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ON AN EXPONENTIAL INEQUALITY AND A STRONG LAW OF LARGE NUMBERS FOR MONOTONE MEASURES

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An exponential inequality for Choquet expectation is discussed. We also obtain a strong law of large numbers based on Choquet expectation. The main results of this paper improve some previous results obtained by many researchers.

Keywords: Choquet expectation, monotone probability, exponential inequality, a strong law of large numbers

Classification: 60E15, 28A12

1. INTRODUCTION

The exponential inequality for the partial sum $\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i])$ is very useful for proving convergence rate in the strong law of large numbers. Several statisticians attempts to find the fastest convergence rate in the strong law of large numbers on dependent random variables [5] has led to the creation of different types of exponential inequality. This motivates us to improve the inequalities which were obtained by many researchers [4, 8, 9, 13, 15, 16]. On the other hand, some problems in mathematical economics, statistics, quantum mechanics and finance cannot be well analyzed by additive probabilities. In this paper, an exponential inequality and a strong law of large numbers for Choquet integral based on some type of monotone measures are given.

The rest of the paper is organized as follows. Some notions and definitions that are useful in this paper are given in Section 2. In Sections 3,4 we state the main results of this paper.

2. DEFINITIONS AND NOTATIONS

In order to derive our main results, we have to recall here the following definitions and notations. Let $(\Omega, \mathcal{F})$ be a fixed measurable space. A random variable $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-measurable if, for each $B \in \mathcal{B}(\mathbb{R})$, the σ-algebra of Borel subsets of $\mathbb{R}$, the preimage $X^{-1}(B)$ is an element of $\mathcal{F}$. A set function $\mu : \mathcal{F} \to [0, \infty]$ is called a monotone measure whenever $\mu(\emptyset) = 0$, $\mu(\Omega) > 0$ and $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$, moreover, μ is called *real* if $||\mu|| = \mu(\Omega) < \infty$ and μ is said to be an *additive measure* if $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$. μ is also *subadditive* whenever

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$\mu(A \cup B) \leq \mu(A) + \mu(B)$ *for all* $A, B \in \mathcal{F}$. μ is called a *monotone probability* if $\|\mu\| = 1$. Notice that a monotone probability with σ-additivity assumption is called a probability measure. If $\mu(B_n) \downarrow \mu(B)$ for all sequences of measurable sets such that $B_n \downarrow B$ and $\mu(B_n) \uparrow \mu(B)$ for all sequences of measurable sets such that $B_n \uparrow B$, the real monotone measure μ is called *continuous*. The *conjugate* $\overline{\mu}$ of a real monotone measure μ is defined by $\overline{\mu}(A) = \|\mu\| - \mu(\Omega \backslash A)$, $A \in \mathcal{F}$. Note that a monotone measure μ is also *submodular (2-alternating)* whenever $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$ *for all* $A, B \in \mathcal{F}$. If μ is a real submodular continuous measure, then the triple $(\Omega, \mathcal{F}, \mu)$ is also called a *submodular continuous (SC-) measure space*. Throughout this paper, $\mathbb{I}_A$ denotes the indicator function of the set A.

Given a real monotone measure space $(\Omega, \mathcal{F}, \mu)$, we shall denote by ω elements of Ω and we put $\{X \geq t\} = \{\omega : X(\omega) \geq t\}$ for any real t. The (asymmetric) Choquet integral (expectation) of X with respect to a real monotone measure μ is defined as

$$\mathbb{E}_C^{\mu}[X] = \int_{-\infty}^{0} [\mu(\{X \geq t\}) - \|\mu\|] \, \mathrm{d}t + \int_{0}^{+\infty} \mu(\{X \geq t\}) \, \mathrm{d}t. \tag{1}$$

Note that for non-negative X the formula (1) is

$$\mathbb{E}_C^{\mu}[X] = \int_{0}^{+\infty} \mu(\{X \geq t\}) \, \mathrm{d}t. \tag{2}$$

The Choquet integral was introduced in [1], see also [2, 10, 14]. Some basic properties of Choquet expectation are summarized in [2], we cite some of them:

- $\mathbb{E}_C^{\mu}[\mathbb{I}_A] = \mu(A)$;
- $\mathbb{E}_C^{\mu}[\beta X] = \beta \mathbb{E}_C^{\mu}[X]$ for any real $\beta \geq 0$ (positive homogeneity);
- $\mathbb{E}_C^{\mu}[X + \beta] = \mathbb{E}_C^{\mu}[X] + \beta \|\mu\|$ for any real β (traslatability), moreover, by traslatability (put $X = 0$), we have $\mathbb{E}_C^{\mu}[\beta \mathbb{I}_\Omega] = \beta \|\mu\|$ for any real β;
- $\mathbb{E}_C^{\mu}[-X] = -\mathbb{E}_C^{\overline{\mu}}[X]$ (asymmetry);
- $\mathbb{E}_C^{\mu}[X] \leq \mathbb{E}_C^{\mu}[Y]$ whenever $X \leq Y$ (monotonicity).

