KYBERNETIKA — VOLUME *51* (2015), NUMBER 1, PAGES 137–149

# FINITE-TIME SYNCHRONIZATION OF CHAOTIC SYSTEMS WITH NOISE PERTURBATION

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In this paper, we investigate the finite-time stochastic synchronization problem of two chaotic systems with noise perturbation. We propose new adaptive controllers, with which we can synchronize two chaotic systems in finite time. Sufficient conditions for the finite-time stochastic synchronization are derived based on the finite-time stability theory of stochastic differential equations. Finally, some numerical examples are examined to demonstrate the effectiveness and feasibility of the theoretical results.

*Keywords:* synchronization, finite-time, noise perturbation, adaptive feedback controller

*Classification:* 34F05, 34H10

## 1. INTRODUCTION

Christian Huygens firstly picked up the appearance of synchronization about 350 years ago. In fact, the significance of synchronization, especially for chaotic systems, was not completely achieved until Pecora and Carroll published their pioneering work on chaos synchronization in 1990 [31]. After that, a large variety of synchronization phenomena, in many chaotic systems [7] and dynamical networks [12, 18], have been widely investigated in different areas including physics, chemistry, biology, etc [28]. And lots of important real applications have been found in many fields, such as information processing, horizontal platform systems [13], secure communication [4, 5], biological system [15], rotating pendulums [14], control processing, chemical reactions and so on [6, 10, 20, 30, 32]. As we all known, a focused problem, in chaos synchronization, is to make the states of the slave system follow the master system with an appropriate controller. Due to a wide variety of applications, many approaches and controllers have been presented, including adaptive control [17, 21, 25], optimal control [9, 34], sliding mode control [33], delayed Lur'e systems control [19], the open-loop-closed-loop coupling technology [16], linearly coupled ordinary differential systems analysis [27] and so on.

Recently, the finite-time synchronization of two chaotic systems has been investigated by many researchers [1, 2, 3, 8, 24, 26, 35, 37, 40]. Finite-time generalized synchronization of chaotic systems with different order has been studied in Ref. [8]. The adaptive feedback controller was proposed to realize the finite-time synchronization for a class

DOI: 10.14736/kyb-2015-1-0137

of chaotic and hyperchaotic systems [35]. In Ref. [1], a robust adaptive controller was introduced to realize finite-time chaos synchronization between two different chaotic systems in the presence of model uncertainties, external disturbances, fully unknown parameters, and input nonlinearities. In Ref. [26], the author implemented and tested experimentally a four-dimensional hyperchaotic system and investigated the synchronization of the system in a finite time, based on the finite-time stability theory. In Ref. [37], the authors investigated the global finite-time synchronization of a class of second-order nonautonomous chaotic systems via a master-slave coupling. In Ref. [40], the adaptive finite-time synchronization of different coupled chaotic systems with unknown parameters was explored.

However, the problems of finite-time synchronization in Refs. [1, 8, 26, 35, 37, 40] did not take noise perturbation into consideration. It deserves pointing out that noise perturbation is widespread in both natural and artificial systems. For instance, because the atmospheric effects and processes such as cloud cover, pollution, etc., are seasonal and stochastic in nature, sunshine duration and solar irradiation are modeled in a stochastic way. Therefore, it has more practical value to explore the influence of circumstance noise on the finite-time synchronization of chaotic systems. The main contribution of this paper is to propose an adaptive feedback controller, which can realize the finite-time stochastic synchronization between two chaotic systems with noise perturbation. Based on the finite-time stability theory for stochastic differential equations, sufficient conditions for the finite-time stochastic synchronization are obtained. Finally, some numerical examples are examined to illustrate the effectiveness of the analytical results. The effects of control parameters and noise intensity on the convergence time are numerically demonstrated.

The rest of this paper is organized as follows. In Section 2, the problem statement and some useful preliminaries are given. Based on the stability theory of stochastic differential equations, sufficient conditions for the finite-time stochastic synchronization are derived analytically in Section 3. In Section 4, some numerical examples are given to show the effectiveness of the theoretical results. Finally, some conclusions are drawn in Section 5.

