KYBERNETIKA — VOLUME 50 (2014), NUMBER 5, PAGES 706–724

RISK AVERSION, PRUDENCE AND MIXED OPTIMAL SAVING MODELS

Irina Georgescu

The paper studies risk aversion and prudence of an agent in the face of a risk situation with two parameters, one described by a fuzzy number, the other described by a fuzzy variable. The first contribution of the paper is the characterization of risk aversion and prudence in mixed models by conditions on the concavity and the convexity of the agent's utility function and its partial derivatives. The second contribution is the building of mixed models of optimal saving and their connection with the concept of prudence and downside risk aversion.

Keywords: possibilistic risk aversion, prudence, optimal saving

Classification: 91B30, 94D05

1. INTRODUCTION

Risk aversion, prudence and optimal saving are important topics in probability theory of risk [1, 11, 17]. The first two regard attitudes of an agent in the face of risk and the third one studies the way the presence of risk influences the choice of level of optimal saving. These topics have been tackled both in case of a single risk parameter (represented by a random variable) and in case of several risk parameters (represented by a random vector).

In the last years, risk phenomena have been also studied in the context of Zadeh's possibility theory [28]. This approach assumes risk modeling by possibility distributions (in particular, by fuzzy numbers) and usual probabilistic indicators (expected value, variance, covariance, etc.) are replaced by corresponding possibilistic indicators [2], [3, 8, 13, 29].

The study of risk by mixed models appears in [16]: some risk parameters are fuzzy numbers and other parameters are random variables. The building of mixed models is based on mixed expected utility, concept which encompasses both the probabilistic expected utility [11, 17] and possibilistic expected utility [3, 14, 15].

This paper aims to approach risk aversion, prudence and optimal saving in the framework of mixed models, in which the agent will be represented by a bidimensional utility

DOI: 10.14736/kyb-2014-5-0706

function [1]. We will define risk aversion and prudence of an agent in the face of risk with two parameters (a fuzzy number and a random variable), then we will characterize these notions by conditions on partial derivatives of the utility function. We will build mixed models of optimal saving which we later relate to the notions of prudence and downside risk aversion.

In Section 2 possibilistic expected utility and two indicators of fuzzy numbers are recalled: the expected value and variance (cf. [2, 3, 13, 29]). Section 3 deals with bivariate mixed expected utility, concept introduced in [16]. The main results are three Jensen–type theorems (Propositions 3.9, 3.11 and 3.13) which represent the mathematical instrument to prove the results of the next sections.

Risk aversion is a topic whose study begins with papers by Arrow [1] and Pratt [26]. In case of risk with several parameters, the probabilistic theory of risk aversion has been developed in [9, 18, 19, 20]. Papers [14, 15] propose an approach to risk aversion in the context of possibility theory. In these papers risk is modeled by a fuzzy number and the definition and evaluation of risk aversion are done by the possibilistic indicators mentioned in Section 2. Section 4 approaches the risk aversion when the agent is represented by a bidimensional utility function and the risk by a mixed vector (a component is a fuzzy number, the other is a random variable). For such a model, three notions of risk aversion are defined: mixed risk aversion, possibilistic risk aversion and probabilistic risk aversion. Then three notions of risk premium are defined as a measure of the three types of risk aversion and formulas for their approximate calculation are proved.

Optimal saving under uncertainty is a topic which starts with the papers by Leland [22] and Sandmo [27], in which the notion of precautionary saving is introduced as a measure of the change in optimal saving by adding risk.

Kimball [21] connects the optimal saving and the agent's prudence in the face of risk. Optimal saving models for probabilistic risk with two parameters can be found in [4, 23] and the n–dimensional case is treated in [19].

Section 5 deals with optimal saving in case of risk represented by a mixed vector. After four optimization problems are formulated, three notions of precautionary saving are defined, indicators which measure the action of various types of risk (mixed, possibilistic, probabilistic) on the level of optimal saving. Using these notions of precautionary saving, three types of consumer's prudence are defined: mixed prudence, possibilistic prudence and probabilistic prudence. The main results of the section (Propositions 5.2, 5.3, 5.4) characterize the three types of prudence by positivity conditions of third–order partial derivatives of the utility function.

In [24], Menezes et al. have characterized prudence by the notion of downside risk aversion. The study of the relation between prudence and downside risk aversion has been deepened in [6, 10], and in [19] these results have been extended to the multidimensional case.

In Section 6 the connection between mixed prudence and a notion of downside risk aversion defined in mixed models is studied. Two new concepts are introduced: mixed utility premium and mixed prudence utility premium. The first notion is similar to

[1]The notions and results of the paper can be extended to a multidimensional utility function. For the clarity of presentation we chose to approach the bidimensional case.

utility premium defined by Friedman and Savage [12], and the second one is similar to prudence utility premium of [5]. Using these notions one defines what means that an agent displays mixed downside risk aversion (MDRA), then one proves that this condition is equivalent to mixed prudence.

2. INDICATORS OF FUZZY NUMBERS

In this section we recall the definition of fuzzy numbers and two of their indicators: the possibilistic expected value and variance [2, 3, 7, 8, 13, 29].

Let A be a fuzzy subset of $\mathbf{R}$. The support of A is the crisp subset of $\mathbf{R}$ defined by $\text{supp}(A) = \{x \in \mathbf{R} | A(x) > 0\}$. A is normal if $A(x) = 1$ for some $x \in \mathbf{R}$.

Let $\gamma \in [0, 1]$. The γ–level set of A is the crisp subset of $\mathbf{R}$ defined by:

$$[A]^\gamma = \begin{cases} \{x \in \mathbf{R} | A(x) \geq \gamma\} & \text{if} \quad \gamma > 0 \\ cl(\text{supp}(A)) & \text{if} \quad \gamma = 0. \end{cases}$$

(Note that $cl(\text{supp}(A))$ is the topological closure of $\text{supp}(A)$.)

A is said to be fuzzy convex if $[A]^\gamma$ is convex for all $\gamma \in [0, 1]$.

A fuzzy number[2] A is a fuzzy subset of $\mathbf{R}$ which is normal, fuzzy convex, upper semi–continuous and with bounded support.

Let A be a fuzzy number and $\gamma \in [0, 1]$. Then $[A]^\gamma$ is a closed and convex subset of $\mathbf{R}$. We denote $a_1(\gamma) = \min[A]^\gamma$ and $a_2(\gamma) = \max[A]^\gamma$. Hence $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for all $\gamma \in [0, 1]$.

A function $f : [0, 1] \to \mathbf{R}$ is a weighting function if it is non–negative, monotone increasing and satisfies the normalization condition $\int_0^1 f(\gamma) \, d\gamma = 1$.

