KYBERNETIKA — VOLUME *50* (2014), NUMBER 1, PAGES 5–18

OPTIMAL CONTROL PROCESSES ASSOCIATED WITH A CLASS OF DISCONTINUOUS CONTROL SYSTEMS: APPLICATIONS TO SLIDING MODE DYNAMICS

Vadim Azhmyakov, Michael V. Basin and Arturo E. Gil García

This paper presents a theoretical approach to optimal control problems (OCPs) governed by a class of control systems with discontinuous right-hand sides. A possible application of the framework developed in this paper is constituted by the conventional sliding mode dynamic processes. The general theory of constrained OCPs is used as an analytic background for designing numerically tractable schemes and computational methods for their solutions. The proposed analytic method guarantees consistency of the resulting approximations related to the original infinite-dimensional optimization problem and leads to specific implementable algorithms.

Keywords: sliding mode, nonlinear systems, absolute continuous approximations

Classification: 93E12, 62A10

1. INTRODUCTION AND MOTIVATION

The study of discontinuous and sliding mode control processes in the framework of the general switched dynamic systems has gained interest in recent years (see e.g., [5, 6, 14, 15, 17, 19, 21, 23, 27, 35, 36, 38]). The complex real-world models described by the differential equations with discontinuous right-hand sides or the corresponding differential inclusions are broadly used in modern engineering. An important case of the above-mentioned discontinuous processes is constituted by a family of the conventional sliding mode control systems. Recall that these control systems are motivated by many significant real world applications of the systems theory. We refer to [3, 12, 14, 15, 25, 34, 36, 42, 45] for engineering applications of the sliding mode approach. It is necessary to stress that the control design technique based on the traditional sliding mode technologies (see [45]) is nowadays a mature and relative simple methodology for synthesis of robust controllers.

In this paper, we study the following system with the affine structure

$$\begin{aligned} \dot{x}(t) &= a\big(t, x(t)\big) + b\big(t, x(t)\big)u(t) \text{ a.e. on } [0, t_f], \\ x(0) &= x_0, \end{aligned} \tag{1}$$

DOI: 10.14736/kyb-2014-1-0005

where $x_0 \in \mathbb{R}^n$ and functions $a : (0, t_f) \times \mathcal{R} \to \mathbb{R}^n$, $b : (0, t_f) \times \mathcal{R} \to \mathbb{R}^{n \times m}$ are Caratheodory functions (see [24, 29, 38] for theoretical details), i. e., measurable in t and continuous in x. The general existence/uniqueness theory for nonlinear ordinary differential equations implies that for every $u(\cdot) \in \mathbb{L}^1_m(0, t_f)$, where $\mathbb{L}^1_m(0, t_f)$ is the Lebesgue space of all m-valued integrable functions, the problem (1) has a unique absolutely continuous solution $x(\cdot)$. We consider (1) over a set $\mathcal{U}$ of bounded measurable control functions and assume that this set has a simple box-like structure

$$\begin{aligned} \mathcal{U} &:= \{v(\cdot) \in \mathbb{L}^1_m(0, t_f) \bigm| v(t) \in U \text{ a.e. on } [0, t_f]\}, \\ U &:= \{u \in \mathbb{R}^m \ : \ v^j_- \leq u_j \leq v^j_+, \ j = 1, \ldots, m\}, \end{aligned} \tag{2}$$

where $v^j_-, v^j_+, j = 1, \ldots, m$ are certain constants. It is well known that the affine dynamic models represent an important class of control systems see, for instance, ([31, 45]). The sliding mode motion of the dynamic system (1) is determined by the following specific feedback control:

$$w(t, x) := \tilde{w}(t, \sigma(t, x)), \tag{3}$$

where $\tilde{w} : \mathbb{R}_+ \times \mathbb{R}^s \to \mathbb{R}^m$ is a bounded measurable (feedback) control function and $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^s$ is a continuously differentiable s-valued function. We also introduce the additional notation for the composite function $u_w(t) \equiv w(t, x(t))$, where $x(\cdot)$ is a trajectory of (1). Since $w(\cdot, \cdot)$ is a bounded function and $x(\cdot)$ an absolutely continuous one, we have $u_w(\cdot) \in \mathcal{U}$ for the box-like set U from (2).

