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BASIC BOUNDS OF FRÉCHET CLASSES

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Algebraic bounds of Fréchet classes of copulas can be derived from the fundamental attributes of the associated copulas. A minimal system of algebraic bounds and related basic bounds can be defined using properties of pointed convex polyhedral cones and their relationship with non-negative solutions of systems of linear homogeneous Diophantine equations, largely studied in Combinatorics. The basic bounds are an algebraic improving of the Fréchet–Hoeffding bounds. We provide conditions of compatibility and propose tools for an explicit description of the basic bounds of simple Fréchet classes.

Keywords: algebraic bound, basic bound, copula, Diophantine equation, Fréchet class, pointed convex polyhedral cone

Classification: 60E05, 62H20, 11D75

1. INTRODUCTION

A copula can be regarded as an abstract structure which fully represents relationship of random variables. As mentioned in [5], there are two main reasons to be interested in copulas. Firstly, it is a way of studying scale-free measures of dependence. Secondly, copulas are starting points for constructing families of multivariate distributions, sometimes with a view to simulation (see e. g. [2]). Recent interest in copulas was prompted by applicability in finance and insurance.

Let $\mathbb{R}$ and $\mathbb{N}$ represent the sets of all real and all natural numbers (including 0). Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be members of $\mathbb{R}^n$ and $\zeta \subseteq [n] = \{1, 2, \ldots, n\}$. We will denote $\mathrm{sel}_\zeta(\boldsymbol{a}, \boldsymbol{b})$ a vector $(c_1, \ldots, c_n)$ such that $c_i = \begin{cases} a_i \text{ if } i \in \zeta \\ b_i \text{ if } i \notin \zeta \end{cases}$. For example, $\mathrm{sel}_{\{1,3\}}((0,\ 0.2,\ 0.5),\ (0.4,\ 0.3,\ 1)) = (0,\ 0.3,\ 0.5)$. For an n-copula C and $\boldsymbol{a}, \boldsymbol{b} \in \langle 0, 1\rangle^n$, $\boldsymbol{a} \leq \boldsymbol{b}$ in component-wise ordering, the *C-volume* of an *n-box* $\langle \boldsymbol{a}, \boldsymbol{b}\rangle = \prod_{i=1}^n \langle a_i, b_i\rangle$ (Cartesian product) is given by

$$V_C(\langle \boldsymbol{a}, \boldsymbol{b}\rangle) = \sum_{\zeta \subseteq [n]} (-1)^{|\zeta|}\, C\left(\mathrm{sel}_\zeta(\boldsymbol{a}, \boldsymbol{b})\right) \tag{1}$$

where $|\zeta|$ is the *cardinality* of the set ζ.

An n-copula is fully determined by its performance on $\langle 0, 1\rangle^n$. So, we can consider the n-copula as a function $C : \langle 0, 1\rangle^n \to \langle 0, 1\rangle$ endowed with the next properties:

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(i) $C(\boldsymbol{x}) = 0$ whenever $\boldsymbol{x} = (x_1, \ldots, x_n) \in \langle 0,1\rangle^n$ has at least one component equal to 0 (C is grounded),

(ii) $C(\boldsymbol{x}) = x_k$ whenever all components of $\boldsymbol{x} \in \langle 0,1\rangle^n$ are equal to 1 except for the kth one (uniformity of 1-margins),

(iii) $V_C(\langle \boldsymbol{a}, \boldsymbol{b}\rangle) \geq 0$ for all $\boldsymbol{a}$ and $\boldsymbol{b}$ from $\langle 0,1\rangle^n$ such that $\boldsymbol{a} \leq \boldsymbol{b}$ (C is n-increasing).

Any k-margin of a copula is again a k-copula. For a nonempty set $\upsilon = \{j_1, j_2, \ldots, j_k\} \subseteq [n]$ with $j_1 < j_2 < \ldots < j_k$ and an n-copula C, a υ-margin of C is its k-margin $C_\upsilon : \langle 0,1\rangle^k \to \langle 0,1\rangle$, defined by $C_\upsilon(\boldsymbol{x}_\upsilon) = C\left(\mathrm{sel}_\upsilon(\boldsymbol{x}, \mathbf{1}_n)\right)$ where $\boldsymbol{x}_\upsilon = (x_{j_1}, x_{j_2}, \ldots, x_{j_k})$ and $\mathbf{1}_n \in \mathbb{R}^n$ is the row vector of ones. Let us broaden the definition to $C_\varnothing = 1$. We will call the set υ, associated with C_υ, the *determinative set* of the copula C_υ.

Let $\boldsymbol{S}$ be a system of subsets of $[n]$. The *Fréchet class* $\mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$ is the set of all n-copulas $\tilde{C}(\boldsymbol{x})$ with given υ-margin $\tilde{C}_\upsilon(\boldsymbol{x}_\upsilon)$ equal to $C_\upsilon(\boldsymbol{x}_\upsilon)$ for each $\upsilon \in \boldsymbol{S}$. We call the set $\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\}$ of copulas *compatible* if $\mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\}) \neq \varnothing$. Because given margins must be identical on common parts of the determinative sets, the set $\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\}$ will further be expected to satisfy

$$C_\zeta\left(\mathrm{sel}_{\zeta\cap\eta}(\boldsymbol{x}_\zeta, \mathbf{1}_{|\zeta|})\right) = C\eta\left(\mathrm{sel}_{\zeta\cap\eta}(\boldsymbol{x}_\eta, \mathbf{1}_{|\eta|})\right) \text{ for any } \zeta,\ \eta \in \boldsymbol{S} \tag{2}$$

in context of Fréchet classes. For any family $\boldsymbol{S} \subseteq \mathcal{P}([n])$ (power set of $[n]$), we call the system $\Delta\boldsymbol{S} = \{\zeta \in \mathcal{P}([n]);\ |\zeta| \leq 1 \text{ or } (\exists \eta \in \boldsymbol{S}) : \zeta \subseteq \eta\}$ the *downward closure* of $\boldsymbol{S}$ in $\mathcal{P}([n])$ (ordered by inclusion). Together with given margins C_υ, their margins and all 1-margins are in fact fixed too. If the set $\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\}$ is compatible then, in addition to defining all υ-margins C_υ for $\upsilon \in \boldsymbol{S}$, there are also clearly identified all ζ-margins C_ζ for $\zeta \in \Delta\boldsymbol{S}$. For example, in the Fréchet class $\mathcal{F}_4(C_{\{1,2,3\}}, C_{\{2,3,4\}})$, all four 1-margins are uniform by definition of copula and $\{1,2\}$-, $\{1,3\}$-, $\{2,3\}$-, $\{2,4\}$- and $\{3,4\}$-margins are also uniquely given.

Fréchet classes are studied largely in context of construction of multivariate distributions. The most frequent are questions of uniqueness, subfamilies with desirable properties, boundaries and their nature (see e. g. [4]). We strive after description of special upper and lower bounds of general Fréchet classes, providing results of the paper [7] in detail.