Definition 2.1. Let $(\Omega, \mathcal{F}, \mu)$ be a monotone measure space. Random variables $Y_1, \ldots, Y_n :$ $\Omega \to \mathbb{R}$ are identically distributed if, for each $n, m \geq 1$ and each Borel set B,

$$\mu\{Y_n \in B\} = \mu\{Y_m \in B\}.$$

Proposition 2.2. (Mesiar et al. [7]) Let $X, Y : \Omega \to [0, \infty)$ be two random variables such that $\mathbb{E}_C^{\mu}[XY]$, $\mathbb{E}_C^{\mu}[X^p]$ and $\mathbb{E}_C^{\mu}[Y^q]$ exist for all $p, q > 1$. If μ is a real submodular monotone measure, then
(i) Hölder's inequality

$$\mathbb{E}_C^{\mu}[XY] \leq \left(\mathbb{E}_C^{\mu}[X^p]\right)^{\frac{1}{p}} \left(\mathbb{E}_C^{\mu}[Y^q]\right)^{\frac{1}{q}}, \tag{3}$$

holds where $\frac{1}{p} + \frac{1}{q} = 1, p > 1$.

(ii) Minkowski's inequality

$$\left(\mathbb{E}_C^{\mu}\left[(X+Y)^p\right]\right)^{\frac{1}{p}} \leq \left(\mathbb{E}_C^{\mu}\left[X^p\right]\right)^{\frac{1}{p}} + \left(\mathbb{E}_C^{\mu}\left[Y^p\right]\right)^{\frac{1}{p}}, \tag{4}$$

holds where $p \geq 1$.

Definition 2.3. (Li et al. [6]) Let $(\Omega, \mathcal{A})$ be a measurable space and μ be a continuous real submodular monotone measure. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables.

(I) We say that X_n converges in μ to X and write $X_n \xrightarrow{\mu} X$ if for all $\epsilon > 0$,

$$\mu\left[|X_n - X| > \epsilon\right] \to 0 \quad \text{as } n \to \infty.$$

(II) We say that X_n is μ-almost everywhere convergent to X and write $X_n \xrightarrow{\mu\text{-}a.e.} X$ if there is subset $E \subset \Omega$ such that $\mu(E) = 0$,

$$\forall \omega \in \Omega \backslash E : X_n(\omega) \to X(\omega) \quad \text{as } n \to \infty.$$

3. AN EXPONENTIAL INEQUALITY

Lemma 3.1. If μ is a real submodular monotone measure, then for any $\beta > 1$, $\delta > 0$ and for any $0 < \lambda \leq \frac{\delta}{2}$,

$$\mathbb{E}_C^{\mu}\left[\exp \lambda \left(X - \frac{\mathbb{E}_C^{\mu}[X]}{\|\mu\|}\right)\right] \leq \|\mu\| \exp\left(\lambda^2 T_{\beta,\delta}\right), \tag{5}$$

where

$$T_{\beta,\delta} = 2\left(\frac{1}{\|\mu\|}\mathbb{E}_C^{\mu}\left[|X|^{2\beta}\right]\right)^{\frac{1}{\beta}} \left(\frac{1}{\|\mu\|}\mathbb{E}_C^{\mu}\left[\exp\left(\frac{\beta\delta}{2(\beta-1)}|X|\right)\right]\right)^{\frac{2(\beta-1)}{\beta}}, \tag{6}$$

such that the Choquet expectations in (6) exist.