We fix a weighting function f and a fuzzy number A. If $g : \mathbf{R} \to \mathbf{R}$ is a continuous utility function then the *possibilistic expected utility* $E(f, g(A))$ associated with A, g and f is defined by:

$$(1) \qquad E(f, g(A)) = \frac{1}{2} \int_0^1 [g(a_1(\gamma)) + g(a_2(\gamma))] f(\gamma) \, d\gamma.$$

If $g = 1_{\mathbf{R}}$ then $E(f, g(A))$ is exactly the *possibilistic expected value* $E(f, A)$ defined in [2, 3]:

$$(2) \qquad E(f, A) = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma) \, d\gamma.$$

If $g(x) = (x - E(f, A))^2$ for any $x \in \mathbf{R}$ then $E(f, g(A))$ is the possibilistic variance $Var(f, A)$ defined in [2, 29]:

$$(3) \quad Var(f, A) = \frac{1}{2} \int_0^1 [(a_1(\gamma) - E(f, A))^2 + (a_2(\gamma) - E(f, A))^2] f(\gamma) \, d\gamma.$$

If $a \in \mathbf{R}$ then the characteristic function $\bar{a}$ of the crisp set $\{a\}$ is called *fuzzy point* ([3], p. 10). We will identify $\bar{a}$ with a.

---

[2]For the definition of fuzzy numbers and their properties we refer to [7, 8].

3. BIVARIATE MIXED EXPECTED UTILITY

The notion of (bidimensional) mixed vector describes uncertainty situations with two risk parameters: one parameter is described by a random variable, the other by a fuzzy number.

Let (A, X) be a mixed vector with A a fuzzy number and X a random variable. [3]We denote by $M(X)$ the expected value of X and by $Var(X)$ its variance. If $g : \mathbf{R} \to \mathbf{R}$ is a continuous utility function then $M(g(X))$ will be the probabilistic expected utility of X w.r.t. g.

Assume that the level sets of A are $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$.

We fix a weighting function $f : [0, 1] \to \mathbf{R}$ and a bidimensional utility function $g : \mathbf{R}^2 \to \mathbf{R}$ of class C^2. For any $a \in \mathbf{R}$, $u(a, X) : \Omega \to \mathbf{R}$ will be the random variable defined by:

$u(a, X)(w) = u(a, X(w))$ for any $w \in \Omega$.

Definition 3.1. (Georgescu and Kinnunen [16]) The mixed expected utility associated with f, u and the mixed vector (A, X) is defined by:

$$E(f, u(A, X)) = \frac{1}{2} \int_0^1 [M(u(a_1(\gamma), X)) + M(u(a_2(\gamma), X))] f(\gamma) \, \mathrm{d}\gamma. \tag{4}$$

For $b \in \mathbf{R}$ let us denote also by b the constant random variable always equal to b.

Remark 3.2.

(i) If the fuzzy number A is the fuzzy point $\bar{a}$ with $a \in \mathbf{R}$ then
$E(f, u(A, X)) = M(u(a, X))$.

(ii) If the random variable X is $b \in \mathbf{R}$ then
$E(f, u(A, X)) = \frac{1}{2} \int_0^1 [u(a_1(\gamma), b) + u(a_2(\gamma), b)] f(\gamma) \, \mathrm{d}\gamma$.

(iii) If A is the fuzzy point $\bar{a}$ and X is the constant random variable b then
$E(f, u(A, X)) = u(a, b)$.

Proposition 3.3. (Georgescu and Kinnunen [16]) Let g, h be two bidimensional utility functions and $\alpha, \beta \in \mathbf{R}$. If $u = \alpha g + \beta h$ then:

(i) $E(f, u(A, X)) = \alpha E(f, g(A, X)) + \beta E(f, h(A, X))$

(ii) $g \le h$ implies $E(f, g(A, X)) \le E(f, h(A, X))$.

Proposition 3.4. If $u(y, x) = (y - E(f, A))(x - M(X))$ for any $y, x \in \mathbf{R}$ then $E(f, u(A, X)) = 0$.

Proof. For any $\gamma \in [0, 1]$ and i=1, 2 we have $u(a_i(\gamma), X) = (a_i(\gamma) - E(f, A))(X - M(X))$, therefore $M(u(a_i(\gamma), X)) = (a_i(\gamma) - E(f, A)) M(X - M(X)) = 0$. From this it follows immediately that $E(f, u(A, X)) = 0$. □

Next we will use the usual notations:

$$u_1 = \frac{\partial u}{\partial y}, \ u_2 = \frac{\partial u}{\partial x}, \ u_{11} = \frac{\partial^2 u}{\partial y^2}, \ u_{22} = \frac{\partial^2 u}{\partial x^2}, \ u_{12} = u_{21} = \frac{\partial^2 u}{\partial y \partial x}, \text{ etc.}$$

[3]All the random variables which appear in the paper will be defined on a fixed probability space $(\Omega, \mathcal{K}, P)$, with $\Omega \subseteq \mathbf{R}$.

Proposition 3.5. $E(f, u(A, X)) \approx u(E(f, A), M(X)) + \frac{1}{2} u_{11}(E(f, A), M(X)) Var(f, A) + \frac{1}{2} u_{22}(E(f, A), M(X)) Var(X)$.

Proof. The second–order Taylor approximation formula, Propositions 3.3 and 3.4 are applied. □

Proposition 3.6. (Niculescu and Perrson [25]) A continuous function $g : \mathbf{R} \to \mathbf{R}$ is convex if and only if for any $a, b \in \mathbf{R}$, $g(\frac{a+b}{2}) \leq \frac{g(a)+g(b)}{2}$.

Proposition 3.7. (Niculescu and Perrson [25]) Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous function. The following are equivalent:

(i) g is convex;

(ii) $g(M(X)) \leq M(g(x))$ for any random variable X.

Proposition 3.8. (Niculescu and Perrson [25]) (Jensen's Inequality) Let $u : \mathbf{R} \to \mathbf{R}$ be a convex function and $[a, b]$ a real interval. If $h : [a, b] \to \mathbf{R}$ is integrable on $[a, b]$ then $u(\int_a^b h(x) f(x)\,\mathrm{d}x) \leq \int_a^b u(h(x)) f(x)\,\mathrm{d}x$.

The versions of Propositions 3.6, 3.7, 3.8 for concave functions are formulated similarly.

We consider now a bidimensional utility function $u : \mathbf{R}^2 \to \mathbf{R}$. We say that the function $u(y, x)$ is convex in y if for any $x \in \mathbf{R}$, the unidimensional function $u(., x)$ is convex. Analogously one defines what means that $u(y, x)$ is convex in x, what means that $u(y, x)$ is concave in y, resp. in x, etc.

