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TRANSFORMATION OF NONLINEAR STATE EQUATIONS INTO THE OBSERVER FORM: NECESSARY AND SUFFICIENT CONDITIONS IN TERMS OF ONE-FORMS

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Necessary and sufficient conditions are given for the existence of state and output transformations, that bring single-input single-output nonlinear state equations into the observer form. The conditions are formulated in terms of differential one-forms, associated with an input-output equation of the system. An algorithm for transformation of the state equations into the observer form is presented and illustrated by an example.

Keywords: nonlinear control system, state and output transformations, observer form, differential one-form

Classification: 93C10, 93B10, 93B17

1. INTRODUCTION

The observer form plays an important role in nonlinear control theory. Once a system is in the observer form, the design of the nonlinear observer with linearizable error dynamics is relatively easy. The earliest methods, relying on a state transformation only (see [2, 5, 10, 15]), provide restrictive conditions for the existence of the observer form for a nonlinear control system. This fact motivates various extensions and generalizations to enlarge the class of systems for which the observer with linear error dynamics can be constructed. In [1, 3, 6, 16], for instance, in addition to a state transformation also an output transformation is allowed. Although both papers [1] and [6] deal with single-output systems, the approaches suggest different observer forms. In [1] the matrix A in the observer form is allowed to depend on control variable u, whereas in [6] the observer form is generalized by allowing input-output injections to depend, besides an input and an output, also on a finite number of derivatives of the input. Multi-output systems were considered in [3, 16]. Other approaches allowing to enlarge the class of systems linearizable by input-output injections are an application of output-dependent time scale transformation [7, 19] and system immersion into a higher order system [12, 18].

The purpose of this paper is to present necessary and sufficient conditions allowing to transform single-input single-output state equations into the observer form via both state and output transformations. The conditions are formulated in terms of an unknown single-variable output dependent function and differential one-forms, directly

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computable from an input-output equation, corresponding to the state equations. Thus, in order to verify the conditions, one has to find the unknown function, which requires integration. Though preliminary results were published in the conference article [13], this paper contains additional contribution. First, considering the special case of third-order systems, we show how the main result can be employed to obtain simpler conditions, independent from the unknown function. Moreover, we provide a comparison of our conditions with those presented earlier in [6]. Finally, in this paper we describe how our results were implemented in *Mathematica*-based package NLControl [20] and its online version [9].

The paper is organized as follows. The preliminary information and formulation of the problem are given in Section 2. Section 3, first, presents a direct formula for computation of necessary differential one-forms and then provides necessary and sufficient solvability conditions. An algorithm for transformation of a system into the observer form and its implementation in *Mathematica*-based package NLControl are described in Section 4. In Section 5 a comparison of our conditions with those of [6] is made. An illustrative example and brief conclusions are provided in Sections 6 and 7, respectively.

2. PRELIMINARIES

Note that throughout the paper we use abridged notation. First, in order to simplify the exposition we leave out the time argument t, so $x := x(t)$. Next, we apply Newton's notation for the first and second time derivatives, i.e., $\dot{x} := \mathrm{d}x/\mathrm{d}t$, $\ddot{x} := \mathrm{d}^2x/\mathrm{d}t^2$, and a more general notation $x^{(k)} := \mathrm{d}^kx/\mathrm{d}t^k$ for a time derivative of an arbitrary order. Moreover, for notational convenience we denote $F' := \partial F/\partial y$ and $F'' := \partial^2 F/\partial y^2$ for a function F dependent on y.

2.1. Problem statement

Consider a nonlinear system, described by the state equations

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x), \end{aligned} \tag{1}$$

where $x(t)$ is an n-dimensional state vector belonging to an open set $\mathcal{X} \subset \mathbb{R}^n$, $u(t)$ is an input belonging to an open set $\mathcal{U} \subset \mathbb{R}$ and $y(t) \in \mathcal{Y} \subset \mathbb{R}$ is an output, $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $h : \mathcal{X} \to \mathcal{Y}$ are assumed to be real analytic functions.

Let $\mathcal{S}$ be an open and dense subset of $\mathcal{X} \times \mathcal{U} \times \mathcal{U}^1 \times \cdots \times \mathcal{U}^{n-2}$, where the rank of the observability matrix is n, i.e., where

$$\operatorname{rank}_{\mathbb{R}} \left[\frac{\partial \left(h, \dot{h}, \ldots, h^{(n-1)} \right)}{\partial x} \right] = n. \tag{2}$$

System (1) is locally, around each point of set $\mathcal{S}$, *(single-experiment) observable*, if condition (2) holds on $\mathcal{S}$. Note that under local observability assumption, one may, by the Implicit Function Theorem, solve the system of equations

$$
\begin{array}{rcl}
y & = & h(x), \\
\dot{y} & = & \dot{h}(x,u), \\
& \vdots & \\
y^{(n-1)} & = & h^{(n-1)}\left(x,u,\dot{u},\ldots,u^{(n-2)}\right)
\end{array}
\tag{3}
$$

locally on $\mathcal{S}$ for $x = \gamma\left(y,\dot{y},\ldots,y^{(n-1)},u,\dot{u},\ldots,u^{(n-2)}\right)$. By substituting the solution into

$$
y^{(n)} = h^{(n)}\left(x,u,\dot{u},\ldots,u^{(n-1)}\right)
$$

one may find the input-output (i/o) equation of (1)

$$
y^{(n)} = \phi\left(y,\dot{y},\ldots,y^{(n-1)},u,\dot{u},\ldots,u^{(n-1)}\right),
\tag{4}
$$

that is valid locally on $\tilde{\mathcal{S}}$, being the image of the map (3), extended by n indentity maps $u^{(k)} = u^{(k)}$, $k = 0,\ldots,n-1$.

Our purpose is to find conditions under which there exist an open and dense subsets $\tilde{\mathcal{X}} \subset \mathcal{X}$ and $\tilde{\mathcal{Y}} \subset \mathcal{Y}$ such that around each $x \in \tilde{\mathcal{X}}$ there exists a local state transformation (i. e., real analytic diffeomorphism) $\psi : N(x) \to M(\psi(x))$ defined by

$$
z = \psi(x)
\tag{5}
$$

and around each $y \in \tilde{\mathcal{Y}}$ there exists a local output transformation (i. e., real analytic diffeomorphism) $\Psi : N(y) \to M(\Psi(y))$, defined by

$$
Y = \Psi(y),
\tag{6}
$$

such that in new state and output coordinates the state equations (1) are locally, on $M(\psi(x)) \times \mathcal{U} \times M(\Psi(y))$, in the observer form

$$
\begin{array}{rcl}
\dot{z}_1 & = & z_2 + \varphi_1(Y,u) \\
& \vdots & \\
\dot{z}_{n-1} & = & z_n + \varphi_{n-1}(Y,u) \\
\dot{z}_n & = & \varphi_n(Y,u) \\
Y & = & z_1.
\end{array}
\tag{7}
$$

Note that the state equations (1) can be transformed locally into the observer form (7) with the state transformation (5) and output transformation (6), if the i/o equation (4), corresponding to (1), is locally, on some open and dense subset $\hat{\mathcal{S}}$ of $\hat{\mathcal{Y}} \times \hat{\mathcal{Y}}^1 \times \cdots \times \hat{\mathcal{Y}}^{n-1} \times \mathcal{U} \times \mathcal{U}^1 \times \cdots \times \mathcal{U}^{n-1}$, transformable into the form

$$
Y^{(n)} = \left(\varphi_1\left(Y,u\right)\right)^{(n-1)} + \cdots + \left(\varphi_{n-1}\left(Y,u\right)\right)^{(1)} + \varphi_n\left(Y,u\right)
\tag{8}
$$

via the output transformation (6).

Indeed, if (8) holds, one can define new state variables as

$$\begin{aligned} z_1 &= Y, \\ z_2 &= \dot{Y} - \varphi_1, \\ z_3 &= \ddot{Y} - \dot{\varphi}_1 - \varphi_2, \\ &\vdots \\ z_n &= Y^{(n-1)} - \varphi_1^{(n-2)} - \cdots - \dot{\varphi}_{n-2} - \varphi_{n-1}, \end{aligned} \tag{9}$$

yielding state equations in the observer form (7). Note that, using the output transformation (6) and the equations (3), one can substitute the variables $Y, \dot{Y}, \ldots, Y^{(n-1)}$ in such a way that the right-hand side of equation (9) depends only on x, meaning that (9) is the state transformation (5).

