DIRECT SOLUTION OF NONLINEAR CONSTRAINED QUADRATIC OPTIMAL CONTROL PROBLEMS USING B-SPLINE FUNCTIONS

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In this paper, a new numerical method for solving the nonlinear constrained optimal control with quadratic performance index is presented. The method is based upon B-spline functions. The properties of B-spline functions are presented. The operational matrix of derivative ($\mathbf{D}_\phi$) and integration matrix ($\mathbf{P}$) are introduced. These matrices are utilized to reduce the solution of nonlinear constrained quadratic optimal control to the solution of nonlinear programming one to which existing well-developed algorithms may be applied. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: optimal control problem, B-spline functions, derivative matrix, collocation method

Classification: 49N10, 65D07, 65R10, 65L60

1. INTRODUCTION

One of the widely used methods to solve optimal control problems is the direct method. There is a large number of research papers that employ this method to solve optimal control problems (see for example [1, 2, 5, 8, 14, 15, 17] and the references therein). This method converts the optimal control problem into a mathematical programming problem by using either the discretization technique [1, 2] or the parameterization technique [5, 14, 15, 17].

The discretization technique converts the optimal control problem into a nonlinear programming problem with a large number of unknown parameters and a large number of constraints [2]. On the other hand, parameterizing the control variables [5, 17] requires the integration of the state equations. While the simultaneous parameterization of both the state variables and the control variables [17] results in a nonlinear programming problem with a large number of parameters and a large number of equality constraints.

In [9] Jaddu and Shimemura proposed a method to solve the linear-quadratic and the nonlinear optimal control problems by using Chebyshev polynomials to parameterize some of the state variables, then the remaining state variables and the control variables are determined from the state equations. Yen and Nagurka [27] proposed a method based on the state parameterization, using Fourier series, to solve the linear-quadratic optimal

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control problem (with equal number of state variables and control variables) subject to state and control inequality constraints. Also Razzaghi and Elnagar [21] proposed a method to solve the unconstrained linear-quadratic optimal control problem with equal number of state and control variables. Their approach is based on using the shifted Legendre polynomials to parameterize the derivative of each of the state variables. The approach proposed in [17] is based on approximating the state variables and control variables with hybrid functions.

In this paper, we present a computational method for solving nonlinear constrained quadratic optimal control problems by using B-spline functions. The method is based on approximating the state variables and the control variables with a semiorthogonal linear B-spline functions [11]. Our method consists of reducing the optimal control problem to a NLP one by first expanding the state rate $\dot{x}(t)$ and the control $u(t)$ as a B-spline functions with unknown coefficients. These linear B-spline functions are introduced. To approximate the integral in the performance index, the matrix $\mathbf{P}$ is given.

The paper is organized as follows: In Section 2 we describe the basic formulation of the linear B-spline functions required for our subsequent development. We discuss in Section 3 on the convergence of the method. Section 4 is devoted to the formulation of optimal control problems. Section 5 summarizes the application of this method to the optimal control problems, and in Section 6, we report our numerical finding and demonstrate the accuracy of the proposed method. Sections 7 completes this paper with a brief conclusion.

2. PROPERTIES OF B-SPLINE FUNCTIONS

2.1. Linear B-spline functions on [0,1]

The mth-order B-spline $N_m(t)$ has the knot sequence $\{\ldots, -1, 0, 1, \ldots\}$ and consists of polynomials of order m (degree $m-1$) between the knots. Let $N_1(t) = \chi_{[0,1]}(t)$ be the characteristic function of [0,1]. Then for each integer $m \geqslant 2$, the mth-order B-spline is defined, inductively by [7]

$$N_m(t) = (N_{m-1} * N_1)(t) = \int_{-\infty}^{\infty} N_{m-1}(t-\tau) N_1(\tau)\,\mathrm{d}\tau = \int_0^1 N_{m-1}(t-\tau)\,\mathrm{d}\tau. \tag{1}$$

It can be shown [3] that $N_m(t)$ for $m \geqslant 2$ can be achieved using the following formula

$$N_m(t) = \frac{t}{m-1} N_{m-1}(t) + \frac{m-t}{m-1} N_{m-1}(t-1),$$

recursively, and $\operatorname{supp}[N_m(t)] = [0, m]$.

The explicit expressions of $N_2(t)$ (linear B-spline function) are [3, 7]

$$N_2(t) = \begin{cases} t, & t \in [0,1], \\ 2-t, & t \in [1,2], \\ 0, & \text{elsewhere}. \end{cases} \tag{2}$$

Suppose $N_{j,k}(t) = N_2(2^j t - k), j, k \in \mathbb{Z}$ and $B_{j,k} = \operatorname{supp}[N_{j,k}(t)] = \operatorname{clos}\{t : N_{j,k}(t) \neq 0\}$. It is easy to see that

$$B_{j,k} = [2^{-j}k, 2^{-j}(2+k)], \quad j, k \in \mathbb{Z}.$$

To use these functions on $[0,1]$,

$$S_j = \{k : B_{j,k} \cap [0,1] \neq \emptyset\}, \quad j \in \mathbb{Z}.$$

It is easy to see that $\min\{S_j\} = -1$ and $\max\{S_j\} = 2^j - 1, j \in \mathbb{Z}$.