## 2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following system described by:

$$\mathrm{d}x_i = f_i(x)\mathrm{d}t, \qquad i = 1, 2, \ldots, n, \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ is the state vector of the chaotic system, $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T : R^n \to R^n$ is a continuously differentiable nonlinear vector function. To realize the complete synchronization of two chaotic systems, we refer to system (1) as the master system, and the slave system is given by:

$$\mathrm{d}y_i = [f_i(y) + u_i(t)]\mathrm{d}t + \sigma_i(e_i(t))\mathrm{d}W(t), \qquad i = 1, 2, \ldots, n, \tag{2}$$

where $y = (y_1, y_2, \ldots, y_n)^T \in R^n$ is the state vector of the slave system, $e_i(t) = y_i(t) - x_i(t)$ $(i = 1, 2, \ldots, n)$ are the state errors between the master system (1) and the slave system (2), $u_i(t)$ $(i = 1, 2, \ldots, n)$ are the controllers to be designed. The noise term in system (2) is mostly applied to demonstrate the coupling process influenced by surrounding fluctuation, inaccurate design of coupling strength, etc. Where $\sigma_i : R^n \to R^{n\times m}$ is continuous nonlinear matrix-valued function, and $W = (w_1, \dots, w_m)^T$ is an $m$-dimensional Brownian motion which is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Accordingly, $\dot{W}$ is an $m$-dimensional white noise.

Throughout this paper, we here make the following assumption:

**Assumption 2.1.** For function $f(x)$ there exists a nonnegative constant $l$ satisfying

$$[x(t) - y(t)]^T [f(x(t)) - f(y(t))] \leq [x(t) - y(t)]^T l [x(t) - y(t)], \forall\, x, y \in R^n. \tag{3}$$

For the noise intensity function, because the speed of the environmental fluctuations is far less than the change rate of practical systems, we have the following assumption:

**Assumption 2.2.** The noise intensity function $\sigma_i(e_i(t))$ $(i = 1, 2, \dots, n)$ satisfies the Lipschitz condition and there exists a nonnegative constant $q$ such that

$$\mathrm{trace}(\sigma_i^T(e_i(t))\sigma_i(e_i(t))) \leq 2q e_i^T(t) e_i(t).$$

Moreover, $\sigma(0) \equiv 0$.

Consider the following $n$-dimensional stochastic differential equation [29]:

$$\mathrm{d}x = \phi(x)\mathrm{d}t + \psi(x)\mathrm{d}W(t), \tag{4}$$

where $x \in R^n$ is the state vector, and $\phi : R^n \to R^n$ and $\psi : R^n \to R^{n\times m}$ are continuous and satisfy $\phi(0) = 0, \psi(0) = 0$. It is assumed that Eq. (4) has a unique and global solution denoted by $x(t; x(0))$ $(0 \leq t < +\infty)$, where $x(0)$ is the initial state.

For each $V \in C^{2,1}(R^n \times R_+, R_+)$, the operator $\mathcal{L}V$ associated to Eq. (4) is defined as follows:

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot \phi + \frac{1}{2}\mathrm{trace}\Big[\psi^T \frac{\partial^2 V}{\partial^2 x} \cdot \psi\Big], \tag{5}$$

where $\partial V/\partial x = (\partial V/\partial x_1, \dots, \partial V/\partial x_n)$, $\partial^2 V/\partial^2 x = (\partial^2 V/\partial x_i \partial x_j)_{n\times n}$.

For getting our main results in the next section, we state two necessary concepts and a lemma about stochastic differential equation.

**Definition 2.3.** (Yin et al. [38]) The trivial solution of (4) is said to be finite-time stable in probability, if the equation admits a unique solution for any initial data $x(0) \in R^n$, denoted by $x(t; x(0))$, moreover, the following statements hold:

(i) For every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r) > 0$ such that

$$P\{|x(t; x(0))| < r, \text{ for all } t \geq 0\} \geq 1 - \varepsilon,$$

where $|x(0)| < \delta$.