2.2. Algebraic framework

Below we give a brief exposition of the linear algebraic approach based on differential forms [5]. Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of independent system variables from the infinite set $\mathcal{C} = \left\{y, \dot{y}, \ldots, y^{(n-1)}; u^{(k)}, k \geq 0\right\}$. For $F\left(y, \dot{y}, \ldots, y^{(n-1)}, u, \ldots, u^{(k)}\right) \in \mathcal{K}$ a *time derivative operator* $\mathrm{d}/\mathrm{d}t : \mathcal{K} \to \mathcal{K}$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\cdot) := \sum_{l=0}^{n-1} \frac{\partial F(\cdot)}{\partial y^{(l)}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} y^{(l)} + \sum_{k \geq 0} \frac{\partial F(\cdot)}{\partial u^{(k)}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} u^{(k)},$$

where $(\mathrm{d}/\mathrm{d}t)\, y^{(l)} = y^{(l+1)}$, $l = 0, \ldots, n-2$; $(\mathrm{d}/\mathrm{d}t)\, u^{(k)} = u^{(k+1)}$, $k \geq 0$ and, according to (4), $(\mathrm{d}/\mathrm{d}t)\, y^{(n-1)} := \phi(\cdot)$. The pair $(\mathcal{K}, \mathrm{d}/\mathrm{d}t)$ is a *differential field.*

Consider next the infinite set of symbols $\mathrm{d}\mathcal{C} = \left\{\mathrm{d}y, \mathrm{d}\dot{y}, \ldots, \mathrm{d}y^{(n-1)}; \mathrm{d}u^{(k)}, k \geq 0\right\}$ and denote by $\mathcal{E}$ the vector space spanned over the field $\mathcal{K}$ by elements of $\mathrm{d}\mathcal{C}$, namely $\mathcal{E} := \mathrm{span}_{\mathcal{K}} \mathrm{d}\mathcal{C}$. A *differential operator* d brings an element from $\mathcal{K}$ to $\mathcal{E}$:

$$\mathrm{d}F(\cdot) := \sum_{l=0}^{n-1} \frac{\partial F(\cdot)}{\partial y^{(l)}} \mathrm{d}y^{(l)} + \sum_{k \geq 0} \frac{\partial F(\cdot)}{\partial u^{(k)}} \mathrm{d}u^{(k)}.$$

Any element of $\mathcal{E}$, called a *differential one-form*, is not necessarily a differential of a function and has the form

$$\omega = \sum_{l=0}^{n-1} A_l \mathrm{d}y^{(l)} + \sum_{k \geq 0} B_k \mathrm{d}y^{(k)},$$

where $A_l, B_k \in \mathcal{K}$ and only a finite number of coefficients B_k are nonzero. For a one-form $\omega = \sum_i \alpha_i \mathrm{d}\xi_i \in \mathcal{E}$ the time derivative operator $\mathrm{d}/\mathrm{d}t : \mathcal{E} \to \mathcal{E}$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} \omega := \sum_i \left(\dot{\alpha}_i \mathrm{d}\xi_i + \alpha_i \mathrm{d}\dot{\xi}_i\right).$$

Note that the operators d and $\mathrm{d}/\mathrm{d}t$ commute, i. e., $\mathrm{d}/\mathrm{d}t\,(\mathrm{d}\alpha) = \mathrm{d}\,(\mathrm{d}/\mathrm{d}t\alpha) = \mathrm{d}\dot{\alpha}$ for $\alpha \in \mathcal{K}$. The rth time derivative of an arbitrary one-form may be computed as

$$\omega^{(r)} = \sum_{q=0}^{r} C_r^q \sum_i \alpha_i^{(r-q)} \mathrm{d}\xi_i^{(q)}, \tag{10}$$

where C_r^q is the binomial coefficient, i. e., $C_r^q := \frac{r!}{(r-q)!q!}$.

Starting from the space $\mathcal{E}$ it is possible to build up structures used in exterior differential calculus. We refer to [5] for details, whereas here we just recall some basic notions. Define the set $\wedge \mathrm{d}\mathcal{C} := \{\mathrm{d}\zeta \wedge \mathrm{d}\eta \mid \zeta, \eta \in \mathcal{C}\}$, where $\wedge$ denotes the *wedge product* with the standard properties $\mathrm{d}\zeta \wedge \mathrm{d}\eta = -\mathrm{d}\eta \wedge \mathrm{d}\zeta$ and $\mathrm{d}\zeta \wedge \mathrm{d}\zeta = 0$ for $\zeta, \eta \in \mathcal{C}$. Introduce the space $\mathcal{E}^2 := \mathrm{span}_{\mathcal{K}} \wedge \mathrm{d}\mathcal{C}$ of two-forms. An operator $\mathrm{d} : \mathcal{E} \to \mathcal{E}^2$, called an *exterior derivative operator*, is defined for $\omega = \sum_{l=1}^{k} \alpha_l\,(\zeta_1, \ldots, \zeta_k)\,\mathrm{d}\zeta_l \in \mathcal{E}$, where $\zeta_1, \ldots, \zeta_k \in \mathcal{C}$, by the rule $\mathrm{d}\omega := \sum_{l,\bar{l}} (\partial \alpha_l / \partial \zeta_{\bar{l}})\,\mathrm{d}\zeta_l \wedge \mathrm{d}\zeta_{\bar{l}}$. The notion of two-form is generalized to an s-form and the wedge product is defined for arbitrary s-forms.

We say that $\omega \in \mathcal{E}$ is an *exact* one-form, if $\omega = \mathrm{d}\alpha$ for some real analytic function α. A one-form ω for which $\mathrm{d}\omega = 0$ is said to be *closed.* Every exact one-form is closed, but the converse holds only locally, see [5].

2.3. Solution for the case without output transformation

In this subsection we recall from [5] the step-by-step algorithm which provides the differential one-forms ω_i, $i = 1, \ldots, n$, leading to necessary and sufficient solvability conditions for the case when the output transformation Ψ is an identity. In Section 3 the one-forms ω_i will be employed to formulate the main result of this paper. Note, however, that we present the algorithm with slight modifications. The main difference is in skipping the step where the integration of the one-forms is required. Unlike [5], in (11a) instead of functions we use one-forms, which are not required to be integrable.

Consider first one-forms

$$P_i := \sum_{q=0}^{n-1} A_i^q \mathrm{d}y^{(q)} + \sum_{q=0}^{n-1} B_i^q \mathrm{d}u^{(q)}, \quad i = 1, \ldots, n,$$

whose coefficients A_i^q and B_i^q can be found by setting $P_1 = \mathrm{d}\phi$ and then computing recursively, for $i = 1, \ldots, n-1$

$$P_{i+1} := P_i - \omega_i^{(n-i)}, \tag{11a}$$

where $\omega_i^{(n-i)}$ denotes the $(n-i)$th derivative of the one-form ω_i, defined by

$$\omega_i := A_i^{n-i} \mathrm{d}y + B_i^{n-i} \mathrm{d}u, \quad i = 1, \ldots, n. \tag{11b}$$

3. MAIN RESULT

The proposition below provides a supplementary result, which is a direct formula for computation of the one-forms[1] ω_i, $i = 1, \ldots, n$. Hereinafter this formula will be used to

[1] Alternatively the one-forms ω_i can be computed using the approach based on the notion of adjoint polynomial [8].

prove the main result, that is Theorem 3.4.

Proposition 3.1. The one-forms ω_i, $i = 1, \ldots, n$ in (11) can be computed directly from the formula

$$\omega_i = \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^j \left[\left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j)} \mathrm{d}y + \left(\frac{\partial \phi}{\partial u^{(n-i+j)}} \right)^{(j)} \mathrm{d}u \right]. \tag{12}$$

The proof of Proposition 3.1 is given in the Appendix.

Under necessary and sufficient conditions of Theorem 3.4 below one may transform the i/o equation (4) into the form (8) and, as a consequence, the state equations (1) into the observer form (7). In order to prove Theorem 3.4, we need the following proposition and lemma.

Proposition 3.2. ([14]) Assume that $\hat{f}(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))$ is a composite function for which derivatives up to order $a + b$ are defined; then

$$\frac{\partial \left(\hat{f}(\xi_1(t), \xi_2(t), \ldots, \xi_r(t)) \right)^{(a+b)}}{\partial \xi_l^{(a)}(t)} = C_{a+b}^b \left(\frac{\partial \hat{f}(\xi_1(t), \xi_2(t), \ldots, \xi_r(t))}{\partial \xi_l(t)} \right)^{(b)},$$

where $l = 1, 2, \ldots, r$ and a, b are nonnegative integers.

Lemma 3.3.

(i) $\displaystyle\sum_{j=1}^{\varsigma} (-1)^{j-1} C_{\varsigma}^{j-1} = (-1)^{\varsigma-1}$ for $\varsigma \geq 1$,

(ii) $\displaystyle\sum_{j=1}^{\varsigma-s+1} (-1)^{j-1} C_{\varsigma-s}^{j-1} = 0$ for $s = 1, \ldots, \varsigma - 1$ and $\varsigma \geq 2$.

The proof of Lemma 3.3 is given in the Appendix.

Furthermore, denote the composite function of $\varphi_s(Y, u)$ and $\Psi(y)$ for $s = 1, \ldots, n$ as

$$\bar{\varphi}_s(y, u) := \varphi_s(\Psi(y), u). \tag{13}$$

Theorem 3.4. The system (1) that is locally observable on S can be locally transformed by the state transformation (5) and the output transformation (6) into the observer form (7) if and only if there exists a function $\lambda(y)$, such that for $\varsigma = 1, \ldots, n$ the one-forms

$$(-1)^{\varsigma-1} C_n^{\varsigma} \lambda^{(\varsigma)} \mathrm{d}y + \sum_{i=1}^{\varsigma} (-1)^{\varsigma-i} C_{n-i}^{\varsigma-i} \lambda^{(\varsigma-i)} \omega_i, \tag{14}$$

where ω_i's are defined by (12), are closed.