The support of $N_{j,k}(t)$ may be out of interval [0,1], we need that these functions intrinsically defined on [0,1] so we put

$$\phi_{j,k}(t) = N_{j,k}(t)\chi_{[0,1]}(t), \quad j \in \mathbb{Z}, \quad k \in S_j. \tag{3}$$

2.2. The function approximation

Suppose $\Phi_j(t)$ is a $(2^j + 1) \times 1$ vector as

$$\Phi_j(t) = [\phi_{j,-1}(t), \phi_{j,0}(t), \ldots, \phi_{j,2^j-1}(t)]^T, \quad j \in \mathbb{Z}. \tag{4}$$

For a fixed $j = M$, a function $f(t) \in L^2[0,1]$ may be represented by the linear B-spline functions as

$$f(t) \simeq \sum_{k=-1}^{2^M-1} s_k \phi_{M,k}(t) = S^T \Phi_M(t), \tag{5}$$

where

$$S = [s_{-1}, s_0, \ldots, s_{2^M-1}]^T, \tag{6}$$

and

$$s_k = f(\frac{k+1}{2^M}), \quad k = -1, \ldots, 2^M - 1. \tag{7}$$

Note that the functions $\phi_{M,k}(t)$ satisfy in the relation

$$\phi_{M,k}(\frac{i+1}{2^M}) = \delta_{k,i} = \begin{cases} 1, & k = i, \\ 0, & k \neq i, \end{cases} \qquad i = -1, \ldots, 2^M - 1.$$

So we have

$$\Phi_M(t_i) = e_i, \quad t_i = \frac{i+1}{2^M}, \quad i = -1, \ldots, 2^M - 1, \tag{8}$$

where e_i is the ith column of unit matrix of order $2^M + 1$, [11].

2.3. The operational matrix of derivative

The differentiation of vectors Φ_J in (4) can be expressed as

$$\Phi'_M = \mathbf{D}_\phi \Phi_M, \tag{9}$$

where $\mathbf{D}_\phi$ is $(2^M + 1) \times (2^M + 1)$ operational matrix of derivative for the linear B-spline functions on [0,1] as follows:

$$\mathbf{D}_\phi = \int_0^1 \Phi'_M(t) \widetilde{\Phi}_M^T(t) \, \mathrm{d}t, \tag{10}$$

where $\widetilde{\Phi}_M$ is the vector of dual basis of Φ_M which can be obtained using the linear combinations of $\phi_{j,k}$ [12, 13] as

$$\widetilde{\Phi}_M = \mathbf{P}^{-1}\Phi_M, \tag{11}$$

where $\mathbf{P}$ is a $(2^M+1)\times(2^M+1)$ tridiagonal matrices as

$$\mathbf{P} = \int_0^1 \Phi_M(t)\Phi_M^T(t)\,\mathrm{d}t = 2^{-M}\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & \frac{1}{6} & \frac{1}{3} \end{bmatrix}. \tag{12}$$

Replacing (11) in (10) we get

$$\mathbf{D}_\phi = \left(\int_0^1 \Phi_M'(t)\Phi_M^T(t)\,\mathrm{d}t\right)\mathbf{P}^{-1} = \mathbf{E}(\mathbf{P}^{-1}), \tag{13}$$

where

$$\mathbf{E} = \int_0^1 \Phi_M'(t)\Phi_M^T(t)\,\mathrm{d}t. \tag{14}$$

It is shown in [12] that $\mathbf{E}$ is a $(2^M+1)\times(2^M+1)$ tridiagonal matrices as

$$\mathbf{E} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & & & \\ \frac{1}{2} & 0 & -\frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & -\frac{1}{2} \\ & & & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then we can approximate the derivative of $f(t)$ as follows

$$f'(t) = S^T\Phi_M'(t) = S^T\mathbf{D}_\Phi\Phi_M(t). \tag{15}$$

3. ON THE CONVERGENCE OF THE METHOD

In this section, we state the convergence properties of the linear B-spline functions. A detailed proof of the following results can be found in [23].

Theorem 3.1. Let $y_1 \leqslant \cdots \leqslant y_{n+2}$ be a partition of $[0,1]$ such that $y_1 < y_{i+2}$, $i = 1,2,\ldots,n$. Let $N_2^1,\ldots,N_2^n$ be the associated linear B-splines. Then there is a dual set of linear functionals $\lambda_1,\ldots,\lambda_n$ with

$$|\lambda_j f| \leqslant 45 h_j^{-\frac{1}{p}} \parallel f \parallel_{L^p[\tilde{I}_j]}, \qquad 1 \leqslant p \leqslant \infty, \qquad f \in L^p[0,1], \tag{16}$$

where $\tilde{I}_j = (y_j, y_{j+2})$ and $h_j = y_{j+2} - y_j$, $j = 1,2,\ldots,n$.

Lemma 3.2. Suppose $\mathcal{S}_2$ is the space of polynomial splines of order 2 with knots $y_1, \ldots, y_n$. Then

$$d(f, \mathcal{S}_2)_p \leqslant \mathcal{O}(n^{-2}) \tag{17}$$

where $d(f, \mathcal{S}_2)_p = d(f, \mathcal{S}_2)_{L^p[0,1]} = \inf_{s \in \mathcal{S}_2} \parallel f - s \parallel_{L^p[0,1]}$. Here, we have used the symbol $\mathcal{O}(n^{-2})$ to express the fact that the decay exponent for the bound in (17) is at least as small as -2.

Theorem 3.3. Let $\{y_i\}_1^{n+2}$ be an extended partition of $[0,1]$, and let $\{N_2^i\}_1^n$ be the associated B-splines. Let $\{\lambda_i\}_1^n$ be the linear functionals defined in Theorem 3.1 forming a dual basis for $\mathcal{S}_2 = \mathrm{span}\{N_2^i\}_1^n$. Then for any $1 \leqslant p \leqslant \infty$,

$$Qf = \sum_{i=1}^{n} (\lambda_i f) N_2^i(x)$$

defines a bounded linear projector of $L^p[0,1]$ onto $\mathcal{S}_2$. Moreover, for all $f \in L^p[0,1]$,

$$\parallel f - Qf \parallel_p \leqslant C d(f, \mathcal{S}_2)_p.$$

In order to establish convergence of the proposed method, Lemma 3.2 and Theorem 3.3 show that with increase in the number of knots as such as B-spline basis, the error terms tend to zero.