(ii) For every initial value $x(0) \in R^n$, the stochastic setting time $T_0 = \inf\{T : x(t; x(0)) = 0, \forall\, t \geq T\}$ is finite almost surely, that is,

$$P\{|x(t; x(0))| = 0\} = 1, \quad \text{for all } t \geq T_0.$$

**Definition 2.4.** Systems (1) and (2) are said to achieve the finite-time stochastic synchronization if, for any initial states $x_i(0), y_i(0) \in R^n \setminus \{0\}$, there exists a finite time function $T_0$ such that

$$P\{|x_i(t;x_i(0)) - y_i(t;y_i(0))| = 0\} = 1, i = 1,2,\ldots,n, \text{ for all } t \geq T_0, \tag{6}$$

where $T_0 = \inf\{T : x_i(t;x_i(0)) = y_i(t;y_i(0)), \forall\, t \geq T\}$ is called the stochastic setting time.

**Lemma 2.5.** (Yin et al. [39]) For system (4), define $T_0(x_0) = \inf\{T \geq 0 : x(t;x_0) = y(t;y_0), \forall\, t \geq T\}$. Assume that system (4) has the unique global solution. If there exists a positive definite, twice continuously differentiable and radially unbounded Lyapunov function $V : R^n \to R^+$, $K_\infty$ class functions $\mu_1$ and $\mu_2$, positive real numbers $c > 0$ and $0 < \gamma < 1$, such that for all $x \in R^n$ and $t \geq 0$,

$$\mu_1(|x|) \leq V(x) \leq \mu_2(|x|),$$

$$\mathcal{L}V(x) \leq -c \cdot (V(x))^\gamma\,,$$

where $|x|$ denotes the Euclidean norm $|x| = \sqrt{\sum_{i=1}^n x_i^2}$, then the origin of system (4) is globally stochastically finite-time stable, and the stochastic settling time function $T_0$ satisfies

$$E[T_0(x_0)] \leq \frac{(V(x_0))^{1-\gamma}}{c(1-\gamma)}.$$

## 3. SUFFICIENT CONDITIONS FOR FINITE-TIME STOCHASTIC SYNCHRONIZATION

In this section, we will investigate the finite-time stochastic synchronization of chaotic systems, and the main results are given in the following theorem.

**Theorem 3.1.** Suppose that the Assumptions 2.1 and 2.2 hold and there exist a sufficiently large positive constant $L$ satisfying $L \geq l + q$, then systems (1) and (2) can achieve finite-time stochastic synchronization under the following adaptive controllers:

$$\begin{aligned} u_i(t) &= \varepsilon_i e_i + k_i \mathrm{sign}(e_i) - \frac{k_i + \bar{k}}{|e_i|}\mathrm{sign}(e_i), && \text{if } e_i \neq 0, \\ u_i(t) &= 0, && \text{if } e_i = 0, \end{aligned} \tag{7}$$

where $e_i = y_i - x_i$, positive constant $\bar{k} \geq 1$. For $e_i \neq 0$, the feedback gains $\varepsilon_i$ and $k_i$ are adapted according to the following updated laws:

$$\dot{\varepsilon}_i(t) = -e_i^2(t), \dot{k}_i(t) = -|e_i(t)|. \tag{8}$$

For $e_i = 0$, we set $k_i \equiv -\bar{k}$ and $\varepsilon_i \equiv -L$.

Proof. From Eqs. (1) and (2), we can get the error system as follows:

$$\dot{e}_i = f_i(y) - f_i(x) + u_i + \sigma_i(e_i(t))\dot{W}(t), \qquad i = 1,2,\ldots,n. \tag{9}$$

Therefore, under Assumptions 2.1 and 2.2, it is from the theory of stochastic differential equation that the error system (9) possesses a unique global solution on $t \geq 0$, denoted by $e_i(t, e_i(0))$ for any initial data $e_i(0) = y_i(0) - x_i(0)$. And $e_i(t, 0) \equiv 0$ is a trivial solution of the error dynamics (9). Obviously, if this trivial solution is globally stochastically finite-time stable, then the finite-time stochastic synchronization between systems (1) and (2) could be realized for every initial data.