Proof. The proof is carried out in two steps.
(a) Let $\|\mu\| = 1$. By the fact that $e^x \leq 1 + x + \frac{x^2}{2}e^{|x|}$ for all $x \in \mathbb{R}$, and any $0 < \lambda \leq \frac{\delta}{2}$, we have

$$\mathbb{E}_C^{\mu}\left[\exp \lambda \left(X - \mathbb{E}_C^{\mu}[X]\right)\right] \leq \mathbb{E}_C^{\mu}\left[1 + \lambda\left(X - \mathbb{E}_C^{\mu}[X]\right) + \frac{\lambda^2}{2}\left(X - \mathbb{E}_C^{\mu}[X]\right)^2 e^{\lambda\left|X - \mathbb{E}_C^{\mu}[X]\right|}\right].$$

Since μ is a submodular, there holds

$$\mathbb{E}_C^{\mu}\left[\exp \lambda \left(X - \mathbb{E}_C^{\mu}[X]\right)\right] \leq \mathbb{E}_C^{\mu}[1] + \mathbb{E}_C^{\mu}\left[\lambda\left(X - \mathbb{E}_C^{\mu}[X]\right)\right]$$
$$+\mathbb{E}_C^{\mu}\left[\frac{\lambda^2}{2}\left(X - \mathbb{E}_C^{\mu}[X]\right)^2 e^{\lambda\left|X - \mathbb{E}_C^{\mu}[X]\right|}\right].$$

Then, by positive homogeneity and traslatability, we obtain

$$\mathbb{E}_C^{\mu}\left[\exp \lambda \left(X - \mathbb{E}_C^{\mu}[X]\right)\right] \leq 1 + \frac{\lambda^2}{2}\mathbb{E}_C^{\mu}\left[\left(X - \mathbb{E}_C^{\mu}[X]\right)^2 e^{\lambda\left|X - \mathbb{E}_C^{\mu}[X]\right|}\right].$$

So, for all $\beta > 1$,

$$\mathbb{E}_C^\mu\left[\exp\lambda\left(X - \mathbb{E}_C^\mu\left[X\right]\right)\right] \le 1 + \frac{\lambda^2}{2}\left(\mathbb{E}_C^\mu\left[\left(X - \mathbb{E}_C^\mu\left[X\right]\right)^{2\beta}\right]\right)^{\frac{1}{\beta}}\left(\mathbb{E}_C^\mu\left[e^{\frac{\beta\lambda}{\beta-1}\left|X - \mathbb{E}_C^\mu[X]\right|}\right]\right)^{\frac{\beta-1}{\beta}}$$

$$\le 1 + \frac{\lambda^2}{2}\left(\left(\mathbb{E}_C^\mu\left[|X|^{2\beta}\right]\right)^{\frac{1}{2\beta}} + \left(\mathbb{E}_C^\mu\left[\left|\mathbb{E}_C^\mu\left[X\right]\right|^{2\beta}\right]\right)^{\frac{1}{2\beta}}\right)^2\left(\mathbb{E}_C^\mu\left[e^{\frac{\beta\lambda}{\beta-1}|X|}e^{\frac{\beta\lambda}{\beta-1}\mathbb{E}_C^\mu[|X|]}\right]\right)^{\frac{\beta-1}{\beta}}$$

$$= 1 + \frac{\lambda^2}{2}\left(\left(\mathbb{E}_C^\mu\left[X\right]^{2\beta}\right)^{\frac{1}{2\beta}} + \mathbb{E}_C^\mu\left|X\right|\right)^2\left(\mathbb{E}_C^\mu\left[e^{\frac{\beta\lambda}{\beta-1}|X|}e^{\frac{\beta\lambda}{\beta-1}\mathbb{E}_C^\mu[|X|]}\right]\right)^{\frac{\beta-1}{\beta}}$$

$$\le 1 + \frac{\lambda^2}{2}\left(\left(\mathbb{E}_C^\mu\left[|X|^{2\beta}\right]\right)^{\frac{1}{2\beta}} + \left(\mathbb{E}_C^\mu\left[|X|^{2\beta}\right]\right)^{\frac{1}{2\beta}}\right)^2\left(\mathbb{E}_C^\mu\left[e^{\frac{\beta\lambda}{\beta-1}|X|}e^{\frac{\beta\lambda}{\beta-1}\mathbb{E}_C^\mu[|X|]}\right]\right)^{\frac{\beta-1}{\beta}}$$