Proof. *Necessity:* Under the local observability assumption one can find on the set $\tilde{S}$ the i/o equation (4), corresponding to (1). Assume that system (1) is locally transformable into the form (7). Consequently, the i/o equation (4) can be rewritten in the form (8). Complete the following steps:

- Take the partial derivatives of both sides of the i/o equation (8) with respect to $y^{(n-\varsigma+j-1)}$, for $j = 1, \ldots, \varsigma$.
- Next, take the $(j-1)$th time derivative of each expression, obtained in the previous step.
- Denote
$$\alpha_j := (-1)^{j-1} C_{n-\varsigma+j-1}^{j-1} \tag{15}$$
and multiply by α_j both sides of the equalities obtained on the previous step.
- Sum up the obtained equalities over $j = 1, \ldots, \varsigma$.

Repeat the same steps with respect to the control variable u. As a result, one obtains the equalities
$$LY = RY \text{ and } LU = RU, \tag{16}$$
where
$$LY := \sum_{j=1}^{\varsigma} \alpha_j \left(\frac{\partial \Psi^{(n)}}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)}, \qquad RY := \sum_{j=1}^{\varsigma} \sum_{s=1}^{n} \alpha_j \left(\frac{\partial \bar{\varphi}_s^{(n-s)}}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)},$$
$$LU := \sum_{j=1}^{\varsigma} \alpha_j \left(\frac{\partial \Psi^{(n)}}{\partial u^{(n-\varsigma+j-1)}} \right)^{(j-1)}, \qquad RU := \sum_{j=1}^{\varsigma} \sum_{s=1}^{n} \alpha_j \left(\frac{\partial \bar{\varphi}_s^{(n-s)}}{\partial u^{(n-\varsigma+j-1)}} \right)^{(j-1)}.$$

Note that $\Psi^{(n)}$ in LY and LU depends, besides other arguments, on $y^{(n)}$ which, according to (4), must be replaced by the function ϕ. In order to take this replacement into account, consider the explicit formula of the nth derivative of the output transformation $\Psi(y)$, which, according to Faà di Bruno's Formula [11], reads as
$$\Psi^{(n)} = \sum \frac{n!}{k_1! \cdots k_n!} \Psi^{\overline{K}} \prod_{\iota=1}^{n} \left(\frac{y^{(\iota)}}{\iota!} \right)^{k_\iota}, \tag{17}$$
where $\overline{K} = k_1 + \cdots + k_n$ denotes the order of derivative with respect to y and the sum is taken over all possible different sets of nonnegative integers $k_1, \ldots, k_n$ being the solutions of the Diophantine equation $k_1 + 2k_2 + \cdots + nk_n = n$. It is easy to observe that $y^{(n)}$ appears in (17) only in the term defined by $\iota = n$ and $k_n = 1$. In this case $k_1 = \cdots = k_{n-1} = 0$ and the corresponding addend of the sum is $\Psi' y^{(n)}$. In order to take the replacement (4) into account and avoid complications in further transformations of $\Psi^{(n)}$, we add to LY a formal zero term, such that LY now reads as
$$LY = \sum_{j=1}^{\varsigma} \alpha_j \left(\left(\frac{\partial \Psi^{(n)}}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)} + \left(\frac{\partial \left(\Psi' \phi - \Psi' y^{(n)} \right)}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)} \right),$$

where in $\Psi^{(n)}$ we consider $y^{(n)}$ as a symbol which we do not have to replace. This trick simplifies the proof below by allowing to use Proposition 3.2.

By Proposition 3.2 for $r = 1$, $a = n - \varsigma + j - 1$ and $b = \varsigma - j + 1$,

$$\frac{\partial \Psi^{(n)}}{\partial y^{(n-\varsigma+j-1)}} = C_n^{\varsigma-j+1}(\Psi')^{(\varsigma-j+1)},$$

yielding

$$LY = \sum_{j=1}^{\varsigma} \alpha_j \left(C_n^{\varsigma-j+1}(\Psi')^{(\varsigma)} + \left(\frac{\partial\left(\Psi'\phi - \Psi' y^{(n)}\right)}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)} \right).$$

Using the product rule to find the derivative with respect to $y^{(n-\varsigma+j-1)}$, one can write

$$\frac{\partial\left(\Psi'\phi - \Psi' y^{(n)}\right)}{\partial y^{(n-\varsigma+j-1)}} = \Psi' \left(\frac{\partial \phi}{\partial y^{(n-\varsigma+j-1)}} - \frac{\partial y^{(n)}}{\partial y^{(n-\varsigma+j-1)}} \right) + \left(\phi - y^{(n)}\right) \frac{\partial \Psi'}{\partial y^{(n-\varsigma+j-1)}}.$$

Since $n - \varsigma + j - 1 < n$ for $\varsigma = 1, \ldots, n$ and $j = 1, \ldots, \varsigma$, then $\frac{\partial y^{(n)}}{\partial y^{(n-\varsigma+j-1)}} = 0$. Also taking into account that $y^{(n)} = \phi$, one obtains

$$\frac{\partial\left(\Psi'\phi - \Psi' y^{(n)}\right)}{\partial y^{(n-\varsigma+j-1)}} = \Psi' \frac{\partial \phi}{\partial y^{(n-\varsigma+j-1)}},$$

which, using the Leibniz formula for the higher order derivative of the product, yields

$$\left(\frac{\partial\left(\Psi'\phi - \Psi' y^{(n)}\right)}{\partial y^{(n-\varsigma+j-1)}} \right)^{(j-1)} = \sum_{i=0}^{j-1} C_{j-1}^{i} (\Psi')^{(j-1-i)} \left(\frac{\partial \phi}{\partial y^{(n-\varsigma+j-1)}} \right)^{(i)}.$$

Thus, LY can be rewritten as follows

$$LY = \sum_{j=1}^{\varsigma} \alpha_j C_n^{\varsigma-j+1} (\Psi')^{(\varsigma)} + \sum_{j=1}^{\varsigma} \sum_{i=0}^{j-1} \alpha_j C_{j-1}^{i} (\Psi')^{(j-1-i)} \left(\frac{\partial \phi}{\partial y^{(n-\varsigma+j-1)}} \right)^{(i)}.$$

Changing the summation order $\sum_{j=1}^{\varsigma} \sum_{i=0}^{j-1} a_{j,i} = \sum_{i=1}^{\varsigma} \sum_{j=0}^{i-1} a_{\varsigma-i+j+1,j}$ one obtains

$$LY = \sum_{j=1}^{\varsigma} \alpha_j C_n^{\varsigma-j+1} (\Psi')^{(\varsigma)} + \sum_{i=1}^{\varsigma} \sum_{j=0}^{i-1} \alpha_{\varsigma-i+j+1} C_{\varsigma-i+j}^{j} (\Psi')^{(\varsigma-i)} \left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j)}.$$

Using (15) and taking into account that $(-1)^{\varsigma-i+j} = (-1)^{\varsigma-i}(-1)^j$ and that by direct computations $C_{n-\varsigma+j-1}^{j-1} C_n^{\varsigma-j+1} = C_n^{\varsigma} C_{\varsigma}^{j-1}$ and $C_{n-i+j}^{\varsigma-i+j} C_{\varsigma-i+j}^{j} = C_{n-i}^{\varsigma-i} C_{n-i+j}^{j}$ we obtain

$$\begin{aligned} LY = {} & C_n^{\varsigma} (\Psi')^{(\varsigma)} \sum_{j=1}^{\varsigma} (-1)^{j-1} C_{\varsigma}^{j-1} \\ & + \sum_{i=1}^{\varsigma} (-1)^{\varsigma-i} C_{n-i}^{\varsigma-i} (\Psi')^{(\varsigma-i)} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} \left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j)}. \end{aligned}$$

Applying (i) of Lemma 3.3 we obtain

$$LY = (-1)^{\varsigma-1} C_n^{\varsigma} (\Psi')^{(\varsigma)} + \sum_{i=1}^{\varsigma} (-1)^{\varsigma-i} C_{n-i}^{\varsigma-i} (\Psi')^{(\varsigma-i)} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} \left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j)} .$$

Since LY and LU have a similar structure, the transformations made with LY can be made also with LU, yielding

$$LU = (-1)^{\varsigma-1} C_n^{\varsigma} \left(\frac{\partial \Psi}{\partial u} \right)^{(\varsigma)} + \sum_{i=1}^{\varsigma} (-1)^{\varsigma-i} C_{n-i}^{\varsigma-i} (\Psi')^{(\varsigma-i)} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} \left(\frac{\partial \phi}{\partial u^{(n-i+j)}} \right)^{(j)} ,$$

which, taking into account that $\frac{\partial \Psi}{\partial u} = 0$, yields

$$LU = \sum_{i=1}^{\varsigma} (-1)^{\varsigma-i} C_{n-i}^{\varsigma-i} (\Psi')^{(\varsigma-i)} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} \left(\frac{\partial \phi}{\partial u^{(n-i+j)}} \right)^{(j)} .$$