Consider the following Lyapunov function:

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \frac{1}{2}\sum_{i=1}^{n}(k_i + \bar{k})^2 + \frac{1}{2}\sum_{i=1}^{n}(\varepsilon_i + L)^2. \tag{10}$$

Thus the diffusion operator $\mathcal{L}$ defined in (5) onto the function $V$ along the error system (9) gives

$$\begin{aligned} \mathcal{L}V(t) \quad = \quad & \sum_{i=1}^{n} e_i(t)[f(y_i) - f(x_i) + \varepsilon_i e_i(t) + k_i \mathrm{sign}(e_i) \\ & -\frac{k_i + \bar{k}}{|e_i|}\mathrm{sign}(e_i)] + \frac{1}{2}\sum_{i=1}^{n}\mathrm{trace}(\sigma_i^T \sigma_i) \\ & +\sum_{i=1}^{n}(k_i + \bar{k})\dot{k}_i + \sum_{i=1}^{n}(\varepsilon_i + L)\dot{\varepsilon}_i. \end{aligned} \tag{11}$$

Substituting $\dot{k}_i$ and $\dot{\varepsilon}_i$ into the rihgt-hand side of Eq. (11), we have

$$\begin{aligned} \mathcal{L}V(t) \quad = \quad & \sum_{i=1}^{n} e_i(t)[f(y_i) - f(x_i)] + \sum_{i=1}^{n}\varepsilon_i e_i^2 + \sum_{i=1}^{n} k_i|e_i| \\ & -\sum_{i=1}^{n}(k_i + \bar{k}) + \frac{1}{2}\sum_{i=1}^{n}\mathrm{trace}(\sigma_i^T \sigma_i) \\ & -\sum_{i=1}^{n}(k_i + \bar{k})|e_i| - \sum_{i=1}^{n}(\varepsilon_i + L)e_i^2. \end{aligned} \tag{12}$$

Simplify Eq. (12), we get

$$\begin{aligned} \mathcal{L}V(t) \quad = \quad & \sum_{i=1}^{n} e_i(t)[f(y_i) - f(x_i)] - L\sum_{i=1}^{n} e_i^2 - \sum_{i=1}^{n}\bar{k}|e_i| \\ & +\frac{1}{2}\sum_{i=1}^{n}\mathrm{trace}(\sigma_i^T \sigma_i) - \sum_{i=1}^{n}(k_i + \bar{k}). \end{aligned} \tag{13}$$

From Assumptions 2.1 and 2.2, we obtain

$$\mathcal{L}V(t) \quad \leq \quad -\sum_{i=1}^{n}(L - l - q)e_i^2 - \sum_{i=1}^{n}\bar{k}|e_i| - \sum_{i=1}^{n}(k_i + \bar{k}). \tag{14}$$

Since $L \geq l + q$ and $\overline{k} \geq 1$, we have

$$\begin{aligned} \mathcal{L}V(t) &\leq -(\sum_{i=1}^{n} \bar{k}|e_i| + \sum_{i=1}^{n}(k_i + \bar{k})) \\ &\leq -\Big(\sum_{i=1}^{n} |e_i| + \sum_{i=1}^{n}(k_i + \bar{k})\Big). \end{aligned} \tag{15}$$

Using the fact that

$$\Big[\sum_{i=1}^{n} |e_i| + \sum_{i=1}^{n}(k_i + \bar{k})\Big] \geq \Big[\sum_{i=1}^{n} e_i^2 + \sum_{i=1}^{n}(k_i + \bar{k})^2\Big]^{\frac{1}{2}},$$

we obtain

$$\begin{aligned} \mathcal{L}V(t) &\leq -\Big[\sum_{i=1}^{n} e_i^2 + \sum_{i=1}^{n}(k_i + \bar{k})^2\Big]^{\frac{1}{2}} \\ &\triangleq -(2V_1)^{\frac{1}{2}}, \end{aligned}$$

where $V_1 = \frac{1}{2}\Big(\sum_{i=1}^{n} e_i^2 + \sum_{i=1}^{n}(k_i + \bar{k})^2\Big)$.

Note that $\mathcal{L}V \leq 0$. Then

$$E\dot{V} = \mathcal{L}V \leq 0.$$

Thus, $V$ is non-increasing in mean square. Then, there exists a upper bound $V^*$, such that

$$V_1 \leq V \leq V^*.$$

Let $\theta = \frac{V_1}{V^*} \leq 1$, then

$$\theta V \leq \theta V^* = V_1.$$

Thus, we have

$$\mathcal{L}V(t) \leq -\sqrt{2\theta}V^{\frac{1}{2}}. \tag{16}$$

According to Lemma 2.5, the trivial solution of the error system (9) is globally stochastically finite-time stable. This means that the synchronization between systems (1) and (2) could be achieved in finite time for almost every initial data, and the finite time is estimated by

$$E[T_0] \leq T_1 = \sqrt{\frac{2}{\theta}}V^{\frac{1}{2}}(0), \tag{17}$$

where $V(0) = [\sum_{i=1}^{n} e_i(0)^2 + \sum_{i=1}^{n}(k_i(0) + \bar{k})^2\Big]/2$. This completes the proof. □