We assume everywhere in this paper that the utility function u is of the class C^2.

Next we will prove the equivalence between some convexity and concavity conditions of the utility function and some Jensen–type inequalities.

Proposition 3.9. The following statements are equivalent:

(i) $u(y, x)$ is convex in each of the variables y and x;

(ii) For any mixed vector (A, X) the following inequality holds:
$u(E(f, A), M(X)) \leq E(f, u(A, X))$

Proof. (i) $\Rightarrow$ (ii) Assume that $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$ for any $\gamma \in [0, 1]$. By hypothesis, the function $v = u(., M(X))$ is convex. Applying Proposition 3.6 for any $\gamma \in [0, 1]$ we have

$$v\Big(\frac{a_1(\gamma) + a_2(\gamma)}{2}\Big) \leq \frac{v(a_1(\gamma)) + v(a_2(\gamma))}{2}. \tag{5}$$

By Proposition 3.8 and (5) it follows:

$$u(E(f, A), M(X)) = v(E(f, A)) = v\Big(\int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma)\,\mathrm{d}\gamma\Big)$$
$$\leq \int_0^1 v\Big(\frac{a_1(\gamma) + a_2(\gamma)}{2}\Big) f(\gamma)\,\mathrm{d}\gamma \leq \frac{1}{2}\int_0^1 [v(a_1(\gamma)) + v(a_2(\gamma))] f(\gamma)\,\mathrm{d}\gamma$$

which can be written:

$$(6)\qquad u(E(f,A),M(X)) \le \frac{1}{2}\int_0^1 [u(a_1(\gamma),M(X)) + u(a_2(\gamma),M(X))]f(\gamma)\,\mathrm{d}\gamma.$$

For any fixed y, the function $u(y,.)$ is convex. Then, by Proposition 3.7, for any $\gamma \in [0,1]$ the following inequalities hold:

$u(a_1(\gamma),M(X)) \le M(u(a_1(\gamma),X))$
$u(a_2(\gamma),M(X)) \le M(u(a_2(\gamma),X))$.

By the property of monotony of the integral, one obtains:

$$\frac{1}{2}\int_0^1 [u(a_1(\gamma),M(X)) + u(a_2(\gamma),M(X))]f(\gamma)\,\mathrm{d}\gamma$$
$$\le \frac{1}{2}\int_0^1 [M(u(a_1(\gamma),X)) + M(u(a_2(\gamma),X))]f(\gamma)\,\mathrm{d}\gamma = E(f,u(A,X)).$$

From (6) and the previous inequality $u(E(f,A),M(X)) \le E(f,u(A,X))$ follows.

(ii) $\Rightarrow$ (i) We fix $x \in \mathbf{R}$. We will prove that $u(\cdot,x)$ is convex. Let $a,b \in \mathbf{R}$ with $a < b$. Let X be the constant random variable $x \in \mathbf{R}$ and the fuzzy number A defined by:

$$A(t) = \begin{cases} 1 & \text{if} \qquad t \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

Then $M(X) = x$ and $a_1(\gamma) = a, a_2(\gamma) = b$ for any $\gamma \in [0,1]$. Then $E(f,A) = \frac{a+b}{2}$ and by Remark 3.2 (ii):

$$E(f,u(A,X)) = \frac{1}{2}\int_0^1 [u(a,x) + u(b,x)]f(\gamma)\,\mathrm{d}\gamma = \frac{u(a,x)+u(b,x)}{2}.$$

The inequality $u(E(f,A),M(X)) \le E(f,u(A,X))$ becomes $u(\frac{a+b}{2},x) \le \frac{u(a,x)+u(b,x)}{2}$. Since this last inequality holds for any $a,b \in \mathbf{R}$, by Proposition 3.6, $u(\cdot,x)$ is convex.

We fix $a \in \mathbf{R}$ and we prove that $u(a,\cdot)$ is convex. Let X be an arbitrary random variable. We consider that A is the fuzzy point $\bar{a}$. Then $E(f,A) = a$ and by Remark 3.2 (i), $E(f,u(A,X)) = M(u(a,X))$. The inequality $u(E(f,A),M(X)) \le E(f,u(A,X))$ becomes $u(a,M(X)) \le M(u(a,X))$. Since this last inequality holds for any random variable X, from Proposition 3.7 it follows that $u(a,\cdot)$ is convex. □

Corollary 3.10. The following statements are equivalent:

(i) $u(y,x)$ is concave in each of the variables y and x;

(ii) For any mixed vector (A,X), $u(E(f,A),M(X)) \ge E(f,u(A,X))$.

Proposition 3.11. The following statements are equivalent:

(i) u is convex in y;

(ii) For any mixed vector (A,X), $M(u(E(f,A),X)) \le E(f,u(A,X))$.

Proof. (i) $\Rightarrow$ (ii) Let (A,X) be an arbitrary mixed vector. We consider the function $v : \mathbf{R} \to \mathbf{R}$ defined by $v(y) = M(u(y,X))$ for any $y \in \mathbf{R}$. For any $a,b \in \mathbf{R}$ the following holds:

$$v(\frac{a+b}{2}) = M\Big(u(\frac{a+b}{2}, X)\Big) \leq M\Big(\frac{u(a,X)+u(b,X)}{2}\Big)$$
$$= \frac{1}{2}(M(u(a,X)) + M(u(b,X))) = \frac{v(a)+v(b)}{2}$$

thus, by Proposition 3.6, v is convex. By formula (2) of Section 2,

$$\begin{aligned} E(f, v(A)) &= \frac{1}{2}\int_0^1 [v(a_1(\gamma)) + v(a_2(\gamma))]f(\gamma)\,\mathrm{d}\gamma \\ &= \frac{1}{2}\int_0^1 [M(u(a_1(\gamma), X)) + M(u(a_2(\gamma), X))]f(\gamma)\,\mathrm{d}\gamma \\ &= E(f, u(A, X)). \end{aligned}$$

Also, $v(E(f,A)) = M(u(E(f,A),X))$, thus the inequality $M(u(E(f,A),X)) \leq E(f,u(A,X))$ gets the form $v(E(f,A)) \leq E(f,v(A))$. This last inequality follows using v's convexity and Propositions 3.8 and 3.6:

$$v(E(f,A)) = v\Big(\int_0^1 \frac{a_1(\gamma)+a_2(\gamma)}{2} f(\gamma)\,\mathrm{d}\gamma\Big) \leq \int_0^1 v\Big(\frac{a_1(\gamma)+a_2(\gamma)}{2}\Big) f(\gamma)\,\mathrm{d}\gamma$$
$$\leq \frac{1}{2}\int_0^1 [v(a_1(\gamma)) + v(a_2(\gamma))]f(\gamma)\,\mathrm{d}\gamma = E(f, v(A)).$$