$$\le 1 + \frac{\lambda^2}{2}\left(2\left(\mathbb{E}_C^\mu\left[|X|^{2\beta}\right]\right)^{\frac{1}{2\beta}}\right)^2\left(\mathbb{E}_C^\mu\left[e^{\frac{\beta\lambda}{\beta-1}|X|}\right]\mathbb{E}_C^\mu\left[e^{\frac{\beta\lambda}{\beta-1}|X|}\right]\right)^{\frac{\beta-1}{\beta}}$$

$$= 1 + 2\lambda^2\left(\mathbb{E}_C^\mu\left[|X|^{2\beta}\right]\right)^{\frac{1}{\beta}}\left(\mathbb{E}_C^\mu\left[\exp\left(\frac{\beta\lambda}{\beta-1}|X|\right)\right]\right)^{\frac{2(\beta-1)}{\beta}}$$

$$\le 1 + 2\lambda^2\left(\mathbb{E}_C^\mu\left[|X|^{2\beta}\right]\right)^{\frac{1}{\beta}}\left(\mathbb{E}_C^\mu\left[\exp\left(\frac{\beta\delta}{2(\beta-1)}|X|\right)\right]\right)^{\frac{2(\beta-1)}{\beta}}$$

$$\le \exp\left(2\lambda^2\left(\mathbb{E}_C^\mu\left[|X|^{2\beta}\right]\right)^{\frac{1}{\beta}}\left(\mathbb{E}_C^\mu\left[\exp\left(\frac{\beta\delta}{2(\beta-1)}|X|\right)\right]\right)^{\frac{2(\beta-1)}{\beta}}\right).$$

(b) Let $\|\mu\| \neq 1$ and $\mu' = \frac{\mu}{\|\mu\|}$. Part (a) implies that

$$\frac{1}{\|\mu\|}\mathbb{E}_C^\mu\left[\exp\lambda\left(X - \frac{\mathbb{E}_C^\mu\left[X\right]}{\|\mu\|}\right)\right] = \frac{1}{\|\mu\|}\mathbb{E}_C^{\mu'\|\mu\|}\left[\exp\lambda\left(X - \frac{\mathbb{E}_C^{\mu'\|\mu\|}\left[X\right]}{\|\mu\|}\right)\right]$$

$$= \mathbb{E}_C^{\mu'}\left[\exp\lambda\left(X - \mathbb{E}_C^{\mu'}\left[X\right]\right)\right]$$

$$\le \exp\left(2\lambda^2\left(\mathbb{E}_C^{\mu'}\left[|X|^{2\beta}\right]\right)^{\frac{1}{\beta}}\left(\mathbb{E}_C^{\mu'}\left[\exp\left(\frac{\beta\delta}{2(\beta-1)}|X|\right)\right]\right)^{\frac{2(\beta-1)}{\beta}}\right)$$

$$= \exp\left(2\lambda^2\left(\mathbb{E}_C^{\frac{\mu}{\|\mu\|}}\left[|X|^{2\beta}\right]\right)^{\frac{1}{\beta}}\left(\mathbb{E}_C^{\frac{\mu}{\|\mu\|}}\left[\exp\left(\frac{\beta\delta}{2(\beta-1)}|X|\right)\right]\right)^{\frac{2(\beta-1)}{\beta}}\right)$$

$$= \exp\left(2\lambda^2\left(\frac{1}{\|\mu\|}\mathbb{E}_C^\mu\left[|X|^{2\beta}\right]\right)^{\frac{1}{\beta}}\left(\frac{1}{\|\mu\|}\mathbb{E}_C^\mu\left[\exp\left(\frac{\beta\delta}{2(\beta-1)}|X|\right)\right]\right)^{\frac{2(\beta-1)}{\beta}}\right).$$

□

Theorem 3.2. Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed random variables. If there exists $\delta > 0$ such that for any real λ such that $|\lambda| \le \delta$,

$$\mathbb{E}_C^\mu\left[\exp\left(\lambda\sum_{i=1}^n X_i\right)\right] \le C\prod_{i=1}^n \mathbb{E}_C^\mu\left[\exp\left(\lambda X_i\right)\right] \tag{7}$$

where C is a positive constant and μ is a real submodular monotone measure, then for any $\beta > 1$, $\delta > 0$ such that $T_{\beta,\delta}$ given by (6) exist, and for any $0 < \epsilon \leq \delta T_{\beta,\delta}$, it holds

$$\mu\left(\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right) > n\epsilon\right) \leq C\|\mu\|\exp\left(\frac{-n\epsilon^2}{4T_{\beta,\delta}}\right),$$

$$\mu\left(-\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\overline{\mu}}[X_i]}{\|\mu\|}\right) > n\epsilon\right) \leq C\|\mu\|\exp\left(\frac{-n\epsilon^2}{4T_{\beta,\delta}}\right).$$

