g(A), \mathrm{m}_g(B)) < \varepsilon$, which proves the continuity of m_g in $A \in \mathfrak{F}_+$.

Then, by Theorem 3.6,

$$\widetilde{\mathrm{d}}(\langle A \rangle, \langle B \rangle) = \sup_{t \in [0,1]} \left|\sqrt{x_A^-(t) \cdot x_A^+(t)} - \sqrt{x_B^-(t) \cdot x_B^+(t)}\right| \tag{7}$$

is a metric on $\widetilde{\mathfrak{F}}_+$ and $\left(\widetilde{\mathfrak{F}}_+, \odot\right)$ is a topological group with $\left(\widetilde{\mathfrak{F}}_+, \odot\right) \cong_{\mathrm{top}} (\mathrm{BVC}_+[0,1], \cdot)$. $\square$

Similarly, as in [6], we can consider the fuzzy logarithm function $\mathrm{Ln} : \mathfrak{F}_+ \to \mathfrak{F}$, defined by

$$A = [x_A^-, x_A^+] \longmapsto \mathrm{Ln}\, A = \left[\ln(x_A^-), \ln(x_A^+)\right]$$

and its inverse, the fuzzy exponential function, $\mathrm{Exp} : \mathfrak{F}_+ \to \mathfrak{F}$, defined by

$$A = [x_A^-, x_A^+] \longmapsto \mathrm{Exp}\, A = \left[e^{x_A^-}, e^{x_A^+}\right].$$

They establish the algebraic isomorphism between the monoids $\left(\mathfrak{F}, +, \overline{0}, -\right) \in \mathfrak{M}$ and $\left(\mathfrak{F}_+, \cdot, \overline{1}, ^{-1}\right) \in \mathfrak{M}$. Since the real-valued functions ln and exp are continuous, we infer that Ln and Exp are continuous. Thus, we have:

Theorem 4.6. $\left(\mathfrak{F}, +, \overline{0}, -, \tau_{\mathrm{d}}\right)$ and $\left(\mathfrak{F}_+, \cdot, \overline{1}, ^{-1}, \tau_{\mathrm{d}}\right)$ are topologically isomorphic monoids with continuous involution.

This theorem generalizes Bica's result. Consequently, we can complete [6, Corollary 24], from the framework of Abelian groups to the framework of topological Abelian groups, as follows:

Theorem 4.7. $\left(\widehat{\mathfrak{F}}, \oplus, \tau_{\widehat{\mathrm{d}}}\right) \cong_{\mathrm{top}} \left(\widetilde{\mathfrak{F}}_+, \odot, \tau_{\widetilde{\mathrm{d}}}\right)$.

Proof. It is easy to see that the correspondence $[A] \mapsto \langle e^A \rangle$, where $A = [x_A^-, x_A^+] \in \mathfrak{F}$ and $e^A = \left[e^{x_A^-}, e^{x_A^+}\right] \in \mathfrak{F}_+$, is a topological isomorphism. □

Remark 4.8. The equivalence class $[A] \in \widehat{\mathfrak{F}}$ of a fuzzy number $A \in \mathfrak{F}$, is defined by the arithmetic mean of the sides of A, respectively, the equivalence class $\langle A \rangle \in \widetilde{\mathfrak{F}}_+$ of a fuzzy number $A \in \mathfrak{F}_+$, is defined by the geometric mean of the sides of A. According to Remark 2.3, we infer that these arithmetic and geometric means are functions with bounded variation of $[0,1]$. We conclude that the quotient sets $\widehat{\mathfrak{F}}$ and $\widetilde{\mathfrak{F}}_+$ are spaces of certain real-valued (crisp) functions (being BVC $[0,1]$ and BVC$_+$ $[0,1]$, respectively), and in this context, we can consider that Theorems 4.3 and 4.5 complete the results obtained in [6, Corollary 24 and Remark 25], such that the quotient sets are completely determined. The equivalence classes $[A]$ and $\langle A \rangle$ are illustrated (for a positive fuzzy number A) in Figure 1:

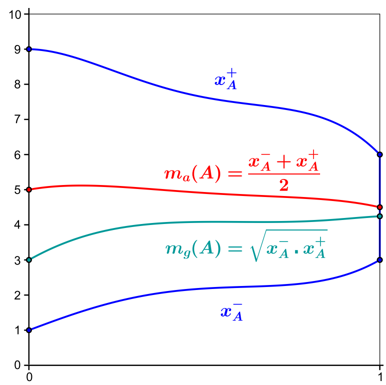

Fig. 1. The arithmetic and geometric means of a fuzzy number.

CONCLUSIONS

In this paper, we have characterized the additive and multiplicative quotient groups of the set of continuous fuzzy numbers, by the topological groups (BVC $[0,1]$, $+$) and (BVC$_+$ $[0,1]$, $\cdot$), respectively, of the continuous functions with bounded variation on $[0,1]$, improving the results from [6, Corollary 24 and Remark 25], and completing these results in the framework of metrizable Abelian groups. So, we have completely answered to the open problem left by Bica in [6, Corollary 24 and Remark 25], concerning the nature of the quotient groups of continuous fuzzy numbers.

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