(ii) $\Rightarrow$ (i) Let $x \in \mathbf{R}$. We will prove that the function $u(\cdot, x)$ is convex. Let $a, b \in \mathbf{R}$ such that $a < b$. As in the proof of Proposition 3.9, we consider the constant random variable X being $x \in \mathbf{R}$ and a fuzzy number A such that $a_1(\gamma) = a$ and $a_2(\gamma) = b$ for any $\gamma \in [0,1]$. Then $E(f,A) = \frac{a+b}{2}$, $E(f,u(A,X)) = \frac{u(a,x)+u(b,x)}{2}$ and $M(u(E(f,A),X)) = u(\frac{a+b}{2}, x)$. This way the inequality $M(u(E(f,A),X)) \leq E(f,u(A,X))$ becomes $u(\frac{a+b}{2}, x) \leq \frac{u(a,x)+u(b,x)}{2}$, thus, by Proposition 3.6, the function $u(\cdot, x)$ is convex. □

Corollary 3.12. The following statements are equivalent:

(i) u is concave in y;

(ii) For any mixed vector (A,X), $M(u(E(f,A),X) \geq E(f,u(A,X))$.

Proposition 3.13. The following statements are equivalent:

(i) u is convex in x;

(ii) For any mixed vector (A,X), $E(f,u(A,M(X)) \leq E(f,u(A,X))$.

P r o o f. (i) $\Rightarrow$ (ii) Let (A,X) be an arbitrary mixed vector. For any $\gamma \in [0,1]$, the unidimensional functions $u(a_1(\gamma),\cdot)$ and $u(a_2(\gamma),\cdot)$ are convex, thus by Proposition 3.7, $u(a_i(\gamma), M(X)) \leq M(u(a_i(\gamma), X))$ for i=1, 2.

Then

$$\begin{aligned} E(f, u(A, M(X)) &= \frac{1}{2}\int_0^1 [u(a_1(\gamma), M(X)) + u(a_2(\gamma), M(X))]f(\gamma)\,\mathrm{d}\gamma \\ &\leq \frac{1}{2}\int_0^1 [M(u(a_1(\gamma), X)) + M(u(a_2(\gamma), X))]f(\gamma)\,\mathrm{d}\gamma \\ &= E(f, u(A, X)). \end{aligned}$$

(ii) $\Rightarrow$ (i) Let $y \in \mathbf{R}$. We prove that the function $u(y, \cdot)$ is convex. Let X be an arbitrary random variable. Taking $A = \bar{y}$, by Remark 3.2 (i), we will have $E(f, u(A, M(X)) = u(y, M(X))$ and $E(f, u(A, X)) = M(u(y, X))$. Then the inequality $E(f, u(A, M(X)) \leq E(f, u(A, X))$ becomes $u(y, M(X)) \leq M(u(y, X))$. By Proposition 3.7, $u(y, \cdot)$ is convex. □

Corollary 3.14. The following statements are equivalent:

(i) u is concave in x;

(ii) For any mixed vector (A, X), $E(f, u(A, M(X)) \geq E(f, u(A, X))$.

4. RISK AVERSION IN A MIXED FRAMEWORK

In this section we will consider an agent represented by a utility function $u : \mathbf{R}^2 \to \mathbf{R}$ in the face of a risk situation with two parameters: one described by a fuzzy number A and another by a random variable X.

We fix a weighting function f and we assume that u is of the class C^2 and $u_1 > 0$, $u_2 > 0$.

Proposition 4.1. Let $(y, x) \in \mathbf{R}^2$ and the mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$. Then:

$$(7) \qquad E(f, u(y + A, x + X)) \approx u(y, x) + \frac{1}{2}u_{11}(y, x)Var(f, A) + \frac{1}{2}u_{22}Var(X)$$

$$(8) \qquad E(f, u(y + A, x)) \approx u(y, x) + \frac{1}{2}u_{11}(y, x)Var(f, A)$$

$$(9) \qquad M(u(y, x + X)) \approx u(y, x) + \frac{1}{2}u_{22}(y, x)Var(X)$$

$$(10) \qquad E(f, u(y + A, x + X)) - E(f, u(y + A, x)) \approx \frac{1}{2}u_{22}(y, x)Var(f, A)$$

$$(11) \qquad E(f, u(y + A, x + X)) - M(u(y, x + X)) \approx \frac{1}{2}u_{11}(y, x)Var(X).$$

P r o o f. By applying Proposition 3.5 to the mixed vector $(y + A, x + X)$ one obtains:

$$E(f, u(y + A, x + X)) \approx u(E(f, y + A), M(x + X))$$
$$+\frac{1}{2}u_{11}(E(f, y{+}A), M(x{+}X))Var(f, y{+}A) + \frac{1}{2}u_{22}(E(f, y{+}A), M(x{+}X))Var(x{+}X).$$

Taking into account that $E(f, y + A) = y$, $M(x + X) = x$, $Var(f, y + A) = Var(f, A)$ and $Var(x + X) = Var(X)$, (7) is immediately obtained.

(8) and (9) are particular cases of (7), (10) is obtained from (7) and (8), and (11) is obtained from (7) and (9). □

Next we will identify the agent with its utility function u.

Definition 4.2. We say that the agent u is

(a) *mixed risk averse* if for any $(x, y) \in \mathbf{R}^2$ and for every mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$ we have $E(f, u(y + A, x + X)) \leq u(y, x)$.

(b) *possibilistically risk averse* if for any $(x, y) \in \mathbf{R}^2$ and for every mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$ we have $E(f, u(y + A, x + X)) \leq M(u(y, x + X))$.

(c) *probabilistically risk averse* if for any $(x, y) \in \mathbf{R}^2$ and for every mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$ we have $E(f, u(y + A, x + X)) \leq E(f, u(y + A, x))$.

According to the previous definition, the agent u is mixed averse if for any wealth level (y, x), it prefers the sure value (y, x) to the average gain obtained by adding a mixed risk (A, X) with $E(f, A) = M(X) = 0$. The other two notions of risk aversion from Definition 4.2 regard the possibilistic, resp. probabilistic component of mixed risk and have similar interpretations.

The following three propositions characterize the notions of risk aversion of Definition 4.2 in terms of the concavity of the agent's utility function.