Finally, if μ is additive, then we get the following exponential inequality

$$\mu\left(\left|\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right)\right| > n\epsilon\right) \leq 2C\|\mu\|\exp\left(\frac{-n\epsilon^2}{4T_{\beta,\delta}}\right).$$

P r o o f. Let $0 < \epsilon \leq \delta T_{\beta,\delta}$. For any $0 < \lambda \leq \frac{\delta}{2}$, Markov's inequality for Choquet expectation implies that

$$\begin{aligned}
&\mu\left(\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right) > n\epsilon\right) = \mu\left(\exp\left(\lambda\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right)\right) > \exp(\lambda n\epsilon)\right)\\
&\leq \exp(-\lambda n\epsilon)\,\mathbb{E}_C^{\mu}\left(\exp\left(\lambda\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right)\right)\right)\\
&= \exp(-\lambda n\epsilon)\exp\left(\frac{-\lambda}{\|\mu\|}\sum_{i=1}^{n}\mathbb{E}_C^{\mu}[X_i]\right)\mathbb{E}_C^{\mu}\left(\exp\left(\lambda\sum_{i=1}^{n}X_i\right)\right)\\
&= \exp(-\lambda n\epsilon)\left(\prod_{i=1}^{n}\exp\left(\frac{-\lambda}{\|\mu\|}\mathbb{E}_C^{\mu}[X_i]\right)\right)\mathbb{E}_C^{\mu}\left(\exp\left(\lambda\sum_{i=1}^{n}X_i\right)\right)\\
&\leq C\exp(-\lambda n\epsilon)\left(\prod_{i=1}^{n}\exp\left(\frac{-\lambda}{\|\mu\|}\mathbb{E}_C^{\mu}[X_i]\right)\right)\prod_{i=1}^{n}\mathbb{E}_C^{\mu}(\exp(\lambda X_i)) \quad \text{(by (7))}\\
&= C\exp(-\lambda n\epsilon)\prod_{i=1}^{n}\mathbb{E}_C^{\mu}\left(\exp\lambda\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right)\right)\\
&\leq C\|\mu\|\exp\left(-\lambda n\epsilon + n\lambda^2 T_{\beta,\delta}\right) \qquad \text{(by (5))}.
\end{aligned} \tag{8}$$

Taking $\lambda = \frac{\epsilon}{2T_{\beta,\delta}}$, since $0 < \epsilon \leq \delta T_{\beta,\delta}$, we derive from (8) that

$$\mu\left(\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right) > n\epsilon\right) \leq C\|\mu\|\exp\left(\frac{-n\epsilon^2}{4T_{\beta,\delta}}\right).$$

Now, we can replace X_i by $-X_i$ in the above statement, we have

$$\begin{aligned}
&\mu\left(\sum_{i=1}^{n}\left(-X_i - \frac{\mathbb{E}_C^{\mu}[-X_i]}{\|\mu\|}\right) > n\epsilon\right) = \mu\left(-\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\overline{\mu}}[X_i]}{\|\mu\|}\right) > n\epsilon\right)\\
&\leq C\|\mu\|\exp\left(\frac{-n\epsilon^2}{4T_{\beta,\delta}}\right).
\end{aligned}$$

Here we have exploited the fact that the term $T_{\beta,\delta}$ given by (6) remains unchanged if we replace X by $-X$. Finally, if μ is additive ($\mu = \overline{\mu}$), then we have

$$\mu\left(\left|\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right)\right| > n\epsilon\right) \le 2C\,\|\mu\| \exp\left(\frac{-n\epsilon^2}{4T_{\beta,\delta}}\right).$$

□

Theorem 3.3. Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed random variables such that conditions (7) hold. Let $\{\epsilon_n, n \ge 1\}$ be a sequence of positive real numbers such that $\epsilon_n \to 0$ as $n \to \infty$. If μ is a submodular monotone probability, then for all sufficiently large n, and any $\beta > 1$, $\delta > 0$ such that $T_{\beta,\delta}$ given by (6) exist, it holds

$$\mu\left(\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right) > n\epsilon_n\right) \le C\,\|\mu\| \exp\left(\frac{-n\epsilon_n^2}{4T_{\beta,\delta}}\right),$$

$$\mu\left(-\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\overline{\mu}}[X_i]}{\|\mu\|}\right) > n\epsilon_n\right) \le C\,\|\mu\| \exp\left(\frac{-n\epsilon_n^2}{4T_{\beta,\delta}}\right).$$