Note that the more general (high order) sliding mode control approach to systems with a relative degree $l \in \mathbb{N}$ is related to a family of functions $\sigma(\cdot, \cdot)$ and a specific form of the feedback control law. In this case the traditional sliding mode notation is represented by $(l-1)$ derivatives of $\sigma(\cdot, \cdot)$. A conventional sliding mode dynamic process (1) is associated with a sliding manifold, namely, a smooth surface $\sigma(t, x) = 0$, which defines the main strategy of the corresponding feedback control in the following sense: a trajectory $x(\cdot)$ generated by (1) with an implemented control function (3) guarantees the prescribed "sliding condition" $\sigma(t, x(t)) = 0$ for the implemented trajectory $x(\cdot)$.

In our contribution, we interpret the closed-loop sliding mode system (2) – (3) as a switched system. We refer to [4, 6, 7, 8, 9, 18, 35, 44] for some theoretical issues and valuable examples of the switched systems. Note that the preceding dynamic systems are often considered as control systems with variable structures, whose continuous and discrete dynamics interact. In view of many formal models of variable structure systems, the above interpretation is strongly motivated. Indeed, the sliding mode control systems (considered as a subclass of the switched systems) possess a specific mechanism of discrete transitions [7]. This prescribed mechanism is specified by a discontinuous feedback (3). In this paper, we propose a theoretical approach to optimal control of the generalized closed-loop system (2) – (3). This analytical method is complemented with an introduced computational technique. Evidently, the ability to operate a sliding mode process in an optimal manner remains a challenging task as the computational complexity associated with such problems often proves to be a bottleneck [7]. Some constructive implementable algorithms for numerical treatment of switched/hybrid systems are discussed in [5, 16, 44].

This paper is an extended version of the previous conference presentation [11]. The paper gives complete proofs of all theorems and lemmas and a simulation example demonstrating application of the developed approximation technique to an optimal control problem for a nonlinear system, whose optimal trajectory belongs to a certain sliding mode manifold.

The remainder of this paper is organized as follows: Section 2 presents some basic mathematical concepts and preliminary theoretical facts. We also give a formulation of the sliding mode-related OCP under consideration. Section 3 proposes a constructive approximative scheme for the original OCPs governed by the discontinuous sliding mode-type dynamic system (2) – (3). We also briefly discuss some numerical aspects of the obtained analytical results that can potentially lead to a computational solution procedure for the optimal control problem. Section 4 summarizes this paper.

2. OPTIMIZATION OF SLIDING MODE CONTROL PROCESSES

The closed-loop realization of the affine control system (1) with (3) can be represented as a system of ordinary differential equations with discontinuous right-hand sides. The discontinuity effect is caused by an assumed class of the feedback control functions. A suitable control synthesis of the type (3) implies certain stability properties of the generated sliding modes. Usually, this stability requirement is determined by the classic asymptotic Lyapunov-type [33] or newly developed finite-time stability properties [28]. Note that the stability here is understood with respect to a prescribed sliding surface of co-dimension $s \in \mathbb{N}$ (see [24]). Our approach to (1) – (3) is based on the following idea: we consider the corresponding closed-loop system governed by a general differential equation with a discontinuous right hand side. This modeling framework makes it possible to apply the celebrated Filippov approach (see [14, 24, 36]) and consider the associated differential inclusion of the following form

$$\begin{aligned} &\dot{\tilde{x}}(t) \in \mathcal{K}[a,b](\tilde{x}(t)) \text{ a.e. on } [0,t_f],\\ &\tilde{x}(0) = (x_0, 0), \end{aligned} \tag{4}$$

where $\tilde{x} := (x^T, t)^T$ is the extended state vector,

$$\mathcal{K}[a,b](\tilde{x}) := \overline{\text{co}}\Big\{ \lim_{j\to\infty} [a(\tilde{x}_j) + b(\tilde{x}_j)w(\tilde{x}_j)] \mid \tilde{x}_j \to \tilde{x},\ \tilde{x}_j \notin \mathcal{S} \bigcup \mathcal{W} \Big\}$$

and $\mathcal{S}, \mathcal{W} \subset \mathbb{R}^n$ are sets of zero measure. We are interested to develop an adequate numerically stable approximation for the differential inclusion (4).