**Remark 3.2.** From the inequality (14) we can see that, for any high level noise, there exits a sufficiently large positive constant $L$ such that the finite-time stochastic synchronization is realized in probability. Hence the synchronization is robust to the noise perturbation. The convergence time of proposed algorithm is closely related to the violation of noise intensities. From the inequality (14) and the Itô formula, one can also see that for fixed $V(0)$, the synchronization time is inversely proportionally to the noise intensity.

If system (4) is free of noise perturbation, namely $\sigma_i(e_i(t)) \equiv 0 (i = 1, 2, \ldots, n)$, from Theorem 3.1, we have the following corollary:

**Corollary 3.3.** Let Assumption 2.1 holds. If $\sigma_i(e_i(t)) \equiv 0$ in system (2) and $L \geq l$ , then the chaotic systems (1) and (2) can achieve finite-time synchronization under the following control scheme:

$$u_i(t) = \varepsilon_i e_i + k_i \text{sign}(e_i) - \frac{k_i + \bar{k}}{|e_i|} \text{sign}(e_i), \qquad e_i \neq 0, i = 1, 2, \ldots, n.$$

For $e_i \neq 0$, the feedback gains $\varepsilon_i$ and $k_i$ are adapted according to the following updated laws $\dot{\varepsilon}_i(t) = -e_i^2(t), \dot{k}_i(t) = -|e_i(t)|$; For $e_i = 0$, we set $k_i \equiv -\bar{k}$ and $\varepsilon_i \equiv -L$.

## 4. SIMULATION RESULTS

In this section, a three-dimensional chaotic system and a hyperchaotic system are performed to verify the feasibility and effectiveness of the above analytical results.

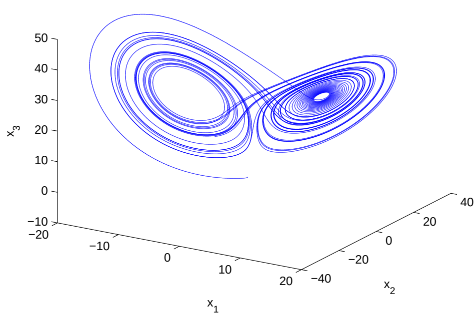


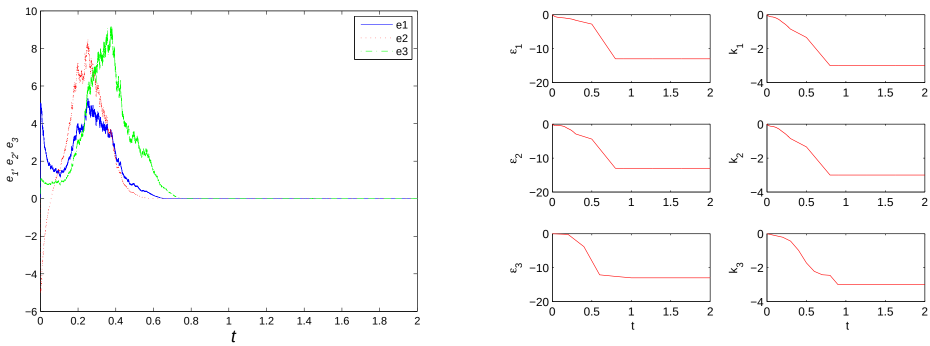

**Fig. 1.** Chaotic attractor generated by the system (18) when $a = 10, b = 8/3$, and $c = 28$.

**Example 4.1.** We take the Lorenz system as the first example, which can be described as follows [11]:

$$\begin{cases} \dot{x}_1 = -ax_1 + ax_2 \\ \dot{x}_2 = cx_1 - x_2 - x_1 x_3 \\ \dot{x}_3 = -x_1 x_2 - bx_3, \end{cases} \tag{18}$$

where $x = (x_1, x_2, x_3)^T \in R^3$ is the state vector. System has a double-scrolling chaotic attractor when $a = 10$, $b = 8/3$, and $c = 28$ as shown in Figure 1.