4. PROBLEM STATEMENT

The problem we are treating is to find the optimal control $\mathbf{u}^*(t)$ and the corresponding optimal state trajectory $\mathbf{x}^*(t)$ that minimizes the performance index

$$J = \frac{1}{2}\mathbf{x}^T(t_f)\mathbf{Z}\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(\mathbf{x}^T(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(t)\mathbf{u}(t)\right) \mathrm{d}t, \tag{18}$$

subject to the following constraints

- System state equations
$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{19}$$
- Boundary condition
$$\Psi(\mathbf{x}(t_0), t_0, \mathbf{x}(t_f), t_f) = 0, \tag{20}$$
- State and control inequality constraints
$$\mathbf{g}_i(\mathbf{x}(t), \mathbf{u}(t), t) \leqslant 0, \quad i = 1, 2, \ldots, w, \tag{21}$$

where $\mathbf{Z}$ and $\mathbf{Q}(t)$ are positive semidefinite matrices, $\mathbf{R}(t)$ is a positive definite matrix, t_0 and t_f are known initial and terminal time respectively, $\mathbf{x}(t) \in \mathbb{R}^l$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^q$ is the control vector, and $\mathbf{f}$ and $\mathbf{g}_i$, $i = 1, 2, \ldots, w$, are nonlinear functions. This problem is defined on the time interval $t \in [t_0, t_f]$. Certain numerical techniques (like B-spline functions) require a fixed time interval, such as $[0,1]$. The independent variable can be mapped to the general interval $\tau \in [0,1]$ via the affine transformation

$$\tau = \frac{t - t_0}{t_f - t_0}. \tag{22}$$

Note that this mapping is still valid with free initial and final times. Using Eq. (22), this problem can be redefined as follows. Minimize the performance index

$$J = \frac{1}{2}\mathbf{x}^T(1)\mathbf{Z}\mathbf{x}(1) + \frac{1}{2}(t_f - t_0)\int_0^1 \big(\mathbf{x}^T(\tau)\mathbf{Q}(\tau)\mathbf{x}(\tau) + \mathbf{u}^T(\tau)\mathbf{R}(\tau)\mathbf{u}(\tau)\big)\, \mathrm{d}\tau, \tag{23}$$

subject to the constraints

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = (t_f - t_0)\mathbf{f}(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau; t_0, t_f), \tag{24}$$

$$\Psi(\mathbf{x}(0), t_0, \mathbf{x}(1), t_f) = 0, \tag{25}$$

$$\mathbf{g}_i(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau; t_0, t_f) \leqslant 0, \quad i = 1, 2, \ldots, w, \quad \tau \in [0,1]. \tag{26}$$

5. THE PROPOSED METHOD

We approximate the system dynamics as follows: let

$$\mathbf{x}(t) = [\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_l(t)]^T, \tag{27}$$

$$\dot{\mathbf{x}}(t) = [\dot{\mathbf{x}}_1(t), \dot{\mathbf{x}}_2(t), \ldots, \dot{\mathbf{x}}_l(t)]^T, \tag{28}$$

$$\mathbf{u}(t) = [\mathbf{u}_1(t), \mathbf{u}_2(t), \ldots, \mathbf{u}_q(t)]^T, \tag{29}$$

$$\hat{\Phi}_{M,l}(t) = I_l \otimes \Phi_M(t), \tag{30}$$

$$\hat{\Phi}'_{M,l}(t) = I_l \otimes \mathbf{D}_\Phi \Phi_M(t), \tag{31}$$

$$\hat{\Phi}_{M,q}(t) = I_q \otimes \Phi_M(t), \tag{32}$$

where I_l and I_q are $l \times l$ and $q \times q$ dimensional identity matrices, $\Phi_M(t)$ is $(2^M + 1) \times 1$ vector, $\otimes$ denotes Kronecker product [10], $\hat{\Phi}_{M,l}(t)$ and $\hat{\Phi}'_{M,l}(t)$ are matrices of order $l(2^M + 1) \times l$ and $\hat{\Phi}_{M,q}(t)$ is a matrix of order $q(2^M + 1) \times q$. Assume that each of $\mathbf{x}_i(t)$ and each of $\mathbf{u}_j(t)$, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, q$, can be written in terms of B-spline functions as

$$\mathbf{x}_i(t) \simeq \Phi_M^T(t)\mathbf{X}_i,$$
$$\dot{\mathbf{x}}_i(t) \simeq \Phi_M^T \mathbf{D}_\Phi^T \mathbf{X}_i,$$
$$\mathbf{u}_j(t) \simeq \Phi_M^T(t)\mathbf{U}_j.$$

Then using Eqs. (30), (31) and (32) we have

$$\mathbf{x}(t) \simeq \hat{\Phi}_{M,l}^T(t)\mathbf{X}, \tag{33}$$

$$\dot{\mathbf{x}}(t) \simeq \hat{\Phi}_{M,l}^{\prime T}(t)\mathbf{X}, \tag{34}$$

$$\mathbf{u}(t) \simeq \hat{\Phi}_{M,q}^T(t)\mathbf{U}, \tag{35}$$

where X and U are vectors of order $l(2^M+1)\times 1$ and $q(2^M+1)\times 1$, respectively, given by

$$\mathbf{X} = \left[\mathbf{X}_1^T, \mathbf{X}_2^T, \dots, \mathbf{X}_l^T\right]^T,$$
$$\mathbf{U} = \left[\mathbf{U}_1^T, \mathbf{U}_2^T, \dots, \mathbf{U}_q^T\right]^T.$$