Proposition 4.3. The following assertions are equivalent:

(i) The agent u is mixed risk averse.

(ii) The function u is concave in each of the variables y and x.

P r o o f. We will prove that the following assertions are equivalent:

(a) The agent u is mixed risk averse.

(b) For any $(x, y) \in \mathbf{R}^2$ and for every mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$ we have $E(f, u(y + A, x + X)) \leq u(E(f, y + A), M(x + X))$.

(c) For every mixed vector (B, Y) we have $E(f, u(B, Y)) \leq u(E(f, B), M(Y))$.

(d) u is concave in each of the variables y and x.

One notices that (b) is a rewriting of (a) and the equivalence (c) $\Leftrightarrow$ (d) follows from Corollary 3.10. To prove (b) $\Rightarrow$) (c), let (B, Y) be an arbitrary mixed vector. Denoting $A = B - E(f, B)$ and $X = Y - M(Y)$ we have $B = E(f, B) + A$, $Y = M(Y) + X$, $E(f, A) = 0$ and $M(X) = 0$. Applying (b) to the pair $(E(f, B), M(B)) \in \mathbf{R}^2$ and to the mixed vector (A, X) (c) is obtained. The implication (c) $\Rightarrow$ (b) is immediate. □

Proposition 4.4. The following assertions are equivalent:

(i) The agent u is possibilistically risk averse.

(ii) The function u is concave in y.

P r o o f. Similar to the proof of Proposition 4.3, applying Corollary 3.12. □

Proposition 4.5. The following assertions are equivalent:

(i) The agent u is probabilistically risk averse.

(ii) The function u is concave in x.

Proof. Corollary 3.14 is applied. □

Proposition 4.6. The agent u is mixed risk averse iff it is simultaneously possibilistically risk averse and probabilistically risk averse.

Proof. By Propositions 4.3 – 4.5. □

In the probability theory of risk, risk premium is the main indicator of risk aversion [1, 26]. In case of risk represented by a mixed vector (A, X) we will define three notions of risk premium: one refers to the possibilistic component, the second refers to the probabilistic component and the third to the overall risk aversion.

For the rest of this section we assume that the utility function u is of the class C^2, is strictly increasing in each argument and strictly concave in each argument.

Definition 4.7. Let $(x, y) \in \mathbf{R}^2$ and the mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$. Then we define:

- *the possibilistic risk premium* $\pi = \pi(y, x, A, X, u)$ as the unique solution of the equation

$$E(f, u(y + A, x + X)) = u(y - \pi, x). \tag{12}$$

- *the probabilistic risk premium* $\rho = \rho(y, x, A, X, u)$ as the unique solution of the equation

$$E(f, u(y + A, x + X)) = u(y, x - \rho). \tag{13}$$

- *a mixed risk premium vector* (π, ρ) as a solution of the equation:

$$E(f, u(y + A, x + X)) = u(y - \pi, x - \rho). \tag{14}$$

Remark 4.8. The uniqueness of the solutions of equations (12) and (13) results from the injectivity of u in y. Equation (14) may have several solutions (π, ρ).

Proposition 4.9. Let $(x, y) \in \mathbf{R}^2$ and the mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$. Then:

(a) the possibilistic risk premium π can be approximated by

$$\pi \approx -\frac{1}{2} \frac{u_{11}(y, x)Var(f, A) + u_{22}(y, x)Var(X)}{u_1(y, x)} \tag{15}$$

(b) the probabilistic risk premium ρ can be approximated by

$$\rho \approx -\frac{1}{2} \frac{u_{11}(y, x)Var(f, A) + u_{22}(y, x)Var(X)}{u_2(y, x)} \tag{16}$$

(c) any mixed risk premium vector (π, ρ) can be approximated by

$$\pi \approx -\frac{1}{4} \frac{u_{11}(y, x)Var(f, A) + u_{22}(y, x)Var(X)}{u_1(y, x)} \tag{17}$$

$$(18) \qquad \rho \approx -\frac{1}{4}\frac{u_{11}(y,x)Var(f,A)+u_{22}(y,x)Var(X)}{u_2(y,x)}.$$

P r o o f. (a) The first–order Taylor formula gives:

$$(19) \qquad u(y-\pi,x) \approx u(y,x) - \pi u_1(y,x).$$

Replacing in (12) $E(f,u(y+A,x+X))$ and $u(y-\pi,x)$ with their approximate values given by (7) and (19) it follows:

$$\frac{1}{2}u_{11}(y,x)Var(f,A) + \frac{1}{2}u_{22}(y,x)Var(X) \approx -\pi u_1(y,x)$$

from where one obtains (15).

(b) Similar to (a).

(c) Let π^0, ρ^0 be the members of the right hand side of (17) and (18). The first–order Taylor formula gives:

$$\begin{aligned} u(y-\pi^0, x-\rho^0) &\approx u(y,x) - \pi^0 u_1(y,x) - \rho^0 u_2(y,x) \\ &= u(y,x) + \tfrac{1}{2}u_{11}(y,x)Var(f,A) + \tfrac{1}{2}u_{22}(y,x)Var(X). \end{aligned}$$

Taking (7) into account, it follows that $E(f,u(y+A,x+X)) = u(y-\pi^0, x-\rho^0)$. □

The formulas of the previous proposition give approximate values for possibilistic risk premium, probabilistic risk premium and mixed risk premium vector in terms of possibilistic variance $Var(f,A)$, the probabilistic variance $Var(X)$, the utility function u and its second–order derivatives. They are similar to approximation formulas of probabilistic risk premium from [9, 18, 19, 20].

Remark 4.10. From Definition 4.7 a way to rank the solutions of equation (14) does not arise. Nevertheless the fact that all these solutions are approximated by the same vector (π^0, ρ^0) (defined by the right hand side member of formulas (17), (18)) suggests to use (π^0, ρ^0) in applications as "unique approximate solution".

5. MIXED MODELS OF OPTIMAL SAVING

In this section we intend to investigate the effect of the risk presence on the optimal saving when we admit the existence of two risk parameters: one represented by a fuzzy number A and another represented by a random variable X.

As a starting point of our approach we indicate the multidimensional optimal saving model of [19] and especially the two-period models from [4, 23]. In these models, both risk parameters are assumed to be random variables.

Our two–period models will be characterized by the following initial data:

- $u(y,x)$ and $v(y,x)$ are consumer's utility functions for period 0, resp. 1.
- for period 0, the variables y and x have sure values y_0 and x_0.
- for period 1, one variable is a fuzzy number and the other is a random variable.

- s is the level of saving.