Finally, if μ is additive, then for all sufficiently large n, we have

$$\mu\left(\left|\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right)\right| > n\epsilon_n\right) \le 2C\,\|\mu\| \exp\left(\frac{-n\epsilon_n^2}{4T_{\beta,\delta}}\right).$$

Proof. The proof follows by Theorem 3.2. Note that n is sufficiently large whenever $\epsilon_n \le \delta T_{\beta,\delta}$. □

Corollary 3.4. Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed random variables such that conditions (7) hold. Let $\epsilon_n = \sqrt{\frac{4T_{\beta,\delta}\alpha \ln n}{n}}$ where $\alpha > 0$ and $T_{\beta,\delta}$ given by (6) exist for some $\beta > 1$ and $\delta > 0$. If μ is a submodular monotone probability, then for all sufficiently large n,

$$\mu\left(\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}[X_i]}{\|\mu\|}\right) > n\epsilon_n\right) \le C\,\|\mu\| \exp(-\alpha \ln n),$$

$$\mu\left(-\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\overline{\mu}}[X_i]}{\|\mu\|}\right) > n\epsilon_n\right) \le C\,\|\mu\| \exp(-\alpha \ln n).$$

Finally, if μ is additive, for all sufficiently large n, we have

$$\mu\left(\left|\sum_{i=1}^{n}\left(X_i - \frac{\mathbb{E}_C^{\mu}(X_i)}{\|\mu\|}\right)\right| > n\epsilon_n\right) \le 2C\,\|\mu\| \exp(-\alpha \ln n).$$

Proof. The proof follows by Theorem 3.3. □

Types	ϵ_n	The upper bound	Condition
[4]	$O\Big(\big(p_n(\ln n)^3/n\big)^{\frac{1}{2}}\Big)$	$\left(4+\frac{\mathbb{E}e^{\delta\lvert X_1\rvert}n^2}{9\alpha^3 p_n(\ln n)^3}\right)e^{-\alpha\ln n}$	$1\le p_n<\frac{n}{2}, 0<\alpha<\delta.$
[8]	$O\Big(\big(p_n(\ln n)^3/n\big)^{\frac{1}{2}}\Big)$	$\left(2\left(1+\frac{\alpha C_0}{4}\right)+\frac{2\mathbb{E}e^{\delta\lvert X_1\rvert}n^2}{9\alpha^3 p_n(\ln n)^3}\right)e^{-\alpha\ln n}$	$1\le p_n<\frac{n}{2}, 0<\alpha<\delta,$ $0<C_0<\infty,$ with an assumption on the covariance.
[16]	$O\Big(\big((\ln n)^3/n\big)^{\frac{1}{2}}\Big)$	$\left(4+\frac{C_0\mathbb{E}e^{\delta\lvert X_1\rvert}}{4\alpha^3(\ln n)^3}\right)e^{-\alpha\ln n}$	$1\le p_n<O(n/\ln n)^{\frac{1}{2}},$ $, C_0>0, 0<\alpha\leqslant\delta.$
[15]	$O\Big((\ln n/n)^{\frac{1}{2}}\Big)$	$\left(4+\frac{C_1\mathbb{E}e^{\delta\lvert X_1\rvert}}{4C_2\alpha^3\ln n}\right)e^{-\alpha\ln n}$	$1\le p_n<O(n/\ln n)^{\frac{1}{2}},$ $C_1>0, C_2>0, \alpha>0.$
[9]	$O\Big(\big(p_n(\ln n)^3/n\big)^{\frac{1}{2}}\Big)$	$\left(2\left(1+\frac{4C_0}{\alpha}\right)+\frac{2\mathbb{E}e^{\delta\lvert X_1\rvert}n^2}{9\alpha^3 p_n(\ln n)^3}\right)e^{-\alpha\ln n}$	$1\le p_n<\frac{n}{2}, 0<\alpha<\delta$ $0<C_0<\infty.$
[13]	$O\Big((\ln n/n)^{\frac{1}{2}}\Big)$	$2\left(1+\frac{\mathbb{E}e^{\delta\lvert X_1\rvert}}{e\delta^3 E\lvert X_1\rvert^2\ln n}\right)e^{-\delta\ln n}$	$\delta>0.$
Our result	$O\Big((\ln n/n)^{\frac{1}{2}}\Big)$	$2e^{-\alpha\ln n}$	$\alpha>0.$

Tab. 1. Some different types of exponential inequalities based on probability measure.