To confirm that the complete synchronization is achieved in finite time, we choose the initial conditions of the Lorenz system as follows: $x(0) = [-2, 3, 1]^T$, $y(0) = [3, -2, 2]^T$; and $\varepsilon_i(0) = k_i(0) = 0$, $L = 13$, $\bar{k} = 3$. For simplicity, we take $\sigma_i(e_i(t)) = \sigma e_i(t)$, $i = 1, 2, \ldots, n$. Additionally, assume that $\dot{W}(t)$ is a one-dimensional white noise. Then, $\sigma_i(e_i(t))$ satisfies the locally Lipschitz condition and the linear growth condition. The corresponding numerical results are shown in Figures 2 (a) and (b). Figure 2 (a) shows the temporal evolutions of synchronization errors between Eqs. (1) and (2), and the temporal evolutions of variable strengths $\varepsilon_i$, $k_i$ are shown in Figure 2 (b). The control parameters $\varepsilon_i$ and $k_i$ converge to -13 and -3 respectively. The simulation results show that the slave system (2) synchronizes the master system (1) after $T_1 = 0.7902$.




(a) (b)

**Fig. 2.** (a) The evolutions of synchronization errors between two coupled Lorenz systems. (b) The evolutions of the feedback strengths $\varepsilon_i$ and $k_i$. The initial values are $[x, y]^T = [-2, 3, 1, 3, -2, 2]^T$, $\varepsilon_i(0) = k_i(0) = 0$. The noise intensity $\sigma = 4$.

To study the effect of the violations of noise intensities $\sigma$ on the settling time, we simulate the evolutions of two chaotic systems with the controllers defined in Eq.(7) through taking different values of $\sigma$. Figure 3 gives the evolutions of the total errors function $E(t)$ with different values of $\sigma$, where $E(t) = \|e(t)\|$. It shows that the synchronization time is inversely proportionally to the noise intensity.

**Example 4.2.** To show the generality of the present method, we take the hyperchaotic Rössler system as the second example. The hyperchaotic Rössler system can be described by a four-dimensional differential equation as follows [22, 23]:

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + ax_2 + x_4 \\ \dot{x}_3 = x_1 x_3 + b \\ \dot{x}_4 = -cx_3 + dx_4, \end{cases} \tag{19}$$

where $x = (x_1, x_2, x_3, x_4)^T \in R^4$ is the state vector. System (19) has a chaotic attractor when $a = 0.25$, $b = 3$, $c = 0.5$, and $d = 0.05$ as shown in Figure 4.

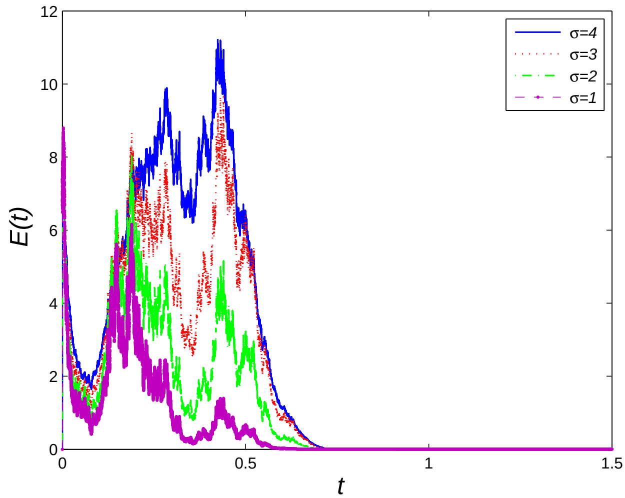



**Fig. 3.** The variations of the total synchronization errors $E(t)$ between chaotic systems (1) and (2) with $\sigma = 1, 2, 3, 4$.