Let us introduce the OCP associated with (1) – (3). Consider a smooth function

$$f^0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}.$$

Given (1), we introduce the OCP as follows:

$$\begin{aligned} &\text{minimize } \tilde{J}(u_w(\cdot)) := \int_0^{t_f} f^0(t, x(t), u_w(t))\,\mathrm{d}t\\ &\text{over all trajectories of } (4) - (3),\\ &\sigma(t, x(t)) = 0\ \forall t \in [t_{sl}, t_f], \end{aligned} \tag{5}$$

where t_{sl} is the time moment of reaching the exact "sliding mode". We now make the conventional technical assumptions, namely, suppose that problem (5) has an optimal solution (control) $u_w^{opt}(\cdot) = w^{opt}(t, x^{opt}(t))$, where $x^{opt}(\cdot)$ is the associated optimal trajectory of (1) generated by (3). From the point of view of the general optimization theory, the sliding condition $\sigma(t,x) = 0$ represents an additional algebraical state constraint for the main OCP (5). Solving a constrained optimal control problem with ordinary differential equations, we deal with functions and systems, which are to be replaced, except for very special cases, by numerically tractable approximations. Implementation of these numerical schemes for control problems with discontinuous dynamics is based on certain approximations of the control sets and finite difference numerical schemes. Some suitable finite-dimensional approximations can be used to design an effective solution procedure for OCPs (5). It is well-known that the celebrated Filippov Selection Lemma (see e. g., [1, 30]) gives rise to an explicit parametrization of convex-valued differential inclusion of the type (7). A modern abstract formulation of this result (in the form of Implicit Function Theorem) is given in [30]. We present here a special case of the general Filippov Selection Lemma (see [26, 37, 38] for more theoretical details).

Proposition 1. A function $z(\cdot)$ is a solution of (4) if and only if it is a solution of the Gamkrelidze system

$$\begin{aligned} &\dot{\eta}(t) = a(t,\eta(t)) + \sum_{j=1}^{n+1} \alpha^j(t) b(t,\eta(t)) u^j(t), \\ &\eta(0) = x_0,\ \alpha(\cdot) \in \Lambda(n+1), \end{aligned} \tag{6}$$

where

$$\alpha(\cdot) := \big(\alpha^1(\cdot), \dots \alpha^{n+1}(\cdot)\big)^T, \quad u^j(\cdot) \in \mathcal{U}\ \forall j = 1, \dots, n+1$$

and

$$\begin{aligned} &\Lambda(n+1) := \{\alpha(\cdot) \mid \alpha^j(\cdot) \in \mathbb{L}_1^1(0,t_f), \\ &\alpha^j(t) \geq 0, \sum_{j=1}^{n+1} \alpha^j(t) = 1, \quad \forall t \in [0,t_f]\}. \end{aligned}$$

Recall that the auxiliary system (6) from the above theorem is called the Gamkrelidze system (see [26]). Evidently, the generalized control system (6) also represents the sliding mode-type dynamics of the main model (1) – (2) with discontinuous control (3). Evidently, the closed-loop realization (1) – (3) can be interpreted as a special case of (4) with a discontinuous right-hand side. Using Proposition 1, we can consider the above-mentioned equivalent Gamkrelidze parametrization (6) for the differential inclusion associated with the original closed-loop system (1) – (3). Moreover, we can also consider the Gamkrelidze-relaxed version of the original OCP (5). This relaxation of the basic OCP (5) occurs if we replace the original affine control system by the corresponding Gamkrelidze system from Proposition 1.

We now use the approximation idea discussed in [10] and consider the so-called "quasi Gamkrelidze" system associated with (6). The discrete approximations of the given control set U can be introduced as follows:

$$\omega^k \in U, \; k = 1, \ldots, M,$$

where $M/(n+1) \in \mathbb{N}$. Assume that for $\epsilon_M > 0$ (accuracy) and for any $\omega \in U$ there exists a point $\omega^k \in U_M$ such that

$$||\omega - \omega^k||_{\mathbb{R}^m} < \epsilon_M.$$

The following auxiliary dynamical system is given by

$$\begin{aligned} \dot{z}(t) &= a(t, z(t)) + \sum_{k=1}^{M} \beta^k(t) b(t, z(t)) \omega^k \text{ a.e. on } [0, t_f], \\ z(0) &= x_0, \end{aligned} \tag{7}$$