Let C be an n-copula and $\boldsymbol{x} \in \langle 0,1\rangle^n$. The requirement of non-negativity of C-volume leads to a series of inequalities

$$V_C\left(\langle \mathrm{sel}_\zeta(\mathbf{0}_n, \boldsymbol{x}), \mathrm{sel}_\zeta(\boldsymbol{x}, \mathbf{1}_n)\rangle\right) = \sum_{\substack{\eta \supseteq \zeta \\ \eta \subseteq [n]}} (-1)^{|\zeta|+|\eta|} C_\eta(\boldsymbol{x}_\eta) \geq 0 \tag{3}$$

for all $\zeta \subseteq [n]$. For example, any 3-copula C must satisfy

$$\begin{array}{cccccccc} 1 & -x_1 & -x_2 & -x_3 & +C_{\{1,2\}} & +C_{\{1,3\}} & +C_{\{2,3\}} & -C \geq 0 \\ & x_1 & & & -C_{\{1,2\}} & -C_{\{1,3\}} & & +C \geq 0 \\ & & x_2 & & -C_{\{1,2\}} & & -C_{\{2,3\}} & +C \geq 0 \\ & & & x_3 & & -C_{\{1,3\}} & -C_{\{2,3\}} & +C \geq 0 \\ & & & & C_{\{1,2\}} & & & -C \geq 0 \\ & & & & & C_{\{1,3\}} & & -C \geq 0 \\ & & & & & & C_{\{2,3\}} & -C \geq 0 \\ & & & & & & & C \geq 0 \end{array} \tag{4}$$

where C_ζ represents everywhere $C_\zeta(\boldsymbol{x}_\zeta)$. A non-negative combination of these inequalities, which eliminate free proper margins of $\tilde{C} \in \mathcal{F}_n(\{C_\upsilon;\, \upsilon \in \boldsymbol{S}\})$, form a bound of the Fréchet class. For example, $\tilde{C}_{\{1,3\}}$ and $\tilde{C}_{\{2,3\}}$ are free proper margins in a Fréchet class $\mathcal{F}_3(C_{\{1,2\}})$. We get

$$1 - x_3 - C_{\{1,2\}} + \tilde{C} \geq 0 \tag{5}$$

by adding the first three inequalities corresponding to (4) which provides the lower bound $-1 + x_3 + C_{\{1,2\}}$ for this class.

This is just the type of boundaries of Fréchet classes that are studied in this article.

2. ALGEBRAIC BOUNDS

In the following, we repeatedly use the identities

$$|\{\theta \in \mathcal{P}([n])\,;\, |\theta| = i \;\;\&\;\; \eta \subseteq \theta \subseteq \nu \subseteq [n]\}| = \binom{|\nu| - |\eta|}{i - |\eta|}$$
$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} = \begin{cases} 0 & \text{for } k > 0 \\ 1 & \text{for } k = 0 \end{cases} \tag{6}$$

where $\eta \subseteq \nu \subseteq [n]$. Let us denote $\operatorname{geq}_\zeta(C)(\boldsymbol{x})$ the C-volume $\sum_{\eta \supseteq \zeta} (-1)^{|\zeta|+|\eta|} C_\eta(\boldsymbol{x}_\eta)$ in (3) for any n-copula C, $\boldsymbol{x} \in \langle 0, 1\rangle^n$ and $\zeta \subseteq [n]$. Relationship between these geq-volumes and margins can be expressed in matrix form by

$$\left(\operatorname{geq}_\zeta(C)(\boldsymbol{x})\right)_{\zeta \subseteq [n]} = \mathbf{G}^{[n]} \cdot \left(C_\zeta(\boldsymbol{x}_\zeta)\right)_{\zeta \subseteq [n]} \tag{7}$$

where

$$\mathbf{G}^{[n]} = \left(g_{\zeta\eta}\right)_{\zeta,\eta \subseteq [n]}, \quad g_{\zeta\eta} = \begin{cases} (-1)^{|\zeta|+|\eta|} & \text{for } \zeta \subseteq \eta, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

is a matrix with Boolean indexed entries and $\left(\operatorname{geq}_\zeta(C)(\boldsymbol{x})\right)_{\zeta \subseteq [n]}$ and $\left(C_\zeta(\boldsymbol{x}_\zeta)\right)_{\zeta \subseteq [n]}$ are considered as column vectors. The matrix $\mathbf{G}^{[n]}$ is regular and the relationship can be reversed

$$\left(C_\zeta(\boldsymbol{x}_\zeta)\right)_{\zeta \subseteq [n]} = \mathbf{H}^{[n]} \cdot \left(\operatorname{geq}_\zeta(C)(\boldsymbol{x})\right)_{\zeta \subseteq [n]} \tag{9}$$

where

$$\mathbf{H}^{[n]} = \left(h_{\zeta\eta}\right)_{\zeta,\eta \subseteq [n]}, \quad h_{\zeta\eta} = \begin{cases} 1 & \text{for } \zeta \subseteq \eta, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

is the inverse matrix of $\mathbf{G}^{[n]}$ and the elements of $\mathbf{G}^{[n]} \cdot \mathbf{H}^{[n]} = (k_{\zeta\eta})_{\zeta,\eta\subseteq[n]}$ are $k_{\zeta\eta} = \sum_{\theta\subseteq[n]} g_{\zeta\theta} h_{\theta\eta} = \begin{cases} 1 & \text{for } \zeta = \eta \\ 0 & \text{otherwise} \end{cases}$ using (6) for $\zeta \subseteq \eta$.

Let us consider relation

$$\sum_{\zeta\subseteq[n]} \alpha_\zeta \mathrm{geq}_\zeta(\tilde{C})(\boldsymbol{x}) = \sum_{\zeta\subseteq[n]} \bar{\alpha}_\zeta \tilde{C}_\zeta(\boldsymbol{x}_\zeta) \tag{11}$$

for some row vector $\boldsymbol{\alpha} = (\alpha_\zeta)^T_{\zeta\subseteq[n]}$ with non-negative components and $\bar{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \cdot \mathbf{G}^{[n]} = (\bar{\alpha}_\zeta)^T_{\zeta\subseteq[n]}$ according to (7), where T denotes transposition. We call the corresponding function $\sum_{\zeta\in\Delta\boldsymbol{S}} \frac{-\bar{\alpha}_\zeta}{\bar{\alpha}_{[n]}} C_\zeta(\boldsymbol{x}_\zeta)$ the *algebraic bound* of the Fréchet class $\mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$ if and only if $\bar{\alpha}_{[n]} \neq 0$ and $\bar{\alpha}_\zeta = 0$ for each $\zeta \in \mathcal{P}([n]) \setminus (\Delta\boldsymbol{S} \cup \{[n]\})$ on the right side of (11). In this case,

$$\sum_{\zeta\in\Delta\boldsymbol{S}} \bar{\alpha}_\zeta C_\zeta(\boldsymbol{x}_\zeta) + \bar{\alpha}_{[n]} \tilde{C}(\boldsymbol{x}) \geq 0 \tag{12}$$

is the associated inequality as $\tilde{C}_\zeta = C_\zeta$ for $\tilde{C} \in \mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$ and $\zeta \in \Delta\boldsymbol{S}$. This bound is *lower algebraic bound* if $\bar{\alpha}_{[n]} > 0$ and *upper algebraic bound* if $\bar{\alpha}_{[n]} < 0$.

The analogue of the following lemma is shown in the paper [7].

Lemma 2.1. Let $\boldsymbol{S} \subseteq \mathcal{P}([n])$ be a family of subsets of $[n]$. If $\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\}$ is a compatible set of copulas then each function of the form $C_\upsilon(\boldsymbol{x}_\upsilon)$ is an upper algebraic bound for any $\upsilon \in \Delta\boldsymbol{S}$, each function of the form $\sum_{i=1}^m C_{\upsilon_i}(\boldsymbol{x}_{\upsilon_i}) - |m| + 1$ is a lower algebraic bound for any partition $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_m\}$ of $[n]$ with $\upsilon_i \in \Delta\boldsymbol{S}$ and 0 is also a lower algebraic bound of $\mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$.

On the other hand, let a function $\tilde{C}(\boldsymbol{x})$ be n-increasing on $\langle 0,1\rangle^n$, bounded above by each function of the form

$$F(\boldsymbol{x}) = C_\upsilon(\boldsymbol{x}_\upsilon) \text{ for some } \upsilon \in \boldsymbol{S} \text{ or } F(\boldsymbol{x}) = x_i \text{ for some } i \notin \cup\boldsymbol{S} \tag{13}$$

and bounded below by each function of the form

$$G(\boldsymbol{x}) = C_\upsilon(\boldsymbol{x}_\upsilon) - n + |\upsilon| + \sum_{i\notin\upsilon} x_i \text{ for some } \upsilon \in \boldsymbol{S} \text{ or } G(\boldsymbol{x}) = 0 \tag{14}$$

on $\langle 0,1\rangle^n$. Then $\tilde{C} \in \mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$.