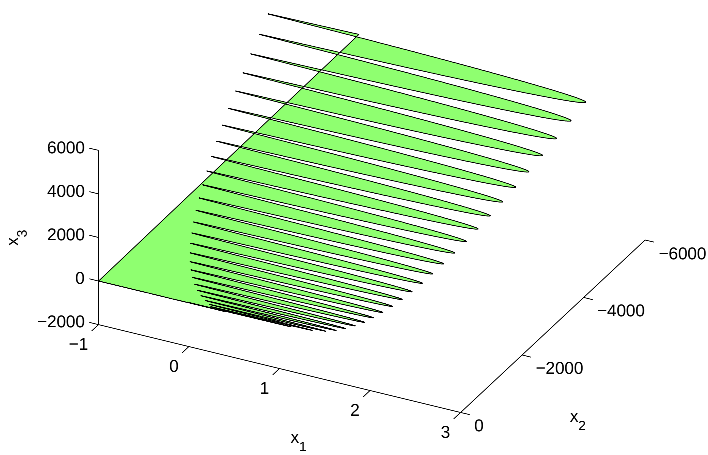
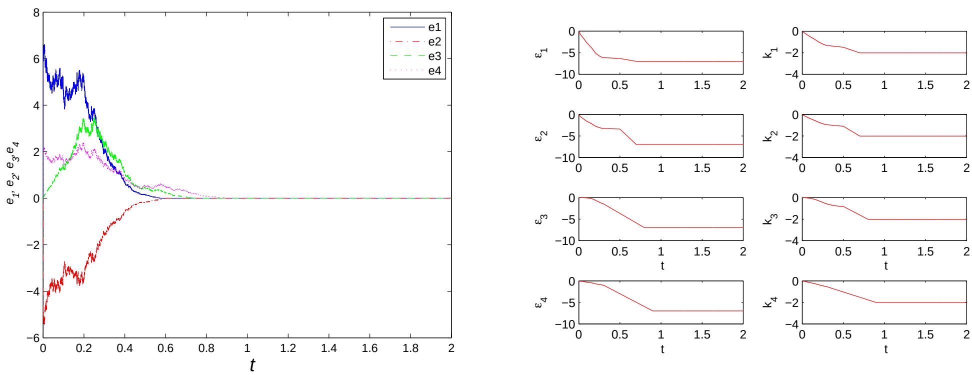



**Fig. 4.** Chaotic attractor generated by the system (19) when $a = 0.25$, $b = 3$, $c = 0.5$, and $d = 0.05$.

For the hyperchaotic Rössler system, we choose the initial conditions of the master system (1) and the controlled slave system (2) as $x(0) = [-4, 1, 3, -1]^T$ and $y(0) = [2, -4, 3, 1]^T$ respectively; and the other parameters as $\varepsilon_i(0) = k_i(0) = 0, L = 7, \bar{k} = 2$. At the same time, take $\sigma_i(e_i(t)) = \sigma e_i(t)$ $(i = 1, 2, \dots, n)$. From Figures 5 (a) and (b), one can find that the slave system (2) synchronizes the master system (1) after $T_1 = 0.8709$. And the control parameters $\varepsilon_i$ and $k_i$ converge to -7 and -2 respectively.



(a) (b)

**Fig. 5.** (a) The evolutions of synchronization errors between two hyperchaotic Rössler systems. (b) The evolutions of the feedback strengths $\varepsilon_i, k_i$. The initial values are $[x, y]^T = [-4, 1, 3, -1, 2, -4, 3, 1]^T$, $\varepsilon_i(0) = k_i(0) = 0$ and the noise intensity $\sigma = 1$.

The above two examples show that chaotic or hyperchaotic synchronization can be quickly achieved by the present method (i. e., the settle time is short) with noise perturbation. The time-varying feedback gains $\varepsilon_i$ and $k_i$ automatically converge to suitable constants.

## 5. CONCLUSIONS

In this paper, we have investigated the finite-time synchronization of chaotic systems with noise perturbation. We proposed an adaptive controller which can synchronize two chaotic or hyperchaotic systems in finite time. In comparison with previous methods, the proposed scheme is simple to implement in practice. Numerical simulations are provided to illustrate the effectiveness and feasibility of the above method. In addition, time delay due to the finite information transmission between two coupled chaotic systems is unavoidable. The present study does not consider the effect of time delay. Therefore, this is our next research topic.

## ACKNOWLEDGEMENT

This work was supported by the National Natural Science Foundation of China (grant nos. 61403393) and the Fundamental Research Funds for the Central Universities (grant no. 2013XK03).

(Received April 28, 2014)

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