Next, consider RY. Note that, if $s > \varsigma - j + 1$, then $n - s < n - \varsigma + j - 1$ and so $\frac{\partial \bar{\varphi}_s^{(n-s)}}{\partial y^{(n-\varsigma+j-1)}} = 0$. Therefore, instead of taking $s = 1, \ldots, n$ we can take $s = 1, \ldots, \varsigma - j + 1$. Moreover, by Proposition 3.2 for $r = 2$, $a = n - \varsigma + j - 1$ and $b = \varsigma - s - j + 1$

$$\frac{\partial \bar{\varphi}_s^{(n-s)}}{\partial y^{(n-\varsigma+j-1)}} = C_{n-s}^{\varsigma-s-j+1} \left(\frac{\partial \bar{\varphi}_s}{\partial y} \right)^{(\varsigma-s-j+1)} .$$

Thus, one can write

$$RY = \sum_{j=1}^{\varsigma} \sum_{s=1}^{\varsigma-j+1} \alpha_j C_{n-s}^{\varsigma-s-j+1} \left(\frac{\partial \bar{\varphi}_s}{\partial y} \right)^{(\varsigma-s)} .$$

Changing the summation order $\sum_{j=1}^{\varsigma} \sum_{s=1}^{\varsigma-j+1} a_{j,s} = \sum_{s=1}^{\varsigma} \sum_{j=1}^{\varsigma-s+1} a_{j,s}$, applying (15) and taking into account that by direct computations $C_{n-\varsigma+j-1}^{j-1} C_{n-s}^{\varsigma-s-j+1} = C_{n-s}^{\varsigma-s} C_{\varsigma-s}^{j-1}$, one obtains

$$RY = \sum_{s=1}^{\varsigma} C_{n-s}^{\varsigma-s} \left(\frac{\partial \bar{\varphi}_s}{\partial y} \right)^{(\varsigma-s)} \sum_{j=1}^{\varsigma-s+1} (-1)^{j-1} C_{\varsigma-s}^{j-1}.$$

Note that for $\varsigma = 1$ $RY = \frac{\partial \bar{\varphi}_1}{\partial y}$. In case $\varsigma \geq 2$, one can separate the last addend of the sum RY, yielding

$$RY = \frac{\partial \bar{\varphi}_\varsigma}{\partial y} + \sum_{s=1}^{\varsigma-1} C_{n-s}^{\varsigma-s} \left(\frac{\partial \bar{\varphi}_s}{\partial y} \right)^{(\varsigma-s)} \sum_{j=1}^{\varsigma-s+1} (-1)^{j-1} C_{\varsigma-s}^{j-1}.$$

By (ii) of Lemma 3.3, $RY = \frac{\partial \bar{\varphi}_\varsigma}{\partial y}$. In the same manner we get $RU = \frac{\partial \bar{\varphi}_\varsigma}{\partial u}$, for $\varsigma = 1, \ldots, n$.

As a result, (16) can be rewritten as

$$\frac{\partial\bar{\varphi}_\varsigma}{\partial y} = (-1)^{\varsigma-1}C_n^\varsigma(\Psi')^{(\varsigma)} + \sum_{i=1}^{\varsigma}(-1)^{\varsigma-i}C_{n-i}^{\varsigma-i}(\Psi')^{(\varsigma-i)}\sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^j\left(\frac{\partial\phi}{\partial y^{(n-i+j)}}\right)^{(j)},$$

$$\frac{\partial\bar{\varphi}_\varsigma}{\partial u} = \sum_{i=1}^{\varsigma}(-1)^{\varsigma-i}C_{n-i}^{\varsigma-i}(\Psi')^{(\varsigma-i)}\sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^j\left(\frac{\partial\phi}{\partial u^{(n-i+j)}}\right)^{(j)}.$$

Adding together the above equalities and taking into account (12) and the notation

$$\lambda := \Psi', \tag{18}$$

we finally obtain the closed differential one-forms

$$\mathrm{d}\bar{\varphi}_\varsigma = \frac{\partial\bar{\varphi}_\varsigma}{\partial y}\mathrm{d}y + \frac{\partial\bar{\varphi}_\varsigma}{\partial u}\mathrm{d}u = (-1)^{\varsigma-1}C_n^\varsigma\lambda^{(\varsigma)}\mathrm{d}y + \sum_{i=1}^{\varsigma}(-1)^{\varsigma-i}C_{n-i}^{\varsigma-i}\lambda^{(\varsigma-i)}\omega_i. \tag{19}$$

Obviously the right-hand side of equality (19) equals (14).

Sufficiency: If there exists a function $\lambda(y)$, such that the one-forms (14), where ω_i's are defined by (12), are closed, then the function $\Psi(y)$ for the output transformation (6) can be calculated as an integral

$$\Psi(y) = \int\lambda(y)\mathrm{d}y. \tag{20}$$

Since the one-forms (14) are closed, locally there exist functions $\bar{\varphi}_\varsigma$ for $\varsigma = 1,\ldots,n$, satisfying (19). Integration of the one-forms allows to find the corresponding functions $\bar{\varphi}_\varsigma$. By means of the functions $\Psi(y)$ and $\bar{\varphi}_\varsigma$ state equations in the observer form (7) can be easily constructed. □

3.1. Simple conditions for the third-order systems

This subsection is intended to show how Theorem 3.4 can be employed to obtain simpler conditions, formulated in terms of partial derivatives such that the verification of the conditions does not depend on an unknown function λ. Here we consider only the case $n = 3$, whereas derivation of similar conditions for an arbitrary n will make the subject for future research.

Denoting the one-forms (14) as

$$\widetilde{\omega}_\varsigma := (-1)^{\varsigma-1}C_n^\varsigma\lambda^{(\varsigma)}\mathrm{d}y + \sum_{i=1}^{\varsigma}(-1)^{\varsigma-i}C_{n-i}^{\varsigma-i}\lambda^{(\varsigma-i)}\omega_i$$

and keeping in mind that a one-form is closed if its exterior derivative is zero, one can represent the conditions of Theorem 3.4 for $n = 3$ as

$$\mathrm{d}\widetilde{\omega}_1 = 0, \quad \mathrm{d}\widetilde{\omega}_2 = 0, \quad \mathrm{d}\widetilde{\omega}_3 = 0.$$

Taking into account that the exterior and time derivative operators commute, one can observe that the conditions above are equivalent to

$$\mathrm{d}\widetilde{\omega}_1 = 0, \quad \mathrm{d}\widetilde{\omega}_2 + 2\mathrm{d}\dot{\widetilde{\omega}}_1 = 0, \quad \mathrm{d}\widetilde{\omega}_3 + \mathrm{d}\dot{\widetilde{\omega}}_2 + \mathrm{d}\ddot{\widetilde{\omega}}_1 = 0. \tag{21}$$

Performing tedious but otherwise direct computations, one obtains

$$\begin{aligned}\mathrm{d}\widetilde{\omega}_1 = &\left(3\lambda' + \lambda\frac{\partial^2\phi}{\partial\dot{y}\partial\ddot{y}}\right)\mathrm{d}\dot{y}\wedge\mathrm{d}y + \lambda\frac{\partial^2\phi}{\partial\ddot{y}\partial\ddot{y}}\mathrm{d}\ddot{y}\wedge\mathrm{d}y\\ &+\lambda\sum_{i=1}^{2}\left(\frac{\partial^2\phi}{\partial u^{(i)}\partial\ddot{y}}\mathrm{d}u^{(i)}\wedge\mathrm{d}y + \frac{\partial^2\phi}{\partial y^{(i)}\partial\ddot{u}}\mathrm{d}y^{(i)}\wedge\mathrm{d}u + \frac{\partial^2\phi}{\partial u^{(i)}\partial\ddot{u}}\mathrm{d}u^{(i)}\wedge\mathrm{d}u\right)\\ &+\left(\lambda'\frac{\partial\phi}{\partial\ddot{u}} + \lambda\left(\frac{\partial^2\phi}{\partial y\partial\ddot{u}} - \frac{\partial^2\phi}{\partial u\partial\ddot{y}}\right)\right)\mathrm{d}y\wedge\mathrm{d}u,\end{aligned} \tag{22a}$$