Proof. The following identities are consequence of the identity $\tilde{C}_\upsilon = \sum_{\zeta\supseteq\upsilon} \mathrm{geq}_\zeta(\tilde{C})$, as a component of (9). For any $\upsilon \in \Delta\boldsymbol{S}$, $C_\upsilon(\boldsymbol{x}_\upsilon)$ is an upper algebraic bound of $\mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$ because

$$0 \leq \sum_{\substack{\zeta\supseteq\upsilon \\ \zeta\subsetneq[n]}} \mathrm{geq}_\zeta(\tilde{C}) = C_\upsilon - \tilde{C}$$

for any $\tilde{C} \in \mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$. For any partition $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_m\}$ of $[n]$ with members from $\Delta\boldsymbol{S}$, $\sum_{i=1}^m C_{\upsilon_i} - |m| + 1$ is a lower algebraic bound as

$$0 \leq \sum_{\substack{\zeta\supseteq\upsilon_1\cap\upsilon_2 \\ \zeta\not\supseteq\upsilon_1,\ \zeta\not\supseteq\upsilon_2}} \mathrm{geq}_\zeta(\tilde{C}) + \sum_{\substack{\zeta\supseteq(\upsilon_1\cup\upsilon_2)\cap\upsilon_3 \\ \zeta\not\supseteq\upsilon_1\cup\upsilon_2,\ \zeta\not\supseteq\upsilon_3}} \mathrm{geq}_\zeta(\tilde{C}) + \sum_{\substack{\zeta\supseteq(\upsilon_1\cup\upsilon_2\cup\upsilon_3)\cap\upsilon_4 \\ \zeta\not\supseteq\upsilon_1\cup\upsilon_2\cup\upsilon_3,\ \zeta\not\supseteq\upsilon_4}} \mathrm{geq}_\zeta(\tilde{C}) + \ldots$$

$$+ \sum_{\substack{\zeta \supseteq \bigcup_{j=1}^{m-2} \upsilon_j \cap \upsilon_{m-1} \\ \zeta \not\supseteq \bigcup_{j=1}^{m-2} \upsilon_j,\ \zeta \not\supseteq \upsilon_{m-1}}} \mathrm{geq}_\zeta(\tilde{C}) + \sum_{\substack{\zeta \supseteq \bigcup_{j=1}^{m-1} \upsilon_j \cap \upsilon_m \\ \zeta \not\supseteq \bigcup_{j=1}^{m-1} \upsilon_j,\ \zeta \not\supseteq \upsilon_m}} \mathrm{geq}_\zeta(\tilde{C})$$

$$= \left[1 - \tilde{C}_{\upsilon_1} - \tilde{C}_{\upsilon_2} + \tilde{C}_{\upsilon_1 \cup \upsilon_2}\right] + \left[1 - \tilde{C}_{\upsilon_1 \cup \upsilon_2} - \tilde{C}_{\upsilon_3} + \tilde{C}_{\upsilon_1 \cup \upsilon_2 \cup \upsilon_3}\right]$$

$$+ \left[1 - \tilde{C}_{\upsilon_1 \cup \upsilon_2 \cup \upsilon_3} - \tilde{C}_{\upsilon_4} + \tilde{C}_{\upsilon_1 \cup \upsilon_2 \cup \upsilon_3 \cup \upsilon_4}\right] + \dots$$

$$+ \left[1 - \tilde{C}_{\bigcup_{j=1}^{m-2} \upsilon_j} - \tilde{C}_{\upsilon_{m-1}} + \tilde{C}_{\bigcup_{j=1}^{m-1} \upsilon_j}\right] + \left[1 - \tilde{C}_{\bigcup_{j=1}^{m-1} \upsilon_j} - \tilde{C}_{\upsilon_m} + \tilde{C}\right]$$

$$= |m| - 1 - \sum_{i=1}^{m} C_{\upsilon_i} + \tilde{C}$$

for any $\tilde{C} \in \mathcal{F}_n(\{C_\upsilon;\, \upsilon \in \boldsymbol{S}\})$. On account of

$$0 \le \mathrm{geq}_{[n]}(\tilde{C}) = \tilde{C},$$

the function 0 is also a lower algebraic bound of $\mathcal{F}_n(\{C_\upsilon;\, \upsilon \in \boldsymbol{S}\})$.

On the other hand, let a function $\tilde{C}(\boldsymbol{x})$ be n-increasing on $\langle 0, 1\rangle^n$ and $G \le \tilde{C} \le F$ on $\langle 0, 1\rangle^n$ for any G of the form (14) and any F of the form (13). The values of $\tilde{C}$ are inside $\langle 0, 1\rangle$ as any value of a function of (13) is less than 1 and the functions (14) contain 0. To show that $\tilde{C}$ is a copula from $\mathcal{F}_n(\{C_\upsilon;\, \upsilon \in \boldsymbol{S}\})$, we are going to prove that $\tilde{C}$ is grounded, its 1-margins are uniform and its υ-margin is C_υ for any $\upsilon \in \boldsymbol{S}$.

Let the kth component x_k of $\boldsymbol{x} \in \langle 0, 1\rangle^n$ be 0. As the union of determinative sets of all given margins of (13) (including the 1-margins) is $[n]$, k belongs to the determinative set υ of some F of the form (13). But since F is an υ-copula and $k \in \upsilon$ then $0 = F(\boldsymbol{x}_\upsilon) \ge \tilde{C}(\boldsymbol{x}) \ge 0$ and $\tilde{C}$ is grounded.

Let $k \in [n]$ and all components of $\boldsymbol{x}$, except for the kth one, be equal to 1. As k belongs to the determinative set τ of some F then $\tilde{C}(\boldsymbol{x}) \le F(\boldsymbol{x}_\tau) = x_k$. On the other hand, let $G = C_\upsilon - n + |\upsilon| + \sum_{i \notin \upsilon} x_i$ be of the form (14) for some $\upsilon \in \boldsymbol{S}$. If $k \in \upsilon$ then $C_\upsilon(\boldsymbol{x}_\upsilon) = x_k$ and $G(\boldsymbol{x}) = C_\upsilon(\boldsymbol{x}_\upsilon) - n + |\upsilon| + \sum_{i \notin \upsilon} x_i = x_k - n + |\upsilon| + n - |\upsilon| = x_k$ (as $x_i = 1$ for $i \notin \upsilon$). If $k \notin \upsilon$, we get $G(\boldsymbol{x}) = C_\upsilon(\boldsymbol{x}_\upsilon) - n + |\upsilon| + \sum_{i \notin \upsilon} x_i = 1 - n + |\upsilon| + n - |\upsilon| - 1 + x_k = x_k$. Consequently, $x_k = G(\boldsymbol{x}) \le \tilde{C}(\boldsymbol{x})$ and thus $\tilde{C}(\boldsymbol{x}) = x_k$. So, $\{k\}$-margin of $\tilde{C}$ is uniform.