5.1. The performance index approximation

By substituting Eqs. (33) – (35) in Eq. (23) we get

$$\begin{aligned} J =& \frac{1}{2}\mathbf{X}^T\hat{\Phi}_{M,l}(1)\mathbf{Z}\hat{\Phi}_{M,l}^T(1)\mathbf{X} + \frac{1}{2}(t_f-t_0)\mathbf{X}^T\left(\int_0^1 \hat{\Phi}_{M,l}(t)\mathbf{Q}(t)\hat{\Phi}_{M,l}^T(t)\,\mathrm{d}t\right)\mathbf{X} \\ &+ \frac{1}{2}(t_f-t_0)\mathbf{U}^T\left(\int_0^1 \hat{\Phi}_{M,q}(t)\mathbf{R}(t)\hat{\Phi}_{M,q}^T(t)\,\mathrm{d}t\right)\mathbf{U}. \end{aligned} \tag{36}$$

Eq. (36) can be computed more efficiently by writing J as

$$\begin{aligned} J =& \frac{1}{2}\mathbf{X}^T\left(\mathbf{Z}\otimes\Phi_M(1)\Phi_M^T(1)\right)\mathbf{X} + \frac{1}{2}(t_f-t_0)\mathbf{X}^T\left(\int_0^1 \mathbf{Q}(t)\otimes\Phi_M(t)\Phi_M^T(t)\,\mathrm{d}t\right)\mathbf{X} \\ &+ \frac{1}{2}(t_f-t_0)\mathbf{U}^T\left(\int_0^1 \mathbf{R}(t)\otimes\Phi_M(t)\Phi_M^T(t)\,\mathrm{d}t\right)\mathbf{U}. \end{aligned} \tag{37}$$

For problems with time-varying performance index, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are functions of time and

$$\int_0^1 \mathbf{Q}(t)\otimes\Phi_M(t)\Phi_M^T(t)\,\mathrm{d}t, \qquad \int_0^1 \mathbf{R}(t)\otimes\Phi_M(t)\Phi_M^T(t)\,\mathrm{d}t$$

can be evaluated numerically. For time-invariant problems, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are constant matrices and can be removed from the integrals. In this case Eq. (37) can be rewritten as

$$\begin{aligned} J(\mathbf{X},\mathbf{U}) =& \frac{1}{2}\mathbf{X}^T\left(\mathbf{Z}\otimes\Phi_M(1)\Phi_M^T(1)\right)\mathbf{X} + \frac{1}{2}(t_f-t_0)\mathbf{X}^T\left(\mathbf{Q}\otimes\mathbf{P}\right)\mathbf{X} \\ &+ \frac{1}{2}(t_f-t_0)\mathbf{U}^T\left(\mathbf{R}\otimes\mathbf{P}\right)\mathbf{U}, \end{aligned} \tag{38}$$

where

$$\mathbf{P} = \int_0^1 \Phi_M(t)\Phi_M^T(t)\,\mathrm{d}t,$$

that this matrix is obtained in Eq.(12).

5.2. The system constraints approximation

We approximate the system constraints as follows:

Using Eqs. (33) – (35) the system constraints (24) – (26) became

- System state equations

$$\hat{\Phi}_{M,l}'^{T}(t)\mathbf{X} = (t_f-t_0)\mathbf{f}(\hat{\Phi}_{M,l}^T(t)\mathbf{X}, \hat{\Phi}_{M,q}^T(t)\mathbf{U}, t; t_0, t_f), \tag{39}$$

- Boundary condition

$$\Psi(\hat{\Phi}^T_{M,l}(0)\mathbf{X}, t_0, \hat{\Phi}^T_{M,l}(1)\mathbf{X}, t_f) = 0, \tag{40}$$

- State and control inequality constraints

$$\mathbf{g}_i(\hat{\Phi}^T_{M,l}(t)\mathbf{X}, \hat{\Phi}^T_{M,q}(t)\mathbf{U}, t; t_0, t_f) \leqslant 0, \qquad i = 1, 2, \dots, w. \tag{41}$$

We collocate Eqs. (39) and (41) at Newton-cotes nodes t_k,

$$t_k = \frac{k-1}{2^M}, \quad k = 1, 2, \dots, 2^M + 1. \tag{42}$$

The optimal control problem has now been reduced to a parameter optimization problem which can be stated as follows. Find $\mathbf{X}$ and $\mathbf{U}$ so that $J(\mathbf{X}, \mathbf{U})$ is minimized (or maximized) subject to Eq. (40) and

$$\hat{\Phi}'^T_{M,l}(t_k)\mathbf{X} = (t_f - t_0)\mathbf{f}(\hat{\Phi}^T_{M,l}(t_k)\mathbf{X}, \hat{\Phi}^T_{M,q}(t_k)\mathbf{U}, t_k), \tag{43}$$

$$\mathbf{g}_i(\hat{\Phi}^T_{M,l}(t_k)\mathbf{X}, \hat{\Phi}^T_{M,q}(t_k)\mathbf{U}, t_k; t_0, t_f) \leqslant 0, \quad i = 1, 2, \dots, w, \quad k = 1, 2, \dots, 2^M + 1. \tag{44}$$

Many well-developed nonlinear programming techniques can be used to solve this extremum problem (see, e. g., [4, 20, 22]).