$$\begin{aligned}\mathrm{d}\widetilde{\omega}_2 + 2\mathrm{d}\dot{\widetilde{\omega}}_1 = &\left(3\lambda' + \lambda\frac{\partial^2\phi}{\partial\dot{y}\partial\ddot{y}}\right)\mathrm{d}\ddot{y}\wedge\mathrm{d}y + 2\lambda\frac{\partial^2\phi}{\partial\ddot{y}\partial\ddot{y}}\mathrm{d}\ddot{y}\wedge\mathrm{d}\dot{y}\\ &+\lambda\sum_{i=1}^{2} i\left(\frac{\partial^2\phi}{\partial u^{(i)}\partial\ddot{y}}\mathrm{d}\ddot{y}\wedge\mathrm{d}u^{(i-1)} + \frac{\partial^2\phi}{\partial y^{(i)}\partial\ddot{u}}\mathrm{d}\ddot{u}\wedge\mathrm{d}y^{(i-1)} + \frac{\partial^2\phi}{\partial u^{(i)}\partial\ddot{u}}\mathrm{d}\ddot{u}\wedge\mathrm{d}u^{(i-1)}\right)\\ &+2\lambda\left(\frac{\partial^2\phi}{\partial\ddot{u}\partial\dot{y}} - \frac{\partial^2\phi}{\partial\dot{u}\partial\ddot{y}}\right)\mathrm{d}\dot{y}\wedge\mathrm{d}\dot{u} - \left(2\lambda'\frac{\partial\phi}{\partial\ddot{u}} + \lambda\left(2\frac{\partial^2\phi}{\partial y\partial\ddot{u}} - \frac{\partial^2\phi}{\partial\dot{u}\partial\dot{y}}\right)\right)\mathrm{d}\dot{u}\wedge\mathrm{d}y\\ &+\lambda\left(\frac{\partial^2\phi}{\partial\dot{u}\partial\dot{u}} - 2\frac{\partial^2\phi}{\partial u\partial\ddot{u}}\right)\mathrm{d}\dot{u}\wedge\mathrm{d}u + \left(\lambda'\frac{\partial\phi}{\partial\dot{u}} + \lambda\left(\frac{\partial^2\phi}{\partial\dot{u}\partial y} - \frac{\partial^2\phi}{\partial u\partial\dot{y}}\right)\right)\mathrm{d}y\wedge\mathrm{d}u\\ &+\lambda\left(\frac{\partial^2\phi}{\partial\dot{u}\partial\dot{y}} - 2\frac{\partial^2\phi}{\partial u\partial\ddot{y}}\right)\mathrm{d}\dot{y}\wedge\mathrm{d}u - \left(2\lambda'\frac{\partial\phi}{\partial\ddot{y}} + \lambda\left(2\frac{\partial^2\phi}{\partial\ddot{y}\partial y} - \frac{\partial^2\phi}{\partial\dot{y}\partial\dot{y}}\right)\right)\mathrm{d}\dot{y}\wedge\mathrm{d}y\end{aligned} \tag{22b}$$

and $\mathrm{d}\widetilde{\omega}_3 + \mathrm{d}\dot{\widetilde{\omega}}_2 + \mathrm{d}\ddot{\widetilde{\omega}}_1 = \mathrm{d}(\mathrm{d}\phi) = 0$. Note that the conditions (21) are satisfied if and only if all the coefficients of the two-forms (22) are equal to zero. Observe that the coefficients in the first two rows of the two-form (22b) as well as the coefficient of $\mathrm{d}\dot{y}\wedge\mathrm{d}\dot{u}$ are equal to zero provided $\mathrm{d}\widetilde{\omega}_1 = 0$. Moreover, the coefficent of $\mathrm{d}\dot{u}\wedge\mathrm{d}y$ in (22b) is equal to the coefficient of $\mathrm{d}\dot{y}\wedge\mathrm{d}u$ in (22b) minus doubled coefficient of $\mathrm{d}y\wedge\mathrm{d}u$ in (22a), therefore it is equal to zero, whenever the coefficient of $\mathrm{d}\dot{y}\wedge\mathrm{d}u$ in (22b) and the coefficient of $\mathrm{d}y\wedge\mathrm{d}u$ in (22a) are equal to zero. Furthermore, taking into account (18) and the fact that, according to the problem statement, Ψ cannot be constant, one can be confident that λ cannot be equal to zero, which allows the division by λ. Thus, equating to zero the remaining coefficients of (22) and dividing both sides of the obtained equalities by λ yield the following conditions for $i = 1, 2$

$$\begin{aligned}&\frac{\partial^2\phi}{\partial\ddot{y}\partial\ddot{y}} = 0, \quad \frac{\partial^2\phi}{\partial u^{(i)}\partial\ddot{u}} = 0, \quad \left(\frac{\partial^2\phi}{\partial\dot{u}\partial\dot{u}} - 2\frac{\partial^2\phi}{\partial u\partial\ddot{u}}\right) = 0,\\ &\frac{\partial^2\phi}{\partial\dot{y}\partial\ddot{u}} = 0, \quad \frac{\partial^2\phi}{\partial u^{(i)}\partial\ddot{y}} = 0, \quad \left(\frac{\partial^2\phi}{\partial\dot{u}\partial\dot{y}} - 2\frac{\partial^2\phi}{\partial u\partial\ddot{y}}\right) = 0,\end{aligned} \tag{23a}$$

$$3\Lambda + \frac{\partial^2\phi}{\partial\dot{y}\partial\ddot{y}} = 0, \qquad 2\Lambda\frac{\partial\phi}{\partial\ddot{y}} + \left(2\frac{\partial^2\phi}{\partial\ddot{y}\partial y} - \frac{\partial^2\phi}{\partial\dot{y}\partial\dot{y}}\right) = 0,$$
$$\Lambda\frac{\partial\phi}{\partial u^{(i)}} + \left(\frac{\partial^2\phi}{\partial y\partial u^{(i)}} - \frac{\partial^2\phi}{\partial u\partial y^{(i)}}\right) = 0, \tag{23b}$$

where $\Lambda := \ln|\lambda|'$. In the general case one can express Λ from the equalities (23b) to obtain

$$-\Lambda = \frac{1}{3}\frac{\partial^2\phi}{\partial\dot{y}\partial\ddot{y}} = \frac{1}{2}\left(\frac{\partial\phi}{\partial\ddot{y}}\right)^{-1}\left(2\frac{\partial^2\phi}{\partial\ddot{y}\partial y} - \frac{\partial^2\phi}{\partial\dot{y}\partial\dot{y}}\right)$$
$$= \left(\frac{\partial\phi}{\partial u^{(i)}}\right)^{-1}\left(\frac{\partial^2\phi}{\partial y\partial u^{(i)}} - \frac{\partial^2\phi}{\partial u\partial y^{(i)}}\right). \tag{24}$$

To verify the conditions above, one has to check whether all the parts of (24) are equal to the same function $-\Lambda(y)$. Note that if $\partial\phi/\partial\ddot{y} = 0$, then, according to the first equality of (23b), $\Lambda = 0$, implying that the output transformation $\Psi = \mathrm{id}$. In this case, instead of (24), conditions (23b) yield

$$\frac{\partial^2\phi}{\partial\dot{y}\partial\dot{y}} = 0, \quad \left(\frac{\partial^2\phi}{\partial y\partial\dot{u}} - \frac{\partial^2\phi}{\partial u\partial\dot{y}}\right) = 0, \quad \frac{\partial^2\phi}{\partial y\partial\ddot{u}} = 0. \tag{25}$$

Furthermore, if $\partial\phi/\partial\ddot{y} \neq 0$ but $\partial\phi/\partial u^{(i)} = 0$ for either $i = 1$ or $i = 2$ or both, then the expression in the last part of (24) for corresponding value of i should be omitted and replaced by the respective condition $\partial^2\phi/\partial u\partial y^{(i)} = 0$.

The conditions (23a) and (24) (or (25), if $\partial\phi/\partial\ddot{y} = 0$) are necessary and sufficient conditions for transformation of the system (4) with $n = 3$ into the observer form (8) and, as a consequence, generically observable system (1) into the observer form (7). If the conditions are satisfied, the output transformation can be found as $\Psi(y) = \int \mathrm{e}^{\int \Lambda\,\mathrm{d}y}\,\mathrm{d}y$.

4. ALGORITHM AND IMPLEMENTATION IN *MATHEMATICA*

In this section we represent an algorithm for transformation of the system (1) into the observer form (7), whenever possible. The algorithm is applied to the i/o representation (4) of the system (1).

In order to present Algorithm 4.1 below, first, we describe the procedure of finding a candidate function λ, necessary for the verification of the conditions (14). Keeping in mind that $\dot{\lambda} = \lambda'\dot{y}$, take the exterior derivative of the one-form (14) for $\varsigma = 1$. For this one-form to be closed its exterior derivative has to equal zero, which yields the differential two-form

$$n\lambda'\mathrm{d}\dot{y} \wedge \mathrm{d}y + \lambda'\mathrm{d}y \wedge \omega_1 + \lambda\mathrm{d}\omega_1 = 0,$$

which, using (12), can be rewritten as

$$\begin{aligned}&\left(\lambda\frac{\partial^2\phi}{\partial\dot y\partial y^{(n-1)}}+n\lambda'\right)\mathrm{d}\dot y\wedge\mathrm{d}y\\&\qquad+\left[\lambda'\frac{\partial\phi}{\partial u^{(n-1)}}+\lambda\left(\frac{\partial^2\phi}{\partial y\partial u^{(n-1)}}-\frac{\partial^2\phi}{\partial u\partial y^{(n-1)}}\right)\right]\mathrm{d}y\wedge\mathrm{d}u\\&\qquad+\lambda\sum_{i=2}^{n-1}\frac{\partial^2\phi}{\partial y^{(i)}\partial y^{(n-1)}}\mathrm{d}y^{(i)}\wedge\mathrm{d}y+\lambda\sum_{i=1}^{n-1}\left(\frac{\partial^2\phi}{\partial u^{(i)}\partial y^{(n-1)}}\mathrm{d}u^{(i)}\wedge\mathrm{d}y\right.\\&\qquad\left.+\frac{\partial^2\phi}{\partial y^{(i)}\partial u^{(n-1)}}\mathrm{d}y^{(i)}\wedge\mathrm{d}u+\frac{\partial^2\phi}{\partial u^{(i)}\partial u^{(n-1)}}\mathrm{d}u^{(i)}\wedge\mathrm{d}u\right)=0.\end{aligned}\tag{26}$$