In the last step, we will prove that υ-margin of $\tilde{C}$ is identical to C_υ for each $\upsilon \in \boldsymbol{S}$. For any $\boldsymbol{x} \in \langle 0, 1\rangle^n$ and $\boldsymbol{y} = \mathrm{sel}_\upsilon(\boldsymbol{x}, \boldsymbol{1}_n)$, one has $\tilde{C}(\boldsymbol{y}) \le C_\upsilon(\boldsymbol{y}_\upsilon) = C_\upsilon(\boldsymbol{x}_\upsilon)$ and $\tilde{C}(\boldsymbol{y}) \ge C_\upsilon(\boldsymbol{y}_\upsilon) - n + |\upsilon| + \sum_{i \notin \upsilon} y_i = C_\upsilon(\boldsymbol{x}_\upsilon) - n + |\upsilon| + n - |\upsilon| = C_\upsilon(\boldsymbol{x}_\upsilon)$ as C_υ comes from (13) and $C_\upsilon - n + |\upsilon| + \sum_{i \notin \upsilon} C_{\{i\}}$ from (14). □

We use knowledge about non-negative solutions of a system of linear homogeneous Diophantine equations as a tool of the following investigation. A linear half-space of $\mathbb{R}^m$ is a subset of $\mathbb{R}^m$ of the form $\{\boldsymbol{\alpha} \in \mathbb{R}^m;\, c_1\alpha_1 + c_2\alpha_2 + \dots + c_m\alpha_m \ge 0\}$ for some fixed nonzero vector $(c_1, c_2, \dots, c_m) \in \mathbb{R}^m$. A *convex polyhedral cone* $\mathcal{C}_m$ in $\mathbb{R}^m$ is defined to be the intersection of finitely many half-spaces. We say that such a cone $\mathcal{C}_m$ is *pointed* if it does not contain a line. In the following, we will always mean a pointed convex polyhedral cone, when we mention a cone. A one-dimensional face of the cone is called an *extreme ray*. A cone $\mathcal{C}_m$ has only finitely many extreme rays $r_1, r_2, \dots, r_k$ and is the convex hull of its extreme rays. Thus, for given nonzero members $\boldsymbol{\beta}_i \in r_i$, each point $\boldsymbol{\alpha} \in \mathcal{C}_m$ can be expressed in the form $\boldsymbol{\alpha} = c_1\boldsymbol{\beta}_1 + c_2\boldsymbol{\beta}_2 + \ldots + c_k\boldsymbol{\beta}_k$, where $c_i \geq 0$ for every $i = 1, 2, \ldots, k$. We call such a set $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_k\}$ *basis* of the cone $\mathcal{C}_m$. Each extreme ray can be expressed as $r_i = \{a\boldsymbol{\beta}_i;\ a$ is non-negative$\}$. Any other basis has the form $\{a_1\boldsymbol{\beta}_1, a_2\boldsymbol{\beta}_2, \ldots, a_k\boldsymbol{\beta}_k\}$ where $a_1,\ a_2, \ldots,\ a_k$ are positive. The basis of the cone $\mathcal{C}_m$ is a minimal (in ordering by inclusion) such subset $\mathcal{B}$ of $\mathcal{C}_m$ that every element of $\mathcal{C}_m$ can be expressed (not necessarily uniquely) as a linear combination of elements of $\mathcal{B}$ with non-negative combination coefficients.

The set of all non-negative solutions of a system of linear homogeneous equations is a cone since every equation can be interpreted as a conjunction of two inequalities. Works [6] and [8] indicate that, if the system is in addition *Diophantine* then there is a basis of all non-negative solutions such that each member of this basis consists of natural components ($\mathbb{N}$-solution) and its support (set of indices of nonzero components) is minimal. Of course, the members of such a basis are minimal in the set of all nonzero $\mathbb{N}$-solutions. Some algorithms to search for such basis are presented, e. g., in papers [1] and [9].

Algebraic bounds are defined through non-negative linear combinations of geq-volumes which eliminate coefficient $\bar{\alpha}_\zeta$ at such $\tilde{C}_\zeta$ in the relation (11) that $\zeta \in \mathcal{P}([n]) \setminus (\Delta\boldsymbol{S} \cup \{[n]\})$. So, all the coefficients $\boldsymbol{\alpha} = (\alpha_\zeta)^T_{\zeta\in\mathcal{P}([n])}$ of such a combination are defined as a non-negative real solution of the system of linear homogeneous Diophantine equations

$$\boldsymbol{\alpha} \cdot G^{[n]}_{\boldsymbol{S}} = \mathbf{0}_{|\mathcal{P}([n])\setminus(\Delta\boldsymbol{S}\cup\{[n]\})|} \tag{15}$$

written in matrix form where $G^{[n]}_{\boldsymbol{S}}$ is the submatrix of $G^{[n]}$ formed just by those (ζ, η)-entries for which $\eta \in \mathcal{P}\left([n]\right) \setminus (\Delta\boldsymbol{S} \cup \{[n]\})$. Let $\mathcal{B}_{\boldsymbol{S}}$ be a basis of all solutions of (15). For an element $\boldsymbol{\beta} \in \mathcal{B}_{\boldsymbol{S}}$ and $\bar{\boldsymbol{\beta}} = \boldsymbol{\beta} \cdot \mathbf{G}^{[n]}$, the inequality

$$\sum_{\zeta\in\Delta\boldsymbol{S}} \bar{\beta}_\zeta C_\zeta(\boldsymbol{x}_\zeta) + \bar{\beta}_{[n]}\tilde{C}(\boldsymbol{x}) \geq 0 \tag{16}$$

of the type (12) is associated with either a lower or an upper algebraic bound or it is just a necessary compatibility condition of the set $\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\}$, depending on whether the coefficient $\bar{\beta}_{[n]}$ at $\tilde{C}(\boldsymbol{x})$ is positive, negative or zero. Let $\boldsymbol{g}_{[n]}$ be the $[n]$-column of $G^{[n]}$. As $\bar{\beta}_{[n]} = \boldsymbol{\beta}\cdot\boldsymbol{g}_{[n]}$, the finite basis $\mathcal{B}_{\boldsymbol{S}}$ can be expressed as the union of three disjoint parts $\mathcal{B}_L = \{\boldsymbol{\beta} \in \mathcal{B}_{\boldsymbol{S}};\ \boldsymbol{\beta}\cdot\boldsymbol{g}_{[n]} > 0\}$, $\mathcal{B}_U = \{\boldsymbol{\beta} \in \mathcal{B}_{\boldsymbol{S}};\ \boldsymbol{\beta}\cdot\boldsymbol{g}_{[n]} < 0\}$ and $\mathcal{B}_0 = \{\boldsymbol{\beta} \in \mathcal{B}_{\boldsymbol{S}};\ \boldsymbol{\beta}\cdot\boldsymbol{g}_{[n]} = 0\}$ separated by the hyperplane

$$\boldsymbol{\alpha} \cdot \boldsymbol{g}_{[n]} = 0. \tag{17}$$

Let us define the function

$$b_L(\boldsymbol{x}) = \max\left\{ \sum_{\zeta\in\Delta\boldsymbol{S}} \frac{-\bar{\beta}_\zeta}{\bar{\beta}_{[n]}} C_\zeta(\boldsymbol{x}_\zeta);\ \boldsymbol{\beta} \in \mathcal{B}_L \right\} \tag{18}$$

as the *basic lower bound* and the function

$$b_U(\boldsymbol{x}) = \min\left\{ \sum_{\zeta\in\Delta\boldsymbol{S}} \frac{-\bar{\beta}_\zeta}{\bar{\beta}_{[n]}} C_\zeta(\boldsymbol{x}_\zeta);\ \boldsymbol{\beta} \in \mathcal{B}_U \right\} \tag{19}$$

as the *basic upper bound* of the Fréchet class $\mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$. It is obvious that these functions are not dependent on the choice of the basis $\mathcal{B}_{\boldsymbol{S}}$ and they are bounds of the Fréchet class $\mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$ if this is nonempty.

The following necessary conditions for compatibility are the result of the previous arguments.

Lemma 2.2. Let the set $\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\}$ of copulas be compatible, $\mathcal{B}_{\boldsymbol{S}} = \mathcal{B}_L \cup \mathcal{B}_U \cup \mathcal{B}_0$ be a basis of all solutions of (15) and $\boldsymbol{x} \in \langle 0,1\rangle^n$. Then each of the following three conditions is valid

$$\begin{aligned}
&(i) && C_\zeta\left(\mathrm{sel}_{\zeta\cap\eta}(\boldsymbol{x}_\zeta, \mathbf{1}_{|\zeta|})\right) = C\eta\left(\mathrm{sel}_{\zeta\cap\eta}(\boldsymbol{x}_\eta, \mathbf{1}_{|\eta|})\right) \text{ for any } \zeta,\eta \in \boldsymbol{S}\\
&(ii) && 0 \le \sum_{\zeta\in\Delta\boldsymbol{S}} \left(\frac{\bar\alpha_\zeta}{\bar\alpha_{[n]}} - \frac{\bar\beta_\zeta}{\bar\beta_{[n]}}\right) C_\zeta(\boldsymbol{x}_\zeta) \text{ for any } \boldsymbol{\alpha} \in \mathcal{B}_L \text{ and } \boldsymbol{\beta} \in \mathcal{B}_U \\
&(iii) && 0 \le \sum_{\zeta\in\Delta\boldsymbol{S}} \bar\beta_\zeta C_\zeta(\boldsymbol{x}_\zeta) \text{ for any } \boldsymbol{\beta} \in \mathcal{B}_0.
\end{aligned} \tag{20}$$

Proof.