Next we will study the case when a fuzzy number A corresponds to y and a random variable X corresponds to x. The other case, when a random variable corresponds to y and a fuzzy number corresponds to x, is tackled similarly.

We will identify the consumer with the pair (u, v) of its utility functions. We assume that u and v are of the class C^3, are strictly increasing in each argument and strictly concave. We shall denote by u_i, u_{ij}, u_{ijk} (resp. v_i, v_{ij}, v_{ijk}) the first, the second and the third partial derivatives of u (resp. v).

We fix a weighting function f. We consider a mixed vector (A, X) and we denote $a = E(f, A)$ and $\bar{x} = M(X)$. The following four situations are possible:

(a) $y = A, x = X$

(b) $y = A, x = \bar{x}$

(c) $y = a, x = X$

(d) $y = a, x = \bar{x}$.

In the probabilistic models of [4, 23], the variable y is interpreted as income risk, and the variable x as non–financial background risk. Then, in our models the fuzzy number A will represent an income risk and the random variable X will be a background risk.

Then, the interpretation of the four situations (a) – (d) is:

	y	x
(a)	possibilistic income risk	probabilistic background risk
(b)	possibilistic income risk	deterministic variable
(c)	deterministic variable	probabilistic background risk
(d)	deterministic variable	deterministic variable

We intend to study the changes of the optimal saving to each of the following three routes: (b) → (a), (c) → (a), (d) → (a). This problem has been inspired by [4, 23], where the change of optimal saving in the presence of a risk with two probabilistic parameters has been analyzed.

For the four situations (a) – (d) above we will consider the following lifetime utilities:

(20) $$V_1(s) = u(y_0 - s, x_0) + E(f, v(A + s, X))$$

(21) $$V_2(s) = u(y_0 - s, x_0) + E(f, v(A + s, \bar{x}))$$

(22) $$V_3(s) = u(y_0 - s, x_0) + M(v(a + s, X))$$

(23) $$V_4(s) = u(y_0 - s, x_0) + v(a + s, \bar{x})$$

and the corresponding optimization problems:

(24) $$\max_s V_1(s)$$

(25) $$\max_s V_2(s)$$

(26) $\max_s V_3(s)$

(27) $\max_s V_4(s).$

The form of the functions $V_1, \ldots, V_4$ is inspired by the expressions of lifetime utilities of the probabilistic models of [4, 23]; the distinction between them consists in the fact that the lifetime utilities from [4, 23] are expressed as probabilistic expected utilities, while $V_1, \ldots, V_4$ use the mixed expected utilities of Section 3.

One notices that in the above models the saving s acts on the “possibilistic” parameter y. We can consider models in which s acts on the ”probabilistic” parameter x, and models in which s acts on both parameters. In the latter case the lifetime utilities will have the form:

(28) $W_1(s) = u(y_0 - s, x_0 - s) + E(f, v(A + s, X + s))$

(29) $W_2(s) = u(y_0 - s, x_0 - s) + E(f, v(A + s, \bar{x} + s))$

(30) $W_3(s) = u(y_0 - s, x_0 - s) + M(v(a + s, X + s))$

(31 (31) $W_4(s) = u(y_0 - s, x_0 - s) + v(a + s, \bar{x} + s).$

We will deal only with the models defined by (20) – (23). For the other cases the theory can be developed similarly. Using Definition 3.1 and Remark 3.2, from (20) – (23) it follows

(32) $V_1'(s) = -u_1(y_0 - s, x_0) + E(f, v_1(A + s, X))$

(33) $V_2'(s) = -u_1(y_0 - s, x_0) + E(f, v_1(A + s, \bar{x}))$

(34) $V_3'(s) = -u_1(y_0 - s, x_0) + M(v_1(a + s, X))$

(35) $V_4'(s) = -u_1(y_0 - s, x_0) + v_1(a + s, \bar{x}).$

From (32) – (35) it easily follows that $V_1, \ldots, V_4$ are strictly concave functions. To exemplify, we will prove formula (32). By Definition 3.1,

$$V_1(s) = u(y_0 - s, x_0) + \frac{1}{2}\int_0^1 [M(v(a_1(\gamma) + s, X)) + M(v(a_2(\gamma) + s, X))] f(\gamma)\, d\gamma,$$

from where, by derivation, one obtains:

$$\begin{aligned} V_1'(s) &= -u_1(y_0 - s, x_0) \\ &+ \frac{1}{2}\int_0^1 [M(v_1(a_1(\gamma) + s, X)) + M(v_1(a_2(\gamma) + s, X))] f(\gamma)\, d\gamma \\ &= -u_1(y_0 - s, x_0) + E(f, v_1(A + s, X)). \end{aligned}$$

Similarly, by deriving (32) one obtains:

$$\begin{aligned} V_1''(s) &= u_{11}(y_0 - s, x_0) \\ &+ \frac{1}{2}\int_0^1 [M(v_{11}(a_1(\gamma) + s, X)) + M(v_{11}(a_2(\gamma) + s, X))] f(\gamma)\, d\gamma. \end{aligned}$$

From this formula it follows immediately that V_1 is strictly concave. One can prove analogously that V_2, V_3 and V_4 are concave.

Hence, the optimal solutions $s_i^* = s_i^*(A, X)$, $i = 1, \ldots, 4$ of problems (24) – (27) will be given by:

(36) $\quad V_i'(s_i^*) = 0,\ i = 1, \ldots, 4.$

By (32) – (35), conditions (36) are written:

(37) $\quad u_1(y_0 - s_1^*, x_0) = E(f, v_1(A + s_1^*, X))$

(38) $\quad u_1(y_0 - s_2^*, x_0) = E(f, v_1(A + s_2^*, \bar{x}))$

(39) $\quad u_1(y_0 - s_3^*, x_0) = M(v_1(a + s_3^*, X))$

(40) $\quad u_1(y_0 - s_4^*, x_0) = v_1(a + s_4^*, \bar{x}).$

We define the following notions of *precautionary saving*:

$s_1^* - s_2^*$ indicates the variation of optimal saving on the route (b) $\to$ (a)

$s_1^* - s_3^*$ indicates the variation of optimal saving on the route (c) $\to$ (a)

$s_1^* - s_4^*$ indicates the variation of optimal saving on the route (d) $\to$ (a)

$s_1^* - s_2^*$ measures the effect of a possibilistic income risk on optimal saving in the presence of a probabilistic background risk. The precautionary saving $s_1^* - s_3^*$ evaluates the change of optimal saving when a background risk is added to a possibilistic income risk model. Finally, $s_1^* - s_4^*$ shows the change of optimal saving when we go from a deterministic situation to a mixed risk situation.