To satisfy the equality, all the coefficients of the two-form on the left-hand side of (26) must be zero. Considering the coefficients of $\mathrm{d}\dot y\wedge\mathrm{d}y$ and $\mathrm{d}y\wedge\mathrm{d}u$ we have

$$\begin{aligned}\lambda\frac{\partial^2\phi}{\partial\dot y\partial y^{(n-1)}}+n\lambda'&=0,\\\lambda'\frac{\partial\phi}{\partial u^{(n-1)}}+\lambda\left(\frac{\partial^2\phi}{\partial y\partial u^{(n-1)}}-\frac{\partial^2\phi}{\partial u\partial y^{(n-1)}}\right)&=0.\end{aligned}\tag{27}$$

Since the output transformation Ψ cannot be constant, the function λ, defined by (18) cannot be equal to zero. Thus, one can divide both sides of the equalities (27) by λ to obtain

$$\begin{aligned}\frac{\partial^2\phi}{\partial\dot y\partial y^{(n-1)}}+n\Lambda&=0,\\\Lambda\frac{\partial\phi}{\partial u^{(n-1)}}+\left(\frac{\partial^2\phi}{\partial y\partial u^{(n-1)}}-\frac{\partial^2\phi}{\partial u\partial y^{(n-1)}}\right)&=0,\end{aligned}\tag{28}$$

where $\Lambda:=\ln|\lambda|'$. Expressing Λ from the equalities (28), one obtains

$$\begin{aligned}&-\Lambda=\\&=\begin{cases}\dfrac{1}{n}\dfrac{\partial^2\phi}{\partial\dot y\partial y^{(n-1)}} & \text{if } \dfrac{\partial\phi}{\partial u^{(n-1)}}=0,\\[2ex]\dfrac{1}{n}\dfrac{\partial^2\phi}{\partial\dot y\partial y^{(n-1)}}=\left(\dfrac{\partial\phi}{\partial u^{(n-1)}}\right)^{-1}\left(\dfrac{\partial^2\phi}{\partial y\partial u^{(n-1)}}-\dfrac{\partial^2\phi}{\partial u\partial y^{(n-1)}}\right) & \text{if } \dfrac{\partial\phi}{\partial u^{(n-1)}}\neq 0.\end{cases}\end{aligned}\tag{29}$$

Next, according to the definition of Λ, the function λ can be found as

$$\lambda=\mathrm{e}^{\int\Lambda\,\mathrm{d}y}.\tag{30}$$

Algorithm 4.1.

Step 1. Using successively (29) and (30), find $\lambda(y)$. If $\lambda(y)$ cannot be found (either the equality in (29) for $\partial\phi/\partial u^{(n-1)}\neq 0$ is not satisfied or the integral in (30) is not computable), the problem is not solvable; stop.

Step 2. Using (12), compute the one-forms ω_i for $i = 1, \ldots, n$.

Step 3. Using ω_i's and $\lambda(y)$ compute the one-forms (14), for $\varsigma = 1, \ldots, n$.

Step 4. Check whether the one-forms (14) are closed or not. If at least one of them is not closed, the problem is not solvable; stop.

Step 5. Rewrite the (closed) one-forms (14) as $\mathrm{d}\bar{\varphi}_\varsigma$ (see (19)) and integrate them, yielding $\bar{\varphi}_\varsigma$ for $\varsigma = 1, \ldots, n$. Using $\lambda(y)$ and (20) one can find the output transformation $\Psi(y)$ and the functions φ_ς in terms of which the system in the observer form (7) can be easily constructed.

Step 6. Using the output transformation (6) and the functions $\varphi_1, \ldots, \varphi_n$, find the system equations in the observer form (7).

In order to facilitate computations, Algorithm 4.1 was implemented within a *Mathematica*-based package NLControl, developed in the Institute of Cybernetics at Tallinn University of Technology. The purpose of the package is to provide the symbolic computational tools that assist the solution of different modeling, analysis, and synthesis problems for nonlinear control systems [20]. The most important functions from the NLControl package are available via the NLControl website [9]. The website is based on the *webMathematica* technology and does not require *Mathematica* software to be installed on a computer. To take advantage of the online tool, which implements Algorithm 4.1, one has to choose the option *Observer Form* in the section *Continuous* and after filling the corresponding text fields, push the button *Evaluate*.

5. COMPARISON WITH EARLIER RESULTS

An alternative solvability condition was given earlier in [6]. We recall this condition in Theorem 5.1.

Theorem 5.1. If the system (1) can be transformed by the state transformation (5) and the output transformation (6) into the observer form (7), then

$$\mathrm{d}\left(\frac{\partial^2 \phi}{\partial \dot{y} \partial y^{(n-1)}}\right) \wedge \mathrm{d}y = 0. \tag{31}$$

Moreover, if (31) is satisfied, then the possible output transformation $\Psi(y)$ is a solution of

$$\Psi' \frac{\partial^2 \phi}{\partial \dot{y} \partial y^{(n-1)}} + n\Psi'' = 0. \tag{32}$$

Using (18), equality (26) can be rewritten as

$$\begin{aligned}
&\left(\Psi' \frac{\partial^2 \phi}{\partial \dot{y} \partial y^{(n-1)}} + n\Psi''\right) \mathrm{d}\dot{y} \wedge \mathrm{d}y \\
&\qquad + \left[\Psi'' \frac{\partial \phi}{\partial u^{(n-1)}} + \Psi' \left(\frac{\partial^2 \phi}{\partial y \partial u^{(n-1)}} - \frac{\partial^2 \phi}{\partial u \partial y^{(n-1)}}\right)\right] \mathrm{d}y \wedge \mathrm{d}u \\
&\quad + \Psi' \sum_{i=2}^{n-1} \frac{\partial^2 \phi}{\partial y^{(i)} \partial y^{(n-1)}} \mathrm{d}y^{(i)} \wedge \mathrm{d}y + \Psi' \sum_{i=1}^{n-1} \left(\frac{\partial^2 \phi}{\partial u^{(i)} \partial y^{(n-1)}} \mathrm{d}u^{(i)} \wedge \mathrm{d}y \right. \\
&\qquad\qquad \left. + \frac{\partial^2 \phi}{\partial y^{(i)} \partial u^{(n-1)}} \mathrm{d}y^{(i)} \wedge \mathrm{d}u + \frac{\partial^2 \phi}{\partial u^{(i)} \partial u^{(n-1)}} \mathrm{d}u^{(i)} \wedge \mathrm{d}u \right) = 0. \qquad (33)
\end{aligned}$$

Recall that the equality above implies that all the coefficients of the two-form on the left-hand side of (33) are zeros. Note that the coefficient of $\mathrm{d}\dot{y} \wedge \mathrm{d}y$ is exactly the left-hand side of condition (32). Thus, (33) obviously implies (32). However, the converse does not hold, since the condition (32) does not guarantee that the other coefficients of the two-form (26) equal to zero. It should be mentioned that (31) is just the exterior derivative of (32) multiplied by $\mathrm{d}y$. To summarize, it can be stated that the conditions from Theorem 5.1 are very mild and far from being sufficient. The output transformation obtained from (32) does not guarantee that the i/o equation (4) can be represented in the form (8). To verify whether the problem is solvable, one has to apply the output transformation to the i/o equation (4) and check whether the obtained i/o equation is transformable into the observer form by the state coordinate transformation only [6].

The following example illustrates the reasoning given above.

Example 5.2. Consider the system

$$\begin{aligned}
\dot{x}_1 &= u x_1 + x_2 \\
\dot{x}_2 &= u^2 x_1 + u x_2 + x_3 \\
\dot{x}_3 &= u x_1 \\
y &= \frac{1}{x_1},
\end{aligned} \qquad (34)$$

which is locally observable on the set $S = \{(x_1; x_2; x_3; u; \dot{u}) \mid x_1 \in \mathbb{R} \setminus \{0\};\ x_2, x_3, u, \dot{u} \in \mathbb{R}\}$. The i/o equation, corresponding to (34), is valid on the set $\tilde{S} = \{(y; \dot{y}; \ddot{y}; u; \dot{u}; \ddot{u}) \mid y \in \mathbb{R} \setminus \{0\};\ \dot{y}, \ddot{y}, u, \dot{u}, \ddot{u} \in \mathbb{R}\}$ and reads as

$$y^{(3)} = 3\dot{u}\dot{y} - \frac{6\dot{y}^3}{y^2} - y\ddot{u} + \frac{6\dot{y}\ddot{y}}{y} + u\left(2\ddot{y} - \frac{4\dot{y}^2}{y} - y\right). \qquad (35)$$