(i) The margins of any Fréchet class must be identical on common parts of the determinative sets.

(ii) For all $\boldsymbol{\alpha} \in \mathcal{B}_L$ and $\boldsymbol{\beta} \in \mathcal{B}_U$, $\sum_{\zeta\in\Delta\boldsymbol{S}} \frac{-\bar\alpha_\zeta}{\bar\alpha_{[n]}} C_\zeta(\boldsymbol{x}_\zeta)$ is a lower bound of the corresponding Fréchet class and thus is smaller than the upper bound $\sum_{\zeta\in\Delta\boldsymbol{S}} \frac{-\bar\beta_\zeta}{\bar\beta_{[n]}} C_\zeta(\boldsymbol{x}_\zeta)$.

(iii) $\sum_{\zeta\in\Delta\boldsymbol{S}} \bar\beta_\zeta C_\zeta(\boldsymbol{x}_\zeta) = \sum_{\zeta\subseteq[n]} \beta_\zeta \mathrm{geq}_\zeta(\tilde C)(\boldsymbol{x}) \ge 0$ for any $\boldsymbol{\beta} \in \mathcal{B}_0$ and $\tilde C \in \mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$. □

Lemma 2.3. Let $\{C_\upsilon(\boldsymbol{x}_\upsilon);\ \upsilon \in \boldsymbol{S}\}$ be a set of copulas satisfying all three conditions (20). Let $b_L(\boldsymbol{x})$ and $b_U(\boldsymbol{x})$ be the corresponding basic bounds defined by (18) and (19). Then $b_L(\boldsymbol{x}) \ge G(\boldsymbol{x})$ for each lower algebraic bound G and $b_U(\boldsymbol{x}) \le F(\boldsymbol{x})$ for each upper algebraic bound F of the Fréchet class $\mathcal{F}_n(\{C_\upsilon;\ \upsilon \in \boldsymbol{S}\})$ and $\boldsymbol{x} \in \langle 0,1\rangle^n$.

Proof. The set $\{C_\zeta\}_{\zeta\in\Delta\boldsymbol{S}}$ is well defined due to *(i)* in (20). Let $\mathcal{B}_{\boldsymbol{S}} = \mathcal{B}_L \cup \mathcal{B}_U \cup \mathcal{B}_0 = \{\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, \ldots, \boldsymbol{\beta}^k\}$ be a basis of all solutions of (15) and $H(\boldsymbol{x}) = \sum_{\zeta\in\Delta\boldsymbol{S}} \frac{-\bar\alpha_\zeta}{\bar\alpha_{[n]}} C_\zeta(\boldsymbol{x}_\zeta)$ be an algebraic bound of the corresponding Fréchet class, where $\bar{\boldsymbol{\alpha}} \cdot \mathbf{H}^{[n]} = \boldsymbol{\alpha} = c_1\boldsymbol{\beta}^1 + c_2\boldsymbol{\beta}^2 + \ldots + c_k\boldsymbol{\beta}^k$ for some non-negative $c_1, c_2, \ldots, c_k$. For any $\boldsymbol{x} \in \langle 0,1\rangle^n$, let y be such that $b_L(\boldsymbol{x}) \le y \le b_U(\boldsymbol{x})$ and consequently $\sum_{\zeta\in\Delta\boldsymbol{S}} \bar\beta^i_\zeta C_\zeta(\boldsymbol{x}_\zeta) + \bar\beta^i_{[n]} y \ge 0$ for each $\boldsymbol{\beta}^i \in \mathcal{B}_L \cup \mathcal{B}_U$. There is always such a number because of *(ii)* in (20). Moreover, $\sum_{\zeta\in\Delta\boldsymbol{S}} \bar\beta^i_\zeta C_\zeta(\boldsymbol{x}_\zeta) + \bar\beta^i_{[n]} y \ge 0$ for each $\boldsymbol{\beta}^i \in \mathcal{B}_0$ on account of *(iii)* in (20). Therefore, $\sum_{\zeta\in\Delta\boldsymbol{S}} \bar\beta^i_\zeta C_\zeta(\boldsymbol{x}_\zeta) + \bar\beta^i_{[n]} y \ge 0$ for each $\boldsymbol{\beta}^i \in \mathcal{B}_{\boldsymbol{S}}$ and $\sum_{i=1}^k c_i \left(\sum_{\zeta\in\Delta\boldsymbol{S}} \bar\beta^i_\zeta C_\zeta(\boldsymbol{x}_\zeta) + \bar\beta^i_{[n]} y\right) = \sum_{\zeta\in\Delta\boldsymbol{S}} \bar\alpha_\zeta C_\zeta(\boldsymbol{x}_\zeta) + \bar\alpha_{[n]} y \ge 0$. It means that $y \ge H(\boldsymbol{x})$ if H is a lower algebraic bound, respectively $y \le H(\boldsymbol{x})$ if H is an upper algebraic bound. □

The following theorem is immediate consequence of Lemmas 2.1, 2.2 and 2.3.

Theorem 2.4. The set $\{C_\upsilon;\, \upsilon \in \boldsymbol{S}\}$ of copulas is compatible if and only if all conditions of (20) are satisfied and there exists such n-increasing function $\tilde{C}(\boldsymbol{x})$ that $b_L(\boldsymbol{x}) \leq \tilde{C}(\boldsymbol{x}) \leq b_U(\boldsymbol{x})$ for each $\boldsymbol{x} \in \langle 0,1\rangle^n$.

3. EXPLICIT DESCRIPTION OF BASIC BOUNDS

In this section we offer a tool that could facilitate the explicit expression of basic bounds of simple Fréchet classes.

Lemma 3.1. Let $\boldsymbol{S}$ be a family of sets over $[n]$, $\boldsymbol{\alpha} = (\alpha_\eta)^T_{\eta\subseteq[n]}$ be a row vector with non-negative components and $\bar{\boldsymbol{\alpha}} = \boldsymbol{\alpha}\cdot \mathbf{G}^{[n]}$ where $\mathbf{G}^{[n]}$ is defined by (8). Then the next two propositions are equivalent

(i) $(\forall \nu \in \mathcal{P}([n])\setminus(\Delta\boldsymbol{S}\cup\{[n]\})) : \bar{\alpha}_\nu = 0$

(ii) $(\forall \nu \in \mathcal{P}([n])\setminus(\Delta\boldsymbol{S}\cup\{[n]\}))$:

$$\alpha_\nu = \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} \Big(\sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subseteq\nu}} (-1)^{|\eta|+|\zeta|+1}\Big)\alpha_\zeta. \tag{21}$$

Proof. As $\bar{\alpha}_\nu = \sum_{\zeta\subseteq\nu}(-1)^{|\nu|+|\zeta|}\alpha_\zeta$, $\bar{\alpha}_\nu = 0$ if and only if

$$\alpha_\nu = \sum_{\zeta\subsetneq\nu}(-1)^{|\nu|+|\zeta|+1}\alpha_\zeta \tag{22}$$

for nonempty ν.

Let $\bar{\alpha}_\nu = 0$ for each $\nu \in \mathcal{P}([n])\setminus(\Delta\boldsymbol{S}\cup\{[n]\})$. This direction of the proof will be performed by structural induction over members of $\mathcal{P}([n])\setminus(\Delta\boldsymbol{S}\cup\{[n]\})$.