Remark 3.5. When μ is a probability measure, Corollary 3.4 improves the corresponding results derived by many researchers [4, 8, 9, 13, 15, 16] (see Table 1). Note that these results were obtained under different constraints. So, for example, in [9], $X_1, X_2, \dots$ are supposed to be strictly stationary and associated random variables, while we have considered a sequence of identically distributed random variables (satisfying (7)).

4. A STRONG LAW OF LARGE NUMBERS

Below we consider a sequence of non-negative random variables $\{X_i\}_{i=1}^{\infty}$ defined on an SC-measure space $(\Omega, \mathcal{F}, \mu)$ and denote $S_n = \sum_{i=1}^{n} X_i$. During this section, we always consider the existence of all Choquet expectations $\mathbb{E}_C^{\mu}[.]$.

Lemma 4.1. (Dvoretzky [3]) Let $\{a_n\}$ be a sequence of nonnegative numbers such that $\sum_{n=1}^{\infty}\frac{a_n}{n}<\infty$. Then there exists a sequence of integers $\{n_k\}$ such that $n_{k+1}>n_k$ for all k, $\frac{n_{k+1}}{n_k}\to 1\,(k\to\infty)$, and $\sum_{k=1}^{\infty} a_{n_k}<\infty$.

Theorem 4.2. Let Φ_1, Φ_2 be two increasing nonnegative-valued functions on $[0,\infty)$ such that Φ_2 is supermultiplicative, (i. e., $\Phi_2(xy)\ge\Phi_2(x)\Phi_2(y)$ for all $x, y\in[0,\infty)$) and $\Phi_1(x)\ge\Phi_2(x)$ for all $x\in[0,\infty)$. If

$$\mathbb{E}_C^{\mu}[S_n]-\mathbb{E}_C^{\mu}[S_m]\le K(n-m) \tag{9}$$

for all sufficiently large $n-m$, where K is a positive constant, and

$$\sum_{n=1}^{\infty}\frac{1}{n\Phi_2(n)}\mathbb{E}_C^{\mu}\left[\Phi_1\left(\left|S_n-\|\mu\|\,\mathbb{E}_C^{\mu}[S_n]\right|\right)\right]<\infty, \tag{10}$$

then $\frac{S_n - \|\mu\| \mathbb{E}_C^\mu [S_n]}{n} \overset{\mu\text{-}a.e.}{\longrightarrow} 0$.

P r o o f. By Lemma 4.1 and (10), there exists a nondecreasing sequence of integers $\{n_k\}$ such that

$$\frac{n_{k+1}}{n_k} \to 1 \qquad (k \to \infty) \tag{11}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{\Phi_2(n_k)} \mathbb{E}_C^\mu \left[\Phi_1 \left(\left|S_{n_k} - \|\mu\| \mathbb{E}_C^\mu \left[S_{n_k}\right]\right|\right)\right] < \infty. \tag{12}$$

Given $\epsilon > 0$. For all $\Phi_2(\alpha) > 0$, we have

$$\begin{aligned}
&\frac{1}{\Phi_2(\alpha)} \mathbb{E}_C^\mu \left[\Phi_1 \left(\left|S_{k_n} - \|\mu\| \mathbb{E}_C^\mu \left[S_{k_n}\right]\right|\right)\right] \\
&\geq \frac{1}{\Phi_2(\alpha)} \mathbb{E}_C^\mu \left[\Phi_1 \left(\left|S_{k_n} - \|\mu\| \mathbb{E}_C^\mu \left[S_{k_n}\right]\right|\right) \mathbb{I}_{\left\{\Phi_1\left(\left|S_{k_n} - \|\mu\|\mathbb{E}_C^\mu[S_{k_n}]\right|\right) \geq \Phi_1(\alpha)\right\}}\right] \\
&\geq \frac{1}{\Phi_2(\alpha)} \mathbb{E}_C^\mu \left[\Phi_1(\alpha) \mathbb{I}_{\left\{\left|S_{k_n} - \|\mu\|\mathbb{E}_C^\mu[S_{k_n}]\right| \geq \alpha\right\}}\right] \geq \mu\left(\left|S_{k_n} - \|\mu\| \mathbb{E}_C^\mu \left[S_{k_n}\right]\right| \geq \alpha\right),
\end{aligned} \tag{13}$$