Definition 5.1. We say that the consumer (u, v) is:

(a) *mixed prudent* if for any mixed vector (A, X) we have
$$s_1^*(A, X) - s_4^*(E(f, A), M(X)) \geq 0$$

(b) *possibilistically prudent* if for any mixed vector (A, X) we have
$$s_1^*(A, X) - s_3^*(E(f, A), X)) \geq 0$$

(c) *probabilistically prudent* if for any mixed vector (A, X) we have
$$s_1^*(A, X) - s_2^*(A, M(X)) \geq 0.$$

A consumer (u, v) is mixed prudent if the presence of mixed risk (A, X) has as a consequence the increase of the level of optimal saving. The other two notions of prudence have a natural interpretation too.

The following propositions characterize the three notions of prudence of Definition 5.1.

Proposition 5.2. The following assertions are equivalent:

(i) The consumer (u, v) is mixed prudent.

(ii) $v_{111} \geq 0$ and $v_{122} \geq 0$.

P r o o f. Since V_4 is strictly concave, the function V_4' is strictly decreasing, hence: $s_1^* \geq s_4^*$ iff $V_4'(s_1^*) \leq V_4'(s_4^*) = 0$.

By (35) and (37):

$$\begin{aligned} V_4'(s_1^*) &= -u_1(y_0 - s_1^*, x_0) + v_1(a + s_1^*, \bar{x}) \\ &= -E(f, v_1(A + s_1^*, X)) + v_1(a + s_1^*, \bar{x}) \\ &= -E(f, v_1(A + s_1^*(A, X), X)) + v_1(E(f, A) + s_1^*(A, X), M(X)). \end{aligned}$$

Then taking into account Proposition 3.9 the following assertions are equivalent:

- Consumer (u, v) is risk averse.
- For any mixed vector (A, X), $v_1(E(f, A + s_1^*(A, X)), M(X)) \leq E(f, v_1(A + s, X))$.
- v_1 is convex in each of the variables y and x.
- $v_{111} \geq 0$ and $v_{122} \geq 0$. □

Proposition 5.3. The following assertions are equivalent:

(i) The consumer (u, v) is possibilistically prudent.

(ii) $v_{111} \geq 0$.

P r o o f. Similar to the proof of Proposition 5.2, using Proposition 3.11. □

Proposition 5.4. The following assertions are equivalent:

(i) The consumer (u, v) is probabilistically prudent.

(ii) $v_{122} \geq 0$.

P r o o f. Similar to the proof of Proposition 5.2, using Proposition 3.13. □

The definition of a consumer's prudence by the positivity of precautionary saving is similar to the approach in [19]. Saying that a consumer is prudent if the occurrence of some type of risk makes it raise its level of optimal saving is very intuitive. At the same time, the characterizations of the three types of prudence by positivity conditions of Propositions 5.2 – 5.4 relate the above approach to that from [4, 21, 23].

6. MIXED DOWNSIDE RISK AVERSION AND PRUDENCE

The probabilistic downside risk aversion, introduced by Menezes et al. [24] is tightly connected to probabilistic prudence (see [6, 10]). Following a parallel line to that from [6, 10] in this section we will define the notion of mixed downside risk aversion and we will establish the relation between this concept and the property of mixed prudence from Section 5.

We will define first the notions of mixed utility premium and mixed prudence utility premium.

Consider an agent with the utility function $u : \mathbf{R}^2 \to \mathbf{R}$ of class C^2, strictly increasing in each argument and strictly concave in each argument.

We fix a weighted function f. We consider $(y, x) \in \mathbf{R}^2$ and a mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$.

By analogy with the notion of utility premium of Friedman and Savage [12], we will define the mixed utility premium.

Definition 6.1. The *mixed utility premium* $w(y, x, A, X, u)$ associated with the pair (y, x), the mixed vector (A, X) and the utility function u is given by

$$w(y, x, A, X, u) = u(y, x) - E(u(y + A, x + X)). \tag{41}$$

By Definition 3.1, if $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$ then

$$w(y, x, A, X, u) = u(y, x) - \frac{1}{2}\int_0^1 [M(u(y + a_1(\gamma), x + X)) + M(u(y + a_2(\gamma), x + X))] f(\gamma)\, \mathrm{d}\gamma. \tag{42}$$

Proposition 6.2. An approximate value of mixed utility premium is given by:

$$w(y, x, A, X, u) \approx \frac{1}{2} u_{11}(y, x) Var(f, A) + \frac{1}{2} u_{22}(y, x) Var(X). \tag{43}$$

Proof. By Proposition 4.1 (7). □

Remark 6.3. By Definition 4.2 (a), the agent u is mixed risk averse iff for any $(y, x) \in \mathbf{R}^2$ and for any mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$, we have $w(y, x, A, X, u) \leq 0$.

The notion of mixed prudence utility premium, similar to the one of (probabilistic) prudence utility premium of [5], is introduced by the following definition:

Definition 6.4. Let $k > 0$. The *mixed prudence utility premium* is defined by

$$S(y, x, k, A, x, u) \approx w(y - k, x, A, X, u) - w(y, x, A, X, u). \tag{44}$$

Now we can define the notion of mixed downside risk aversion.

Definition 6.5. The agent u displays mixed downside risk aversion (MDRA) if for any $(y, x) \in \mathbf{R}^2$, $k > 0$ and for any mixed vector (A, X) with $E(f, A) = 0$ and $M(X) = 0$, $S(y, x, k, A, X, u) \geq 0$ holds.

Remark 6.6. The agent u displays MDRA iff the function $S(y, x, k, A, X, u)$ is decreasing in y.

Proposition 6.7. The following assertions are equivalent:

(i) The agent u displays MDRA.

(ii) $u_{111} \geq 0$ and $u_{122} \geq 0$.

P r o o f. From (42) it follows:

$$\begin{aligned}\frac{\partial w(y,x,A,X,u)}{\partial y} &= u_1(y,x) - \frac{1}{2}\int_0^1 [M(u_1(y+a_1(\gamma),x+X)) \\ &\quad + M(u_2(y+a_1(\gamma),x+X))]f(\gamma)\,\mathrm{d}\gamma \\ &= u_1(y,x) - E(f,u_1(y+A,x+X)) \\ &= u_1(F(f,y+A),M(x+X)) - E(f,u_1(y+A,x+X)).\end{aligned}$$

Applying Remark 6.6 and Proposition 3.9, the following assertions are equivalent:

- The agent u displays MDRA.
- $\frac{\partial w(y,x,A,X,u)}{\partial y} \leq 0$ for any $(y,x) \in \mathbf{R}^2$, $k > 0$ and for any mixed vector (A,X) with $E(f,A) = 0$ and $M(X) = 0$.
- $u_1(E(f,y+A),M(X)) \leq E(f,u_1(y+A,x+X))$ for any $(y,x) \in \mathbf{R}^2$, $k > 0$ and for any mixed vector (A,X) with $E(f,A) = 0$ and $M(X) = 0$.
- $u_1(y,x)$ is convex in each of the variables y and x.
- $u_{111} \geq 0$ and $u_{122} \geq 0$. □

Now we go back to the mixed optimal saving model of Section 5 in which $u(y,x)$ and $v(y,x)$ are consumer's utility functions in period 0, resp. 1.