I. Let $\nu \notin \Delta\boldsymbol{S}\cup\{[n]\}$ and $\zeta \in \Delta\boldsymbol{S}$ for each $\zeta \subsetneq \nu$. In terms (22), there is $\alpha_\nu = \sum_{\zeta\subsetneq\nu}(-1)^{|\nu|+|\zeta|+1}\alpha_\zeta$. As $\{\eta \notin \Delta\boldsymbol{S};\, \zeta\subseteq\eta\subseteq\nu\} = \{\nu\}$, it is just the identity (21).

II. Now, let $\nu \notin \Delta\boldsymbol{S}\cup\{[n]\}$ and for each such $\theta \subsetneq \nu$ that $\theta \notin \Delta\boldsymbol{S}$, (21) is met, i.e. $\alpha_\theta = \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\theta}}\left(\sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subseteq\theta}}(-1)^{|\eta|+|\zeta|+1}\right)\alpha_\zeta$. Then, starting with (22),

$$\alpha_\nu = \sum_{\zeta\subsetneq\nu}(-1)^{|\nu|+|\zeta|+1}\alpha_\zeta = \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}}(-1)^{|\nu|+|\zeta|+1}\alpha_\zeta + \sum_{\substack{\theta\notin\Delta\mathbf{S}\\ \theta\subsetneq\nu}}(-1)^{|\nu|+|\theta|+1}\alpha_\theta$$

$$= \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}}(-1)^{|\nu|+|\zeta|+1}\alpha_\zeta + \sum_{\substack{\theta\notin\Delta\mathbf{S}\\ \theta\subsetneq\nu}}(-1)^{|\nu|+|\theta|+1}\sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\theta}}\sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subseteq\theta}}(-1)^{|\eta|+|\zeta|+1}\alpha_\zeta$$

$$= \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}}(-1)^{|\nu|+|\zeta|+1}\alpha_\zeta + \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}}\sum_{\substack{\theta\notin\Delta\mathbf{S}\\ \zeta\subseteq\theta\subsetneq\nu}}\sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subseteq\theta}}(-1)^{|\eta|+|\zeta|+|\nu|+|\theta|}\alpha_\zeta$$

$$= \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|+1}\alpha_\zeta + \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} \sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subsetneq\nu}} \sum_{\substack{\theta\notin\Delta\mathbf{S}\\ \eta\subseteq\theta\subsetneq\nu}} (-1)^{|\eta|+|\zeta|+|\nu|+|\theta|}\alpha_\zeta$$

$$\overset{(6)}{=} \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|+1}\alpha_\zeta + \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} \sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subsetneq\nu}} (-1)^{|\eta|+|\zeta|} \sum_{i=|\eta|}^{|\nu|-1} (-1)^{|\nu|+i}\binom{|\nu|-|\eta|}{i-|\eta|}\alpha_\zeta$$

$$\overset{(6)}{=} \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|+1}\alpha_\zeta + \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} \sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subsetneq\nu}} (-1)^{|\eta|+|\zeta|+1}\alpha_\zeta$$

$$= \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} \sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subseteq\nu}} (-1)^{|\eta|+|\zeta|+1}\alpha_\zeta.$$

On the other hand, let (21) hold for each $\nu\in\mathcal{P}([n])\setminus(\Delta\boldsymbol{S}\cup\{[n]\})$. Then

$$\bar{\alpha}_\nu = \sum_{\zeta\subseteq\nu}(-1)^{|\nu|+|\zeta|}\alpha_\zeta = \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|}\alpha_\zeta + \sum_{\substack{\theta\notin\Delta\mathbf{S}\\ \theta\subseteq\nu}} (-1)^{|\nu|+|\theta|}\alpha_\theta$$

$$= \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|}\alpha_\zeta + \sum_{\substack{\theta\notin\Delta\mathbf{S}\\ \theta\subseteq\nu}} (-1)^{|\nu|+|\theta|} \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\theta}} \sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subseteq\theta}} (-1)^{|\eta|+|\zeta|+1}\alpha_\zeta$$

$$= \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|}\alpha_\zeta + \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} \sum_{\substack{\theta\notin\Delta\mathbf{S}\\ \zeta\subseteq\theta\subseteq\nu}} \sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subseteq\theta}} (-1)^{|\nu|+|\zeta|+1+|\eta|+|\theta|}\alpha_\zeta$$

$$= \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|}\alpha_\zeta + \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} \sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subseteq\nu}} \sum_{\substack{\theta\subseteq[n]\\ \eta\subseteq\theta\subseteq\nu}} (-1)^{|\nu|+|\zeta|+1+|\eta|+|\theta|}\alpha_\zeta$$

$$\overset{(6)}{=} \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|}\alpha_\zeta + \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|+1} \sum_{\substack{\eta\notin\Delta\mathbf{S}\\ \zeta\subseteq\eta\subseteq\nu}} \sum_{i=|\eta|}^{|\nu|} (-1)^{|\eta|+i}\binom{|\nu|-|\eta|}{i-|\eta|}\alpha_\zeta$$

$$\overset{(6)}{=} \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|}\alpha_\zeta + \sum_{\substack{\zeta\in\Delta\mathbf{S}\\ \zeta\subseteq\nu}} (-1)^{|\nu|+|\zeta|+1}\alpha_\zeta = 0.$$

□

The previous lemma indicates that the cone of non-negative solutions of the equation (15) is isomorphic to a cone in some lower dimensional space.

Theorem 3.2. Let $\boldsymbol{S}$ be a family of subsets of $[n]$ not containing $[n]$. The mapping $\mathrm{proj}:\mathcal{C}_{\boldsymbol{S}}\to\mathcal{C}'_{\boldsymbol{S}}$, $(\alpha_\zeta)^T_{\zeta\in\mathcal{P}([n])}\mapsto(\alpha_\zeta)^T_{\zeta\in\Delta\boldsymbol{S}\cup\{[n]\}}$ is a linear one-to-one correspondence of the cone $\mathcal{C}_{\boldsymbol{S}}$ of all solutions $\boldsymbol{\alpha}=(\alpha_\zeta)^T_{\zeta\in\mathcal{P}([n])}$ of the system

$$\boldsymbol{\alpha}\cdot G^{[n]}_{\boldsymbol{S}} = \mathbf{0}_{|\mathcal{P}([n])\setminus(\Delta\boldsymbol{S}\cup\{[n]\})|} \tag{23}$$

meeting the condition

$$(\forall \zeta \in \mathcal{P}([n])) : \alpha_\zeta \geq 0 \tag{24}$$

and the cone $\mathcal{C}'_{\boldsymbol{S}}$ of all such vectors $\boldsymbol{\alpha}' = (\alpha_\zeta)^T_{\zeta \in \Delta \boldsymbol{S} \cup \{[n]\}}$ that

$$(\forall \zeta \in \Delta \boldsymbol{S} \cup \{[n]\}) : \alpha_\zeta \geq 0 \ \ \& \tag{25}$$

$$(\forall \nu \in \mathcal{P}([n]) \setminus (\Delta \boldsymbol{S} \cup \{[n]\})) : \sum_{\substack{\zeta \in \Delta \mathbf{S} \\ \zeta \subseteq \nu}} \left(\sum_{\substack{\eta \notin \Delta \mathbf{S} \\ \zeta \subseteq \eta \subseteq \nu}} (-1)^{|\eta|+|\zeta|+1} \right) \alpha_\zeta \geq 0.$$

Moreover, this mapping keeps correspondence of the vectors having all components natural.

P r o o f. The function proj is an orthogonal projection, that is linear. The rest of the claim is an immediate consequence of Lemma 3.1. □

According to Theorem 3.2, the cones $\mathcal{C}_{\boldsymbol{S}}$ and $\mathcal{C}'_{\boldsymbol{S}}$ are in some sense similar and they correspond to each other in linear features. We will use this resemblance to give an explicit description of the basic bounds of general Fréchet classes $\mathcal{F}_n(\varnothing)$. Use of this technique appears promising for describing the basic bounds general Fréchet classes, where $\Delta \boldsymbol{S}$ consists of several 2-element subsets of $[n]$. Overall, similar investigations of one higher full horizon $\Delta \boldsymbol{S} = \{\upsilon \subseteq [n];\ |\upsilon| \leq 2\}$ of given margins is technically difficult without a computer. The cardinality of the basis $\mathcal{B}_{\boldsymbol{S}}$ of the relevant cone is 69 for $n = 4$, 694 for $n = 5$ and steeply increases with n.