6. ILLUSTRATIVE EXAMPLES

This section is devoted to numerical examples. We implement the proposed method in last section with Maple 17 software in personal computer.

To illustrate our technique, we present four numerical examples, and make a comparison with some of the results in the literatures.

Example 6.1. This example is studied by using rationalized Haar functions [19] and hybrid of block-pulse and Bernoulli polynomials [17] and hybrid of block-pulse and Legendre polynomials [14]. Find the control vector $u(t)$ which minimizes

$$J = \frac{1}{2}\int_0^1 \left(x_1^2(t) + u^2(t)\right)\, \mathrm{d}t, \tag{45}$$

subject to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{46}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \tag{47}$$

and subject to the following inequality control constraint

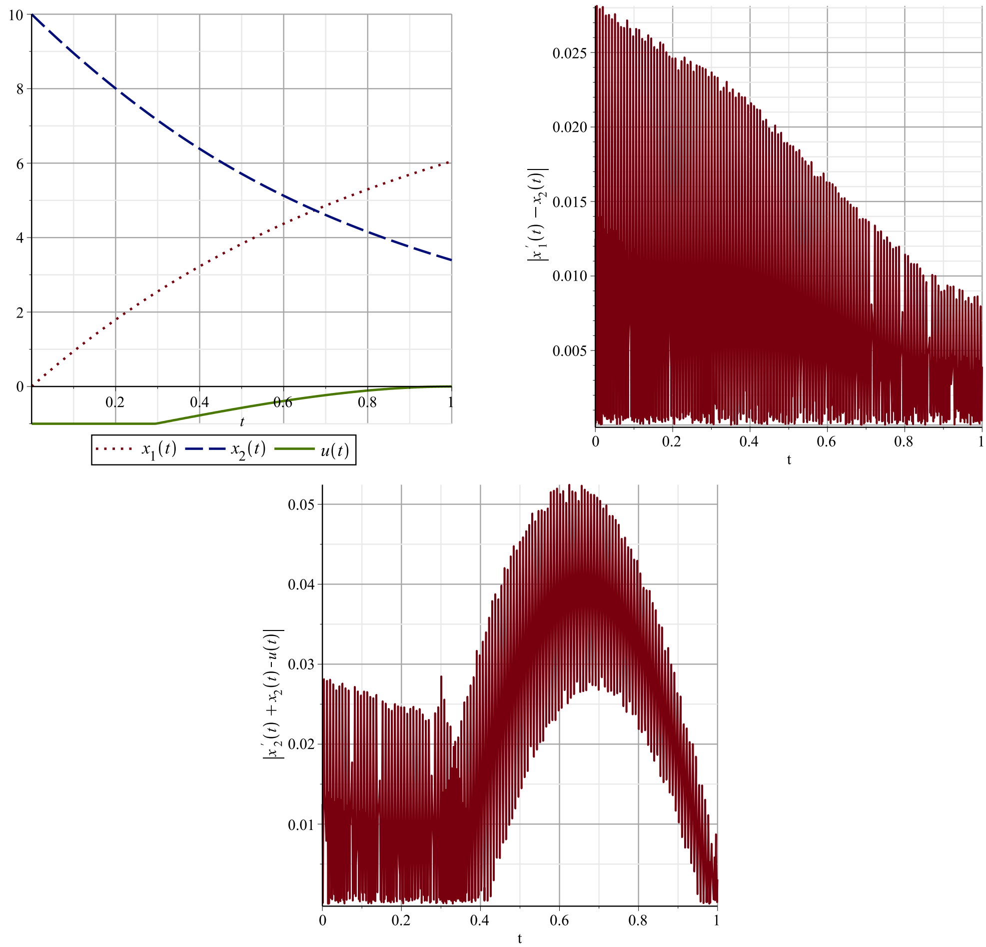

Fig. 1. State and control variables and the constraint errors $|\dot{x}_1(t) - x_2(t)|$ and $|\dot{x}_2(t) + x_2(t) - u(t)|$ for Example 6.1 with $M = 8$.

$$|u(t)| \leqslant 1. \tag{48}$$

This problem can be solved by using Maple 17. In Table 1, parameters are defined as follows: K in [19] is the order of Rationalized Haar functions, N and M_1 in [14] are the order of block-pulse functions and Legendre polynomials, respectively, and N and M in [17] are the order of block-pulse functions and Bernoulli polynomials, respectively. In Figure 1, the control and state variables with the absolute value of constraint's errors, for $M = 8$, are reported.

Methods	J	CPU time
Rationalized Haar functions [19]		
$K = 4$	8.07473	0.389
$K = 8$	8.07065	0.546
Hybrid of block-pulse and Legendre [14]		
$N = 4, M_1 = 3$	8.07059	1.592
$N = 4, M_1 = 4$	8.07056	4.304
Hybrid of block-pulse and Bernoulli [17]		
$N = 4, M = 2$	8.07058	0.858
$N = 4, M = 3$	8.07055	1.155
Present method		
$M = 6$	8.07058	0.780
$M = 7$	8.07056	1.046
$M = 8$	8.07054	1.905
Exact	8.07054	

Tab. 1. Estimated and exact values of J for Example 6.1.