First, we follow the procedure described in [6]. One can verify that the condition (31) is *satisfied* and the solution of differential equation (32) is $\Psi = 1/y$. Nevertheless, as it will be shown below, the system (34) is not transformable into the observer form. According to [6], the next step is to find the i/o equation in terms of new output. Taking into account (6), the output transformation $\Psi = 1/y$ brings the i/o equation (35) into the form

$$Y^{(3)} = 3\dot{Y}\dot{u} + 2\ddot{Y}u + Y\ddot{u} + Yu. \qquad (36)$$

To check, whether the equation (36) can be represented in the observer form (8), one has to compute the one-forms ω_i, $i = 1, 2, 3$ with respect to new output Y and verify whether they are closed or not. Using (12) (or alternatively the algorithm either from Subsection 2.3 or from [6]) with respect to Y, we obtain $\omega_1 = 2u\mathrm{d}Y + Y\mathrm{d}u$, $\omega_2 = -\dot{u}\mathrm{d}Y + \dot{Y}\mathrm{d}u$ and $\omega_3 = u\mathrm{d}Y + Y\mathrm{d}u$. The exterior derivative of the one-forms reads as $\mathrm{d}\omega_1 = \mathrm{d}u \wedge \mathrm{d}Y$, $\mathrm{d}\omega_2 = \mathrm{d}Y \wedge \mathrm{d}\dot{u} - \mathrm{d}u \wedge \mathrm{d}\dot{Y}$ and $\omega_3 = 0$, implying that not all the one-forms are closed and, as a consequence, the system (36) is not transformable into the observer form (8). Indeed, the equation (36) can be rewritten as

$$Y^{(3)} = (Yu)^{(2)} + \left(\dot{Y}u\right)^{(1)} + Yu. \tag{37}$$

It is not hard to observer that, unlike (8), the equation (37) contains the term which depends on $\dot{Y}$. As a consequence, the state equations (34) cannot be transformed into the observer form (7).

Now, we follow Algorithm 4.1. Computing (29) for ϕ equals to the right-hand side of (35), one obtains $-\Lambda = 2/y \neq 3/y$. The obtained inequation means that the condition for existence of Λ is not satisfied, which implies that the state equations (34) are not transformable into the observer form (7).

This example demonstrates clearly that in the case $\partial\phi/\partial u^{(n-1)} \neq 0$ the condition (29) is stronger than (31). Otherwise, they coincide. Note, however, that, like (31), condition (29) is only necessary, meaning that in some cases the function λ, found by (29) and (30), does not guaranty that the system is transformable into the observer form. Nevertheless, unlike those in [6], our necessary and sufficient conditions (14) are formulated in terms of the original system and do not require the intermediate step of finding the i/o equation in terms of new output (like (36) in this example). Thus, once the function λ is found, one can directly check whether it makes the necessary and sufficient conditions (14) fulfilled or not.

6. EXAMPLE

Examine the model of a direct current (DC) motor, described by the equations (see [5])

$$\begin{aligned}
\dot{x}_1 &= -K_m x_1 x_2 - \frac{R_a + R_f}{K} x_1 + u \\
\dot{x}_2 &= -\frac{B}{J} x_2 - x_3 + \frac{K_m}{J} K x_1^2 \\
\dot{x}_3 &= 0 \\
y &= x_1,
\end{aligned} \tag{38}$$

where x_1 denotes the magnetic flux and verifies $x_1 > 0$; x_2 denotes the rotor speed; x_3 denotes the constant load torque; R_a and R_f denote the stator and the inductor resistances, respectively; B is the viscous friction coefficient, and K_m is the constant motor torque. The system (38) is locally observable on the set $S = \{(x_1; x_2; x_3; u; \dot{u}) \mid x_1, x_2, x_3, u, \dot{u} \in \mathbb{R}; x_1 > 0\}$ The input-output equation, corresponding to (38), is valid on the set $\tilde{S} = \{(y; \dot{y}; \ddot{y}; u; \dot{u}; \ddot{u}) \mid y, \dot{y}, \ddot{y}, u, \dot{u}, \ddot{u} \in \mathbb{R}; y > 0\}$ and reads as

$$y^{(3)} = \frac{B}{J}\left(\dot{u} - \ddot{y} + \frac{\dot{y}\,(\dot{y} - u)}{y}\right) - \frac{2KK_m^2 y^2 \dot{y}}{J} + \ddot{u} - \frac{2\dot{u}\dot{y} + \ddot{y}\,(u - 3\dot{y})}{y} + \frac{2\dot{y}^2\,(u - \dot{y})}{y^2}.$$

We will follow Algorithm 4.1.

Step 1. Using (29), one obtains $\Lambda = -1/y$, which, according to (30), yields

$$\lambda = \frac{1}{y}. \tag{39}$$

Step 2. Compute, according to (12),

$$\begin{aligned}
\omega_1 &= \left(-\frac{B}{J} - \frac{u-3\dot{y}}{y}\right)\mathrm{d}y + \mathrm{d}u,\\
\omega_2 &= \left(\frac{B\,(2\dot{y}-u)}{Jy} - \frac{3\ddot{y}}{y} + \frac{2u\dot{y}}{y^2} - \frac{2KK_m^2y^2}{J}\right)\mathrm{d}y + \left(\frac{B}{J} - \frac{2\dot{y}}{y}\right)\mathrm{d}u,\\
\omega_3 &= \left(\frac{(3\dot{y}-2u)\,\ddot{y} - 2\dot{u}\dot{y}}{y^2} + \frac{B}{J}\left(\frac{\dot{u}-2\ddot{y}}{y} + \frac{\dot{y}^2}{y^2}\right) + \frac{2\,(u-\dot{y})\,\dot{y}^2}{y^3} + \frac{\ddot{u}}{y}\right)\mathrm{d}y\\
&\quad + \left(\frac{\ddot{y}}{y} - \frac{B\dot{y}}{Jy}\right)\mathrm{d}u.
\end{aligned} \tag{40}$$

Step 3. For the case $n = 3$ the one-forms (14) read as

$$\begin{aligned}
3\dot{\lambda}\mathrm{d}y + \lambda\omega_1,\\
-3\ddot{\lambda}\mathrm{d}y - 2\dot{\lambda}\omega_1 + \lambda\omega_2,\\
\lambda^{(3)}\mathrm{d}y + \ddot{\lambda}\omega_1 - \dot{\lambda}\omega_2 + \lambda\omega_3,
\end{aligned}$$

which, according to (40) and (39), yield

$$\begin{aligned}
-\frac{Ju+By}{Jy^2}\mathrm{d}y + \frac{1}{y}\mathrm{d}u,\\
-\frac{Bu+2KK_m^2y^3}{Jy^2}\mathrm{d}y + \frac{B}{Jy}\mathrm{d}u,\\
0.
\end{aligned}$$

Step 4. It is not hard to verify that all three one-forms above are closed, meaning that the conditions for transformation of the system (38) into the observer form (7) are satisfied.

Step 5. Now one can define

$$\begin{aligned}
\mathrm{d}\bar{\varphi}_1 &:= -\frac{Ju+By}{Jy^2}\mathrm{d}y + \frac{1}{y}\mathrm{d}u,\\
\mathrm{d}\bar{\varphi}_2 &:= -\frac{Bu+2KK_m^2y^3}{Jy^2}\mathrm{d}y + \frac{B}{Jy}\mathrm{d}u,\\
\mathrm{d}\bar{\varphi}_3 &:= 0,
\end{aligned}$$

integration of which yields

$$\bar{\varphi}_1 = -\frac{B \ln y}{J} + \frac{u}{y},$$
$$\bar{\varphi}_2 = \frac{Bu}{Jy} - \frac{KK_m^2 y^2}{J},$$
$$\bar{\varphi}_3 = 0.$$

Taking into account (6), (20) and (39), one finds the output transformation

$$Y = \Psi(y) = \ln y, \tag{41}$$

which, according to (13), leads to

$$\varphi_1 = -\frac{BY}{J} + \frac{u}{\mathrm{e}^Y},$$
$$\varphi_2 = \frac{Bu}{J\mathrm{e}^Y} - \frac{KK_m^2 \mathrm{e}^{2Y}}{J},$$
$$\varphi_3 = 0.$$

Step 6. By (9) for $n = 3$, one can define new state variables as

$$z_1 = Y,$$
$$z_2 = \dot{Y} + \frac{BY}{J} - \frac{u}{\mathrm{e}^Y},$$
$$z_3 = \ddot{Y} + \frac{B\dot{Y}}{J} - \frac{\dot{u} - u\dot{Y}}{\mathrm{e}^Y} - \frac{Bu}{J\mathrm{e}^Y} + \frac{KK_m^2 \mathrm{e}^{2Y}}{J},$$

which, due to the output transformation (41) and the state equations (38), can be rewritten as

$$z_1 = \ln x_1,$$
$$z_2 = -\frac{R_a + R_f}{K} + \frac{B \ln x_1}{J} - K_m x_2,$$
$$z_3 = -\frac{B\,(R_a + R_f)}{JK} + K_m x_3,$$

that leads to the new state equations in the observer form:

$$\dot{z}_1 = z_2 - \frac{Bz_1}{J} + \frac{u}{\mathrm{e}^{z_1}},$$
$$\dot{z}_2 = z_3 + \frac{Bu}{J\mathrm{e}^{z_1}} - \frac{KK_m^2 \mathrm{e}^{2z_1}}{J},$$
$$\dot{z}_3 = 0,$$
$$Y = z_1.$$

Remark 6.1. Note that in [5] the output of the system (38) was already chosen as $y = \ln x_1$. Such farsighted choice allowed to transform the system into the observer form only by the state transformation and to avoid the necessity in the output transformation, which was not considered in [5]. Our task was to show how the output transformation $Y = \ln x_1$ can be computed. Therefore, we used the output $y = x_1$, which is more natural for the model of a DC motor [4].