Theorem 3.3. (Fréchet–Hoeffding) If C is any n-copula, then

$$\max\{0, x_1 + x_2 + \ldots + x_n - n + 1\} \leq C(\boldsymbol{x}) \leq \min\{x_1, x_2, \ldots, x_n\} \tag{26}$$

for every $\boldsymbol{x} \in \langle 0, 1 \rangle^n$.

Theorem 3.4. The basic bounds of Fréchet class $\mathcal{F}_n(\varnothing)$ are just the Fréchet–Hoeffding bounds.

P r o o f. In our case, $\boldsymbol{S} = \varnothing$, $\Delta \boldsymbol{S}$ consist of the empty set and all singletons under $[n]$ and the condition (25) takes the form

$$\alpha_{[n]} \geq 0 \ \ \& \ \ (\forall \nu \in \mathcal{P}([n]) \setminus \{[n]\}) : \sum_{j \in \nu} \alpha_{\{j\}} \geq (|\nu| - 1)\, \alpha_\varnothing \tag{27}$$

because of the identities (6). First, we show that the set $\mathcal{B}' = \{\boldsymbol{\beta}^{1\prime}, \ldots, \boldsymbol{\beta}^{n\prime}, \boldsymbol{\gamma}^{1\prime}, \ldots, \boldsymbol{\gamma}^{n\prime}, \boldsymbol{\delta}', \boldsymbol{\varepsilon}'\}$ is a basis of the cone $\mathcal{C}'_{\boldsymbol{S}}$ while the $\varnothing$-, singleton and $[n]$-components of its members are

$$\beta^i_\zeta = \begin{cases} 1 & \text{for } \zeta = \{i\} \\ 0 & \text{otherwise} \end{cases}, \quad \gamma^i_\zeta = \begin{cases} 0 & \text{for } \zeta = \{i\} \text{ or } \zeta = [n] \\ 1 & \text{otherwise,} \end{cases}, \tag{28}$$

$$\delta_\zeta = \begin{cases} n-1 & \text{for } \zeta = \varnothing \\ n-2 & \text{for } |\zeta| = 1 \\ 0 & \text{for } \zeta = [n] \end{cases}, \quad \varepsilon_\zeta = \begin{cases} 0 & \text{for } \zeta = \varnothing \text{ or } |\zeta| = 1 \\ 1 & \text{for } \zeta = [n]. \end{cases}.$$

It is obvious that each vector of the set $\mathcal{B}'$ satisfies the condition (27) and thus is member of $\mathcal{C}'_{\boldsymbol{S}}$. Now we are going to demonstrate that every vector $\boldsymbol{\alpha}' = (\alpha_\zeta)^T_{\zeta\in\Delta\boldsymbol{S}\cup\{[n]\}} \in \mathcal{C}'_{\boldsymbol{S}}$ is a non-negative linear combination $\boldsymbol{\alpha}' = b_1\boldsymbol{\beta}^{1\prime}+ \ldots + b_n\boldsymbol{\beta}^{n\prime}+ c_1\boldsymbol{\gamma}^{1\prime}+ \ldots + c_n\boldsymbol{\gamma}^{n\prime}+ d\boldsymbol{\delta}' + e\boldsymbol{\varepsilon}'$ of elements of $\mathcal{B}'$. In addition, a combination with natural coefficients can be found if all components of $\boldsymbol{\alpha}'$ are natural.

Any such combination must have $e = \alpha_{[n]}$ because the only vector from $\mathcal{B}'$ that is nonzero in the $[n]$-component is $\boldsymbol{\varepsilon}$. When investigating the combination, consider simplifications $\boldsymbol{\beta}' = b_1\boldsymbol{\beta}^{1\prime}+ \ldots + b_n\boldsymbol{\beta}^{n\prime}$ and $\boldsymbol{\gamma}' = c_1\boldsymbol{\gamma}^{1\prime} + \ldots + c_n\boldsymbol{\gamma}^{n\prime}$ with components

$$\beta_\zeta = \begin{cases} b_j & \text{for } \zeta = \{j\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_\zeta = \begin{cases} \sum_{i=1}^n c_i & \text{for } \zeta = \varnothing \\ \sum_{i=1}^n c_i - c_j & \text{for } \zeta = \{j\} \\ 0 & \text{for } \zeta = [n] \end{cases} . \tag{29}$$

We distinguish two cases.

I. Let $\sum_{i=1}^n \alpha_{\{i\}} \le (n-1)\alpha_\varnothing$. Then the other combination coefficients can be chosen as

$$b_j = 0, \quad c_j = \sum_{i=1}^n \alpha_{\{i\}} - \alpha_{\{j\}} - (n-2)\alpha_\varnothing \quad \text{for} \quad j \in [n]$$
$$\text{and} \quad d = (n-1)\alpha_\varnothing - \sum_{i=1}^n \alpha_{\{i\}}$$

(all are non-negative on account of (27) and the assumption of this case) as $b_1\boldsymbol{\beta}^{1\prime}+ \ldots + b_n\boldsymbol{\beta}^{n\prime}+ c_1\boldsymbol{\gamma}^{1\prime}+ \ldots + c_n\boldsymbol{\gamma}^{n\prime}+ d\boldsymbol{\delta}' + e\boldsymbol{\varepsilon}'$ is equal to

$$0 + \sum_{j=1}^n \left(\sum_{i=1}^n \alpha_{\{i\}} - \alpha_{\{j\}} - (n-2)\alpha_\varnothing \right) + \left((n-1)\alpha_\varnothing - \sum_{i=1}^n \alpha_{\{i\}} \right)(n-1) + 0 = \alpha_\varnothing \text{ in the } \varnothing\text{-component},$$

$$0 + \sum_{k=1}^n \left(\sum_{i=1}^n \alpha_{\{i\}} - \alpha_{\{k\}} - (n-2)\alpha_\varnothing \right) - \left(\sum_{i=1}^n \alpha_{\{i\}} - \alpha_{\{j\}} - (n-2)\alpha_\varnothing \right) + \left((n-1)\alpha_\varnothing - \sum_{i=1}^n \alpha_{\{i\}} \right)(n-2) + 0 = \alpha_{\{j\}} \text{ in the } \{j\}\text{-component},$$

$$0 + 0 + 0 + \alpha_{[n]} = \alpha_{[n]} \text{ in the } [n]\text{-component},$$

using simplifications (29).

II. Let $\sum_{i=1}^n \alpha_{\{i\}} > (n-1)\alpha_\varnothing$. There exists a disjoint union $\kappa \cup \{k\} \cup \lambda = [n]$ such that $\min\{\alpha_\varnothing, \alpha_{\{k\}}\} \ge \alpha_{\{j\}}$ for all $j \in \kappa$ and $\max\{\alpha_\varnothing, \alpha_{\{k\}}\} \le \alpha_{\{j\}}$ for all $j \in \lambda$. We also accept emptiness of κ or λ. Appropriate coefficients are

$$b_j = 0, \quad c_j = \alpha_\varnothing - \alpha_{\{j\}} \text{ for } j \in \kappa,$$
$$b_k = \sum_{i\in\kappa\cup\{k\}} \alpha_{\{i\}} - |\kappa|\alpha_\varnothing, \quad c_k = \sum_{i\in\kappa} \alpha_{\{i\}} - (|\kappa| - 1)\alpha_\varnothing,$$
$$b_j = \alpha_{\{j\}} - \alpha_\varnothing, \quad c_j = 0 \text{ for } j \in \lambda$$