Example 6.2. Consider the Breakwell problem from [6]. The performance index to be minimized is given by

$$J = \frac{1}{2}\int_0^1 u^2(t)\,\mathrm{d}t, \tag{49}$$

subject to the state equations

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = u(t), \tag{50}$$

with the endpoint conditions

$$x_1(0) = x_1(1) = 0, \qquad x_2(0) = -x_2(1) = 1, \tag{51}$$

and the state constraint

$$x_1(t) \leqslant 0.1. \tag{52}$$

The exact solution to this problem is given by

$$u^*(t) = \begin{cases} \frac{200}{9}t - \frac{20}{3}, & t \in [0, 0.3]\,, \\ 0, & t \in [0.3, 0.7]\,, \\ -\frac{200}{9}t + \frac{140}{9}, & t \in [0.7, 1]\,. \end{cases} \tag{53}$$

This example was studied by using pseudospectral method [6] and ChFD scheme [16]. Here we applied the proposed method to solve this problem. The approximate solutions of $x_1(t)$, $x_2(t)$, and $u(t)$, obtained by the B-spline functions with $M = 8$, and the exact solutions together error bounds $|x_1^*(t) - x_1(t)|$, $|x_2^*(t) - x_2(t)|$ and $|u^*(t) - u(t)|$ are plotted in Figure 2. This results show that accuracy of our method in comparison with ChFD scheme [16] whose result are plotted in Figure 3.

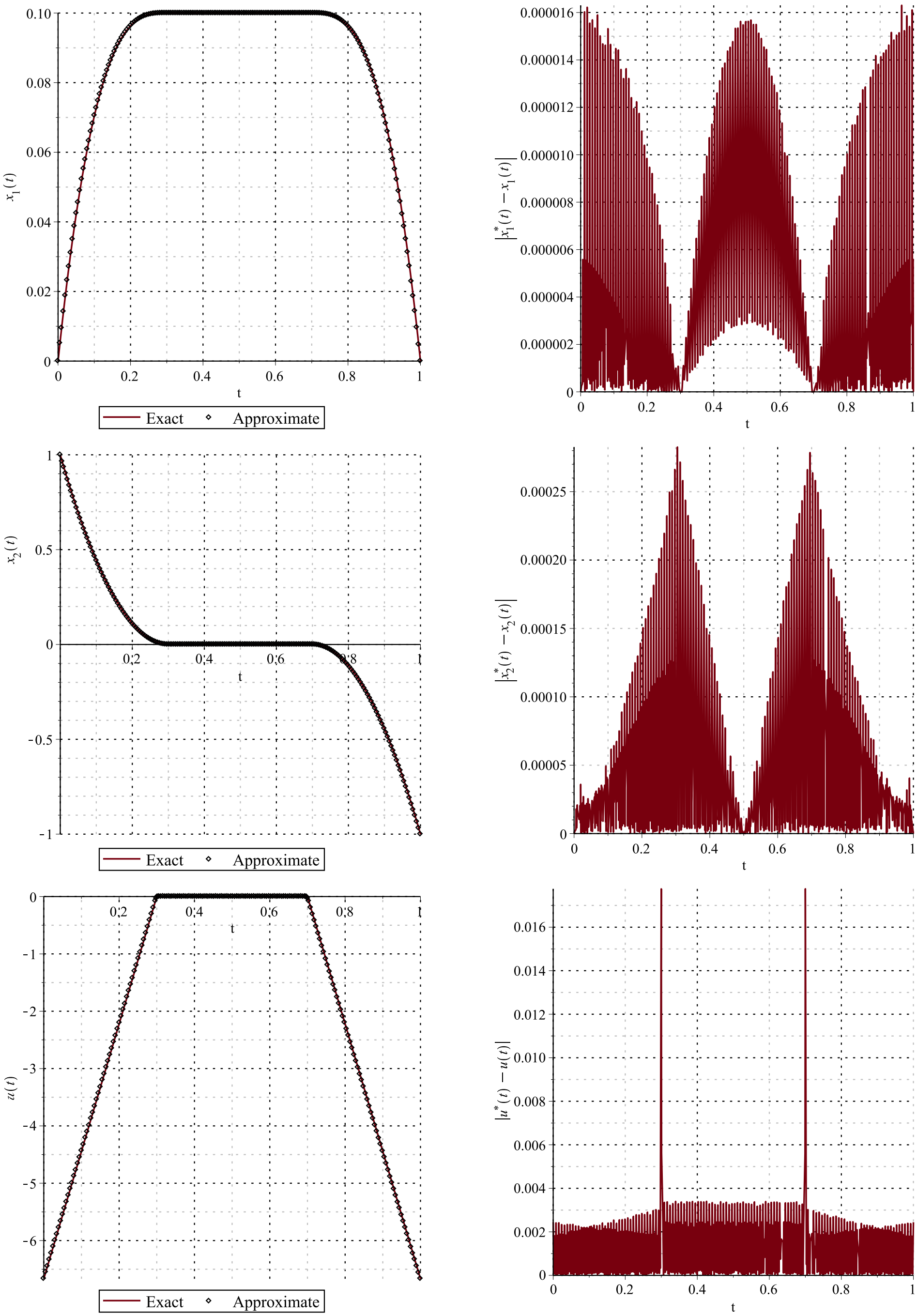

Fig. 2. Exact value and approximation of optimal control and state variables and error bounds using B-spline functions for Example 6.2 with $M = 8$.