One may find more examples in [9], choosing the option *Observer Form* in the section *Continuous* and following the link *Examples.*

7. CONCLUSIONS

The paper provides necessary and sufficient conditions for local transformation of nonlinear state equations into the observer form using both state and output transformations. The conditions require that certain differential one-forms, associated with the input-output equation of the system, are closed. Once the input-output equation is obtained, the conditions can be easily constructed due to the direct formula for computation of the necessary one-forms. However, note that the conditions depend on an unknown single-variable output dependent function. As a consequence, the verification of the conditions requires to solve certain partial differential equation, which can be a difficult task. On the other hand, in the discrete-time case simple necessary and sufficient conditions exist that are directly computable from the input-output equation and do not depend on an unknown output function [17]. These conditions are expressed in terms of exterior derivatives and exterior products of one-forms, similar to those in the continuous-time case. The difference between the discrete- and continuous-time cases is that the output transformation and shift operator commute, whereas this does not hold for the output transformation and derivative operator. Nevertheless, in this paper we showed how the main result can be employed to obtain simpler conditions, formulated solely in terms of partial derivatives of the input-output equation of the system such that the verification of the conditions does not depend on an unknown function. Though we considered only the special case of the third-order system, we expect that the same approach can be used to derive the similar conditions for systems of an arbitrary order. Derivation of such conditions will make the subject for future research. Another future goal is the extension of our conditions to the multi-input case as well as to the more general case, when the input-output injections in the observer form are allowed to depend not only on the input and the output but also on the derivatives of the input [5, 6]. Moreover, we have an intention to compare our results with those presented in [16] and [19], which rely on the tools from differential geometry.

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APPENDIX

For the proof of Lemmas 7.1 and 3.3 we will use the binomial theorem $(a+b)^k = \sum_{l=0}^{k} C_k^l a^l b^{k-l}$, which for $a=-1$, $b=1$ and $k \geq 1$ gives

$$\sum_{l=0}^{k} C_k^l (-1)^l = 0. \tag{42}$$

Separating the last addend of the sum above and placing it into the right-hand side of the equality, yields

$$\sum_{l=0}^{k-1} C_k^l (-1)^l = -(-1)^k. \tag{43}$$

Proof of Proposition 3.1

In order to prove Proposition 3.1, we need Lemma 7.1 below.

Lemma 7.1. For $j \geq 1$ and $r \geq j+1$ the following holds

$$\sum_{i=1}^{j} (-1)^i C_{r-1}^{i-1} C_{r-i}^{r-j-1} = (-1)^j C_{r-1}^{j}. \tag{44}$$

Proof. Take (43) for $k=j$, $l=i-1$ and multiply both sides of the equality by $-C_{r-1}^{j}$, where $r \geq j+1$, to obtain

$$\sum_{i=1}^{j} (-1)^i C_j^{i-1} C_{r-1}^{j} = (-1)^j C_{r-1}^{j}.$$

Taking into account the definition of the binomial coefficient, i.e., $C_n^k = \frac{n!}{(n-k)!k!}$, one can easily verify that $C_j^{i-1} C_{r-1}^{j} = C_{r-1}^{i-1} C_{r-i}^{r-j-1}$, which implies the validity of (44). $\square$

Now we are ready to prove Proposition 3.1.

Proof. The proof is by mathematical induction on i. One can easily verify that (11b) and (12) coincide for $i=1$. Next we assume that the statement of the proposition holds for $i \leq k$ (k is an arbitrary integer from 1 to $n-1$) and show that it is true for $i=k+1$.

From (11a) one obtains

$$P_{k+1} = \mathrm{d}\phi - \sum_{i=1}^{k} \omega_i^{(n-i)}.$$

From the assumption that (12) holds for $i \leq k$ we have

$$P_{k+1} = \mathrm{d}\phi - \sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j C_{n-i+j}^{j} \left[\left(\frac{\partial \phi}{\partial y^{(n-i+j)}} \right)^{(j)} \mathrm{d}y + \left(\frac{\partial \phi}{\partial u^{(n-i+j)}} \right)^{(j)} \mathrm{d}u \right]^{(n-i)}.$$

Using the rule (10), the latter yields

$$P_{k+1} = \mathrm{d}\phi - \sum_{i=1}^{k}\sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^{j} \sum_{q=0}^{n-i} C_{n-i}^{q} \cdot \\ \cdot \left[\left(\frac{\partial\phi}{\partial y^{(n-i+j)}} \right)^{(j+n-i-q)} \mathrm{d}y^{(q)} + \left(\frac{\partial\phi}{\partial u^{(n-i+j)}} \right)^{(j+n-i-q)} \mathrm{d}u^{(q)} \right].$$

From (11b) follows that in order to find ω_{k+1}, only A_{k+1}^{n-k-1} and B_{k+1}^{n-k-1} are necessary. In other words, we are interested only in such elements of P_{k+1} where the order of differentiation of $\mathrm{d}y$ and $\mathrm{d}u$ is $q = n-k-1$. Thus, we have

$$A_{k+1}^{n-k-1} = \frac{\partial\phi}{\partial y^{(n-k-1)}} - \sum_{i=1}^{k}\sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^{j} C_{n-i}^{n-k-1} \left(\frac{\partial\phi}{\partial y^{(n-i+j)}} \right)^{(j+k-i+1)}, \tag{45a}$$

$$B_{k+1}^{n-k-1} = \frac{\partial\phi}{\partial u^{(n-k-1)}} - \sum_{i=1}^{k}\sum_{j=0}^{i-1}(-1)^j C_{n-i+j}^{j} C_{n-i}^{n-k-1} \left(\frac{\partial\phi}{\partial u^{(n-i+j)}} \right)^{(j+k-i+1)}. \tag{45b}$$

Note that (45a) and (45b) have a similar structure. Thus, all the transformations made with one expression will be similar for the other.

Changing the summation order $\sum_{i=1}^{k}\sum_{j=0}^{i-1} a_{i,j} = \sum_{j=1}^{k}\sum_{i=1}^{j} a_{k-j+i,i-1}$, and taking into account that $-(-1)^{i-1} = (-1)^i$, rewrite (45a) as follows

$$\begin{aligned} A_{k+1}^{n-k-1} &= \frac{\partial\phi}{\partial y^{(n-k-1)}} + \sum_{j=1}^{k}\sum_{i=1}^{j}(-1)^i C_{n-k+j-1}^{i-1} C_{n-k+j-i}^{n-k-1} \left(\frac{\partial\phi}{\partial y^{(n-k+j-1)}} \right)^{(j)} \\ &= \frac{\partial\phi}{\partial y^{(n-k-1)}} + \sum_{j=1}^{k} \left[\left(\frac{\partial\phi}{\partial y^{(n-k+j-1)}} \right)^{(j)} \sum_{i=1}^{j}(-1)^i C_{n-k+j-1}^{i-1} C_{n-k+j-i}^{n-k-1} \right]. \end{aligned} \tag{46}$$

Taking into account that $\frac{\partial\phi}{\partial y^{(n-k-1)}} = (-1)^j C_{n-k-1+j}^{j} \left(\frac{\partial\phi}{\partial y^{(n-k-1+j)}} \right)^{(j)}$ for $j = 0$ and using Lemma 7.1 for $r = n-k+j$, one can rewrite (46) as

$$A_{k+1}^{n-k-1} = \sum_{j=0}^{k}(-1)^j C_{n-k-1+j}^{j} \left(\frac{\partial\phi}{\partial y^{(n-k-1+j)}} \right)^{(j)}. \tag{47}$$

Analogously, from (45b) we get

$$B_{k+1}^{n-k-1} = \sum_{j=0}^{k}(-1)^j C_{n-k-1+j}^{j} \left(\frac{\partial\phi}{\partial u^{(n-k-1+j)}} \right)^{(j)}. \tag{48}$$

Using (11b), (47) and (48)

$$\omega_{k+1} = \sum_{j=0}^{k}(-1)^j C_{n-k-1+j}^{j} \left[\left(\frac{\partial\phi}{\partial y^{(n-k-1+j)}} \right)^{(j)} \mathrm{d}y + \left(\frac{\partial\phi}{\partial u^{(n-k-1+j)}} \right)^{(j)} \mathrm{d}u \right]$$

being (12) for $i = k+1$. □

Proof of Lemma 3.3

Proof.

(i) Taking into account that $-(-1)^{\varsigma} = (-1)^{\varsigma-1}$, the equality (43) taken for $k = \varsigma$ and $l = j-1$ confirms statement (i).

(ii) Since $s = 1, \dots, \varsigma - 1$ and $\varsigma \geq 2$, one can conclude that $\varsigma - s \geq 1$ and it is eligible to take (42) for $k = \varsigma - s$, which after replacing the summation index l by $j-1$, leads to (ii). □

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