by monotonicity and positive homogeneity. Since Φ_2 is supermultiplicative [12], by (13) and (12), we have

$$\sum_{k=1}^{\infty} \mu\left(\left|S_{n_k} - \|\mu\| \mathbb{E}_C^\mu \left[S_{n_k}\right]\right| > \epsilon n_k\right) \leq \frac{1}{\Phi_2(\epsilon)} \sum_{n=1}^{\infty} \frac{\mathbb{E}_C^\mu \left[\Phi_1 \left|S_{n_k} - \|\mu\| \mathbb{E}_C^\mu \left[S_{n_k}\right]\right|\right]}{\Phi_2(n_k)} < \infty.$$

Since μ is a real submodular continuous measure, using the Borel–Cantelli lemma, we obtain

$$\frac{S_{n_k} - \|\mu\| \mathbb{E}_C^\mu \left[S_{n_k}\right]}{n_k} \overset{\mu\text{-}a.e.}{\longrightarrow} 0. \tag{14}$$

If $n_k \leq n < n_{k+1}$, then

$$\frac{S_n - \|\mu\| \mathbb{E}_C^\mu [S_n]}{n} \leq \frac{\left|S_{n_{k+1}} - \|\mu\| \mathbb{E}_C^\mu \left[S_{n_{k+1}}\right]\right|}{n_{k+1}} \frac{n_{k+1}}{n_k} + \frac{\|\mu\| \left(\mathbb{E}_C^\mu \left[S_{n_{k+1}}\right] - \mathbb{E}_C^\mu \left[S_{n_k}\right]\right)}{n_k}. \tag{15}$$

The first summand in the right-hand sides of (15) converges μ-almost everywhere to zero by (14) and (11). Similar lower bounds are valid for the left hand side of (15). Therefore, $\limsup_{n\to\infty} \frac{\left|S_n - \|\mu\|\mathbb{E}_C^\mu[S_n]\right|}{n} < \delta$ μ-a.e for any $\delta > 0$, by (11). This completes the proof. □

Let $\Phi_1 = \Phi_2 = \Phi$, then we get the following result.

Corollary 4.3. Let Φ be an increasing nonnegative-valued function on $[0, \infty)$ such that Φ is supermultiplicative. If

$$\mathbb{E}_C^\mu [S_n] - \mathbb{E}_C^\mu [S_m] \leq K (n - m) \tag{16}$$

for all sufficiently large $n - m$, where K is a positive constant, and

$$\sum_{n=1}^{\infty} \frac{1}{n\Phi(n)} \mathbb{E}_C^{\mu}\left[\Phi\left(\left|S_n - \|\mu\| \mathbb{E}_C^{\mu}\left[S_n\right]\right|\right)\right] < \infty, \tag{17}$$

then $\frac{S_n - \|\mu\| \mathbb{E}_C^{\mu}[S_n]}{n} \overset{\mu\text{-}a.e.}{\longrightarrow} 0$.

Corollary 4.4. Let Φ be an increasing nonnegative-valued function on $[0, \infty)$ such that Φ is supermultiplicative. If

$$\mathbb{E}_C^{\mu}\left[S_n\right] - \mathbb{E}_C^{\mu}\left[S_m\right] \leq K(n - m)$$

for all sufficiently large $n - m$, where K is a positive constant, and

$$\mathbb{E}_C^{\mu}\left[\Phi\left(\left|S_n - \|\mu\| \mathbb{E}_C^{\mu}\left[S_n\right]\right|\right)\right] = O\left(\frac{\Phi(n)}{\psi(n)}\right) \text{ for some function } \psi \in \Psi_c,$$

then $\frac{S_n - \|\mu\| \mathbb{E}_C^{\mu}[S_n]}{n} \overset{\mu\text{-}a.e.}{\longrightarrow} 0$. Here Ψ_c is the set of all function $\psi : (0, \infty) \longrightarrow (0, \infty)$ which do not decrease on domain (x_0, ∞) for some constant x_0, and the series $\sum \frac{1}{n\psi(n)}$ is convergent ([11, Chap. 9]).

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