Proposition 6.8. Under the conditions of Section 5 on the mixed optimal saving model the following conditions are equivalent:

(a) Consumer (u,v) is mixed prudent.

(b) $v_{111} \geq 0$ and $v_{122} \geq 0$.

(c) Consumer (u,v) displays MDRA.

P r o o f. (a) ⇔ (b) By Proposition 5.2.
(b) ⇔ (c) By Proposition 6.7. □

7. CONCLUSIONS

For risk situations with mixed parameters, in the paper the following topics have been studied: risk aversion, prudence and optimal saving. These topics are developed in the framework of an expected utility theory whose main concept is mixed expected utility.

We resume the main contributions of the paper:

First we mention the characterization of some concavity and convexity conditions of bivariate utility functions by Jensen-type inequalities (expressed in terms of mixed expected utility). Some notions of risk aversion of an agent in the face of mixed risk are defined; then they are characterized by positivity conditions on third–order partial derivatives of the utility functions. Some indicators of mixed risk aversion are introduced and formulas for their approximate calculation are proved.

Then the effect of mixed risk and its components on the variation of the level of optimal saving is studied. The prudence of an agent (= consumer) faced with mixed risk is defined and is characterized in terms of partial derivatives of the utility function.

Finally the notion of mixed downside risk aversion is introduced and its connection to mixed prudence is analyzed.

We mention the following open problems:

(1) proving some Pratt-type theorems [9, 18, 19, 26] in order to compare the mixed risk aversions of two or more agents.

(2) the study of higher- order attitudes of agents [6, 10] faced with mixed risk.

(3) the definition of a notion similar to stochastic dominance [11, 17] in order to compare the situations of mixed risks.

ACKNOWLEDGMENTS

The author would like to thank the anonymous referees for their valuable comments on improving the paper linguistically as well as technically.

(Received September 17, 2011)

REFERENCES

[1] K. J. Arrow: Essays in the theory of risk bearing. North–Holland, Amsterdam 1970.

[2] C. Carlsson and R. Fullér: On possibilistic mean value and variance of fuzzy numbers. Fuzzy Sets Syst. *122* (2001), 315–326.

[3] C. Carlsson and R. Fullér: Possibility for Decision. Springer 2011.

[4] C. Courbage and B. Rey: Precautionary saving in the presence of other risks. Econom. Theory *32* (2007), 417–424.

[5] C. Courbage and B. Rey: On the shape of non–monetary measures in the face of risk. In: The 36th Seminar of the European Group of Risk and Insurance Economists (EGRIE), Bergen 2009.

[6] D. Crainich and L. Eeckhoudt: On the intensity of downside risk aversion. J. Risk Uncertainty *36* (2008), 267–276.

[7] D. Dubois and H. Prade: Fuzzy Sets and Systems: Theory and Applications. Academic Press, New York 1980.

[8] D. Dubois and H. Prade: Possibility Theory. Plenum Press, New York 1988.

[9] G. T. Duncan: A matrix measure of multivariate local risk aversion. Econometrica *45* (1977), 895–903.

[10] L. Eeckhoudt and H. Schlesinger: Putting risk in its proper place. Amer. Econom. Rev. *96* (2006), 280–289.

[11] L. Eeckhoudt, C. Gollier. and H. Schlesinger: Economic and Financial Decisions under Risk. Princeton University Press, 2005.

[12] M. Friedman and L. Savage: The utility analysis of choices involving risk. J. Polit. Econom. *56* (1948), 279–304.

[13] R. Fullér and P. Majlender: On weighted possibilistic mean and variance of fuzzy numbers. Fuzzy Sets Syst. *136* (2003), 363–374.

[14] I. Georgescu: Possibilistic risk aversion. Fuzzy Sets Syst. *60* (2009), 2608–2619.

[15] I. Georgescu: A possibilistic approach to risk aversion. Soft Comput. *15* (2011), 795–801.

[16] I. Georgescu and J. Kinnunen: Multidimensional risk aversion with mixed parameters. In: The 6th IEEE International Symposium on Applied Computational Intelligence and Informatics (SACI 2011), Timisoara 2011, pp. 63–68.

[17] C. Gollier: The Economics of Risk and Time. MIT Press, 2004.

[18] M. Hellwig: Risk aversion in the small and in the large when outcomes are multidimensional. Preprints of the Max Planck Institute for Research on Collective Goods, Bonn 2006.

[19] E. Jouini, C. Napp, and D. Nocetti: On multivariate prudence. J. Econom. Theory. *148* (2013), 1255–1267.

[20] E. Karni: On multivariate risk aversion. Econometrica *47* (1979), 1391–1401.

[21] M. Kimball: Precautionary saving in the small and in the large. Econometrica *58* (1990), 58–73.

[22] H. E. Leland: Saving and uncertainty: the precautionary demand for saving. Quarterly J. Econom. *82* (1968), 465–473.

[23] M. Menegatti: Optimal saving saving in the presence of two risks. J. Econom. *96* (2009), 277–288.

[24] C. Menezes, C. Geiss, and J. Tressler: Increasing downside risk. Amer. Econom. Rev. *70* (1980), 921–932.

[25] C. P. Niculescu and L. E. Perrson: Convex Functions and their Applications: A Contemporary Approach. Springer, 2005.

[26] J. Pratt: Risk aversion in the small and in the large. Econometrica *32* (1964), 122–130.

[27] A. Sandmo: The effect of uncertainty on saving decision. Rev. Econom. Studies *37* (1970), 353–360.

[28] L. A. Zadeh: Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets Syst. *1* (1978), 3–28.

[29] W. G. Zhang and Y. L. Wang: A comparative study of possibilistic variances and covariances of fuzzy numbers. Fund. Inform. *79* (2007), 257–263.

Irina Georgescu, Academy of Economic Studies, Department of Economic Cybernetics, Piaţa Romana No 6 R 70167, Oficiul Postal 22, Bucharest. Romania.
e-mail: irina.georgescu@csie.ase.ro