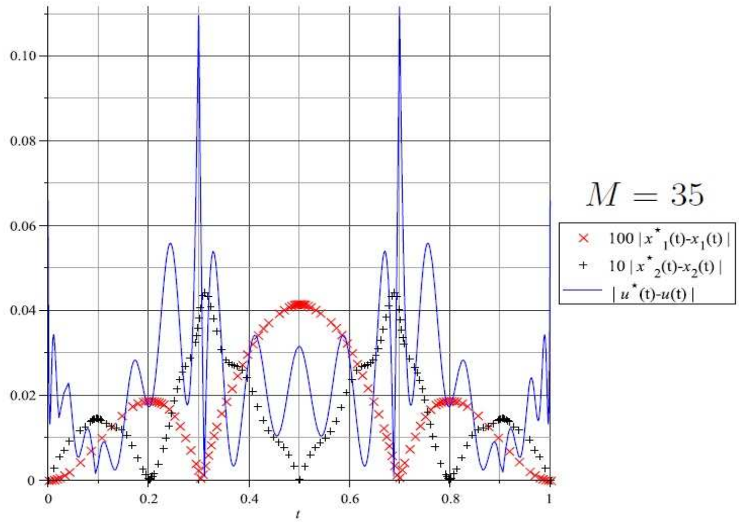

Fig. 3. Exact value and approximation errors of $|x_1^*(t) - x_1(t)|$, $|x_2^*(t) - x_2(t)|$ and $|u^*(t) - u(t)|$ using ChFD scheme [16] for Example 6.2 with $M = 35$.

Example 6.3. This example is studied by using generalized gradient method [18], classical Chebyshev [25], Fourier-based state parametrization [26], rationalized Haar [15], hybrid of block-pulse and Legendre polynomials [14],hybrid of block-pulse and Bernoulli polynomials [17] and interpolating scaling functions [5]. Find the control vector u(t) which minimizes

$$J = \int_0^1 \left(x_1^2(t) + x_2^2(t) + 0.005u^2(t)\right) \,\mathrm{d}t, \tag{54}$$

subject to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{55}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \tag{56}$$

and the following state variable inequality constraint

$$x_2(t) \leqslant r(t), \tag{57}$$

where

$$r(t) = 8(t - 0.5)^2 - 0.5, \quad 0 \leqslant t \leqslant 1.$$

The computational result for $x_2(t)$ for $M = 6$ together with $r(t)$ are given in Figure 4. In Table 2, we compare the minimum of J using the proposed method with other solutions

Methods	J	CPU time
Rationalized Haar functions [15]		
$K = 64, w = 100$	0.170115	1.877
$K = 128, w = 100$	0.170103	1.983
Hybrid of block-pulse and Legendre [14]		
$N = 4, M_1 = 3$	0.17013645	0.951
$N = 4, M_1 = 4$	0.17013640	1.545
Hybrid of block-pulse and Bernoulli [17]		
$N = 4, M = 3$	0.1700305	0.756
$N = 4, M = 4$	0.1700301	0.921
interpolating scaling functions [5]		
$n = 4, r = 5$	0.16982646	2.251
$n = 5, r = 5$	0.16982636	3.175
Present method		
$M = 6$	0.16967511	0.694
$M = 7$	0.16978300	1.151
$M = 8$	0.16981099	1.183

Tab. 2. Results for Example 6.3.

in the literature. In Table 2, parameters are defined as follows: K and w in [15] are the number of terms in Rationalized Haar functions expansion and the number of error repetition for the objective functions, respectively, r in [5] is the order of Legender polynomials and n is the number of terms in Interpolating scaling functions expansion, the other parameters have been defined previously.

Example 6.4. We consider the optimal maneuvers of a rigid asymmetric spacecraft. This example is studied by using quasilinearization and Chebyshev polynomials [8] and hybrid of block-pulse and Bernoulli polynomials [17]. The system state equations are

$$\begin{aligned}
&\dot{x}_1(\tau) = -\frac{I_3 - I_2}{I_1} x_2(\tau)x_3(\tau) + \frac{u_1(\tau)}{I_1},\\
&\dot{x}_2(\tau) = -\frac{I_1 - I_3}{I_2} x_1(\tau)x_3(\tau) + \frac{u_2(\tau)}{I_2},\\
&\dot{x}_3(\tau) = -\frac{I_2 - I_1}{I_3} x_1(\tau)x_2(\tau) + \frac{u_3(\tau)}{I_3},\\
&x_1(\tau) - (5 \times 10^{-6}\tau^2 - 5 \times 10^{-4}\tau + 0.016) \leqslant 0,\\
&x_1(100) = x_2(100) = x_3(100) = 0,
\end{aligned}$$

where $I_1 = 86.24$, $I_2 = 85.07$ and $I_3 = 113.59$. The performance index to be minimized, starting from the initial states $x_1(0) = 0.01$, $x_2(0) = 0.005$ and $x_3(0) = 0.001$ is

$$J = \frac{1}{2}\int_0^{100} \left(u_1^2(\tau) + u_2^2(\tau) + u_3^2(\tau)\right)\,\mathrm{d}\tau.$$

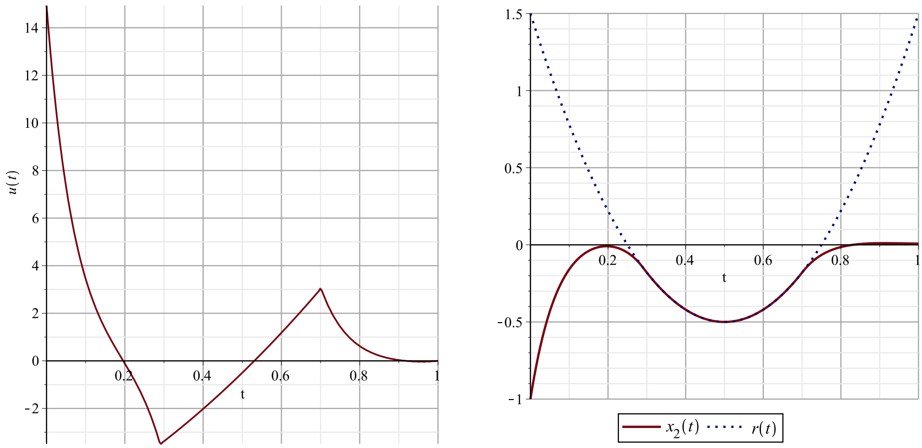

Fig. 4. Control and state variables and constraint boundary for Example 6.3 with $M = 8$.