and $d = 0$. Note that $b_j - c_j = \alpha_{\{j\}} - \alpha_\varnothing$ for each $j \in [n]$. Consequently, $b_1\boldsymbol{\beta}^{1\prime} + \ldots + b_n\boldsymbol{\beta}^{n\prime} + c_1\boldsymbol{\gamma}^{1\prime} + \ldots + c_n\boldsymbol{\gamma}^{n\prime} + d\boldsymbol{\delta}' + e\boldsymbol{\varepsilon}'$ is equal to

$$0 + \sum_{i\in\kappa} \left(\alpha_\varnothing - \alpha_{\{i\}}\right) + \sum_{i\in\kappa} \alpha_{\{i\}} - (|\kappa| - 1)\alpha_\varnothing + 0 + 0 = \alpha_\varnothing$$

in the $\varnothing$-component,

$$b_j + \sum_{i\in\kappa} \left(\alpha_\varnothing - \alpha_{\{i\}}\right) + \sum_{i\in\kappa} \alpha_{\{i\}} - (|\kappa| - 1)\alpha_\varnothing - c_j + 0 + 0 = \alpha_\varnothing + \\ + \alpha_{\{j\}} - \alpha_\varnothing = \alpha_{\{j\}} \text{ in the } \{j\}\text{-component},$$

$$0 + 0 + 0 + \alpha_{[n]} = \alpha_{[n]} \text{ in the } [n]\text{-component.}$$

In the next few lines we will show that that $\mathcal{B}'$ is a minimal set of generators of the cone $\mathcal{C}'_{\boldsymbol{S}}$, i. e. none of its vector can be expressed as a linear combination of the others.

Because $\boldsymbol{\varepsilon}'$ is orthogonal to the other vectors of the set $\mathcal{B}'$, $\boldsymbol{\varepsilon}'$ cannot be expressed as a linear combination of the others and also may not appear with positive coefficient in their expression.

Expression of $\boldsymbol{\beta}^{j\prime}$ using the other generators fails because all its components except one are zero, what would be broken by any positive multiple of another vector from $\mathcal{B}'$.

Since the $\varnothing$-component of $\boldsymbol{\gamma}^{j\prime}$ is nonzero, its expression would have to contain some of the vectors $\boldsymbol{\delta}'$ or $\boldsymbol{\gamma}^{i\prime}$ for $i \neq j$. But each of them violates zero in the $\{j\}$-component of $\boldsymbol{\gamma}^{j\prime}$.

Finally, when

$$\boldsymbol{\delta}' = b_1\boldsymbol{\beta}^{1\prime} + \ldots + b_n\boldsymbol{\beta}^{n\prime} + c_1\boldsymbol{\gamma}^{1\prime} + \ldots + c_n\boldsymbol{\gamma}^{n\prime} \tag{30}$$

for some non-negative coefficients $b_1, \ldots, b_n, c_1, \ldots, c_n$ then $n-1 = c_1 + \ldots + c_n$ in the $\varnothing$-components of vectors on both sides of (30). Thus, the sum of all singleton components of the combination on the right side of (30) is at least $(c_1 + \ldots + c_n)(n-1) = (n-1)^2$ according to (29). But the sum of all singleton components of $\boldsymbol{\delta}'$ is $n(n-2)$, which is less than $(n-1)^2$.

The basis of the corresponding cone $\mathcal{C}_{\boldsymbol{S}}$ is $\mathcal{B}_{\boldsymbol{S}} = \{\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^n, \boldsymbol{\gamma}^1, \ldots, \boldsymbol{\gamma}^n, \boldsymbol{\delta}, \boldsymbol{\varepsilon}\}$. For $\zeta \in \mathcal{P}([n]) \setminus (\Delta\boldsymbol{S} \cup \{[n]\})$, the ζ-components of its members are

$$\beta^i_\zeta = \sum_{j\in\zeta} \beta^i_{\{j\}} - (|\zeta| - 1)\,\beta^i_\varnothing = \begin{cases} 1 & \text{for } i \in \zeta \\ 0 & \text{otherwise} \end{cases},$$

$$\gamma^i_\zeta = \sum_{j\in\zeta} \gamma^i_{\{j\}} - (|\zeta| - 1)\,\gamma^i_\varnothing = \begin{cases} 0 & \text{for } i \in \zeta \\ 1 & \text{otherwise} \end{cases},$$

$$\delta_\zeta = \sum_{j\in\zeta} \delta^i_{\{j\}} - (|\zeta| - 1)\,\delta^i_\varnothing = n - |\zeta| - 1,\ \varepsilon_\zeta = 0$$

by (21). For the full definition of these vectors add the equations in (28).

We get the coefficients $\bar{\boldsymbol{\beta}}^1, \ldots, \bar{\boldsymbol{\beta}}^n, \bar{\gamma}^1, \ldots, \bar{\gamma}^n, \bar{\boldsymbol{\delta}}, \bar{\boldsymbol{\varepsilon}}$ from inequalities of type (16) and the inequalities themselves by multiplying the members of the base $\mathcal{B}_{\boldsymbol{S}}$ by the matrix

$\mathbf{G}^{[n]}$ on the right, which yields

$$\bar{\beta}^j_\zeta = \begin{cases} 1 & \text{for } \zeta = \{j\} \\ -1 & \text{for } \zeta = [n] \\ 0 & \text{otherwise} \end{cases}, \quad x_j - \tilde{C}(\boldsymbol{x}) \geq 0,$$

$$\bar{\gamma}^j_\zeta = \begin{cases} 1 & \text{for } \zeta = \varnothing \\ -1 & \text{for } \zeta = \{j\} \\ 0 & \text{otherwise} \end{cases}, \quad 1 - x_j \geq 0,$$

$$\bar{\delta}_\zeta = \begin{cases} n-1 & \text{for } \zeta = \varnothing \\ -1 & \text{for } |\zeta| = 1 \\ 1 & \text{for } \zeta = [n] \\ 0 & \text{otherwise} \end{cases}, \quad n - 1 - \sum_{i=1}^{n} x_i + \tilde{C}(\boldsymbol{x}) \geq 0,$$

$$\bar{\varepsilon}_\zeta = \begin{cases} 1 & \text{for } \zeta = [n] \\ 0 & \text{otherwise} \end{cases}, \quad \tilde{C}(\boldsymbol{x}) \geq 0.$$

So, $\mathcal{B}_L = \{\boldsymbol{\delta}, \boldsymbol{\varepsilon}\}$ with $b_L(\boldsymbol{x}) = \max\{-n+1+\sum_{i=1}^n x_i, 0\}$ and $\mathcal{B}_U = \{\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^n\}$ with $b_U(\boldsymbol{x}) = \min\{x_1, x_2, \ldots, x_n\}$. □

Let $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ be subsets of $\mathcal{P}([n])$ such that $\Delta\boldsymbol{S}_1 \subseteq \Delta\boldsymbol{S}_2$. Then the basic bounds of $\mathcal{F}_n(\{C_v;\, v \in \boldsymbol{S}_2\})$ are narrower than or equal to the basic bounds of $\mathcal{F}_n(\{C_v;\, v \in \boldsymbol{S}_1\})$ as the system (15) for $\boldsymbol{S} = \boldsymbol{S}_1$ contains all equations of the system for $\boldsymbol{S} = \boldsymbol{S}_2$. So, the Fréchet–Hoeffding bounds are the widest in this hierarchy.

4. CONCLUSION

In this article, we develop understanding of algebraic bounds of general Fréchet classes, initiated by paper [7]. The basic bounds, narrowing the Fréchet–Hoeffding bounds of a Fréchet class, can be starting point for constructions of more accurate boundaries using analytical methods given by [4] and mentioned, e. g., in [7].

Several issues still arises from this text. It is questionable whether the fourth condition of Theorem 2.4 can be concluded from the previous three formulated in (20).

We have demonstrated a use of the tool from Theorem 3.4 for an explicit expression of the basic bounds of simple Fréchet classes over $\varnothing$. It would be interesting to test the efficiency of the method for classes over more complex sets $\{C_v;\, v \in \boldsymbol{S}\}$.

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