Methods	J	CPU time
Quasilinearization and Chebyshev polynomials [8]		
$N = 6$	0.00536584	0.07
$N = 8$	0.00534427	0.10
$N = 10$	0.00534063	0.12
Quasilinearization and Chebyshev polynomials [8] with using 2 subintervals		
$N = 10$	0.00530902	0.36
Hybrid of block-pulse and Bernoulli [17]		
$N = 6, M = 3$	0.00531097	1.89
$N = 6, M = 4$	0.00530263	2.12
$N = 6, M = 5$	0.00530213	2.74
Present method		
$M = 6$	0.00530712	0.11
$M = 7$	0.00530812	0.23
$M = 8$	0.00530838	0.35

Tab. 3. Results for Example 6.4.

We use transformation $\tau = 100t, 0 \leqslant t \leqslant 1$, in order to use our proposed method. In Table 3, parameters are defined as follows: N in [8] is the order of Chebyshev series, the other parameters have been defined previously. Optimal control and state variables and constraint boundary, for $M = 8$, are shown in Figure 5.

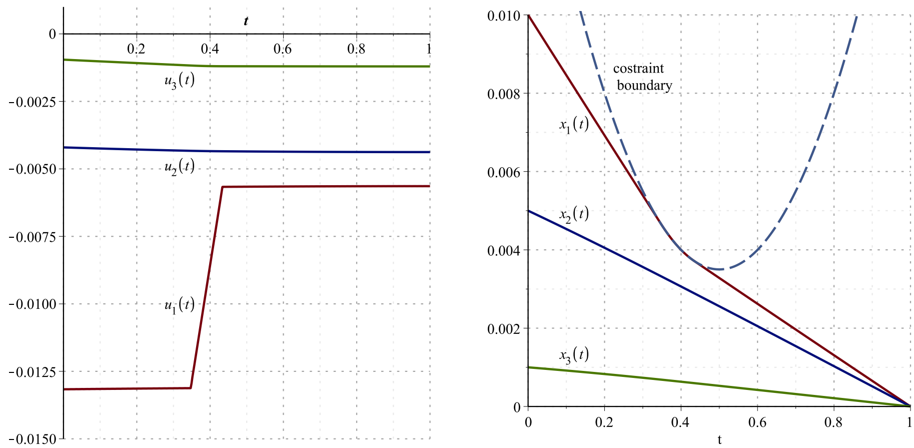

Fig. 5. Control and state variables and constraint boundary for Example 6.4 with $M = 8$.

Methods	J	CPU time
Method of [4]		
$m = 5$	0.5366×10^{-2}	2.589
$m = 7$	0.53614×10^{-2}	2.607
$m = 9$	$0.53610895 \times 10^{-2}$	3.002
$m = 11$	$0.5361102700 \times 10^{-2}$	3.021
Hybrid of block-pulse and Bernoulli [17]		
$N = 2, M = 2$	0.593000×10^{-2}	1.904
$N = 2, M = 3$	0.528915×10^{-2}	2.125
$N = 2, M = 4$	0.521421×10^{-2}	2.305
$N = 2, M = 5$	0.521411×10^{-2}	2.663
Present method		
$M = 5$	0.521184×10^{-2}	1.661
$M = 6$	0.515476×10^{-2}	1.734
$M = 7$	0.515097×10^{-2}	1.909

Tab. 4. Results for Example 6.5.

Example 6.5. Consider the problem of transferring containers from a ship to a cargo truck [24]. The container crane is driven by a hoist motor and a trolley drive motor. The aim is to minimize the swing during and at the end of the transfer. After appropriate normalization, we summarize the problem as follows:

$$J = 4.5 \int_0^1 \left(x_3^2(t) + x_6^2(t)\right) \, \mathrm{d}t,$$

subject to

$$\begin{aligned}
\dot{x}_1(t) &= 9x_4(t),\\
\dot{x}_2(t) &= 9x_5(t),\\
\dot{x}_3(t) &= 9x_6(t),\\
\dot{x}_4(t) &= 9(u_1(t) + 17.2656x_3(t)),\\
\dot{x}_5(t) &= 9u_2(t),\\
\dot{x}_6(t) &= \frac{-9(u_1(t) + 27.0756x_3(t) + 2x_5(t)x_6(t))}{x_2(t)},
\end{aligned}$$

where

$$\begin{aligned}
x(0) &= [0, 22, 0, 0, -1, 0]^T,\\
x(1) &= [10, 14, 0, 2.5, 0, 0]^T,
\end{aligned}$$

and

$$\begin{aligned}
&|u_1(t)| \leqslant 2.83374, \quad t \in [0, 1],\\
&-0.80865 \leqslant u_2(t) \leqslant 0.71265, \quad t \in [0, 1]
\end{aligned}$$

with continuous state inequality constraints,

$$\begin{aligned}
|x_4(t)| &\leqslant 2.5, \quad t \in [0, 1],\\
|x_5(t)| &\leqslant 1.0, \quad t \in [0, 1].
\end{aligned}$$

In Table 4, parameters are defined as follows: m in [4] is the degree of Chebyshev polynomials, the other parameters have been defined previously.

7. CONCLUSION

In this paper we presented a numerical scheme for solving nonlinear constrained quadratic optimal control problems. The method of linear B-spline functions was employed. Also several test problems were used to see the applicability and efficiency of the method. The obtained results show that the new approach can solve the problem effectively.

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