where $\beta^k(\cdot)$ are elements of $\mathbb{L}^1(0, t_f)$, $\beta^k(t) \geq 0$, and

$$\sum_{k=1}^{M} \beta^k(t) = 1 \; \forall t \in [0, 1].$$

Define

$$\beta_M(\cdot) := (\beta^1(\cdot), \ldots, \beta^M(\cdot))$$

and let $\aleph(M)$ denote the set of such admissible functions $\beta_M(\cdot)$ (generalized controls). Note that for a fixed U_M the corresponding quasi Gamkrelidze system (7) has an absolutely continuous solution $z^\beta_M(\cdot)$ for any admissible generalized control $\beta_M(\cdot) \in \aleph(M)$. The quasi Gamkrelidze system (7) allows further approximations summarized in the next result [10].

Theorem 1. Consider the original system (1) – (3) and a solution $\eta(\cdot))$ to the associated Gamkrelidze system. Then, there exists a sequence $\{\beta_M(\cdot)\} \subset \aleph(M)$ of the generalized controls and the corresponding sequence $\{z^\beta_M(\cdot)\}$ of solutions of the given quasi Gankrelidze systems, such that $z^\beta_M(\cdot)$ is a strong approximation of $\eta(\cdot)$

$$\begin{aligned} \lim_{M \to \infty} \|z^\beta_M(\cdot) - \eta(\cdot)\|_{\mathbb{C}_n(0,t_f)} &= 0, \\ \lim_{M \to \infty} \|z^\beta_M(\cdot) - \eta(\cdot)\|_{\mathbb{W}^{1,1}_n(0,t_f)} &= 0. \end{aligned}$$

Here $|| \cdot ||_{\mathbb{W}^{1,1}_n(0,t_f)}$ is the norm in the standard Sobolev space $\mathbb{W}^{1,1}_n(0, t_f)$.

The presented theorem establishes the approximability property of solutions to the classic Gamkrelidze system from Proposition 1. This result can also be interpreted as a specific approximation of the conventional control $u(\cdot)$ by a sequence of admissible generalized controls $\beta(\cdot)$. This last approximability property is understood with respect to the M-convergence of the generated trajectories. As shown in [10], the solution $z^\beta_M(\cdot)$ also satisfies the following specific differential inclusion

$$\dot{z}^\beta_M(t) \in a(t, z^\beta_M(t)) + b(t, z^\beta_M(t)) \times U \text{ a.e. on } [0, t_f]. \tag{8}$$

Evidently, (8) defines a new differential inclusion that approximates the original differential inclusion (4) with the accuracy specified in Theorem 1). We also refer to [10] for a formal proof of this fact.

3. CONSISTENT APPROXIMATIONS OF OCPS ASSOCIATED WITH SLIDING MODE DYNAMICS

The previous section contains some general results that are related to constructive approximations of a large class of discontinuous dynamic systems and sliding mode-type discontinuous control processes (1) – (3). Our objective is to use these strong approximability concepts, namely, the auxiliary system (6), and study the corresponding approximations of the main OCP (5). Using the quasi-Gamkrelidze system (7), we can introduce the following approximation of the main OCP (5)

$$\begin{aligned} &\text{minimize } J(\beta_M(\cdot)) := \int_0^{t_f} f^0(t, z(t))\,\mathrm{d}t \\ &\text{over all trajectories of (7)},\ \beta_M(\cdot) \in \aleph(M) \\ &-\epsilon_s \leq \sigma(t, z(t)) \leq \epsilon_s \ \forall t \in [\tilde{t}_{sl}, t_f], \end{aligned} \tag{9}$$

where ϵ_s is a s-dimensional vector of the prescribed "sliding accuracies" for the corresponding components of $\sigma(\cdot,\cdot)$. Evidently, the parameter ϵ_s characterizes a quasi-sliding mode dynamic process. In general, the trajectory of the approximating system (7) does not possess a "sliding" property with respect to the prescribed sliding manifold from (5). Let us assume that the set of admissible solutions in (9) is non-empty. We are now able to formulate the main result yielding an effective approximation of the original OCP (5).

Theorem 2. Let the original system (1) – (3) satisfy all assumptions of Section 1. For any $\epsilon_s > 0$, there is a number $M_\epsilon \in \mathbb{N}$ such that for any $M \in \mathbb{N}$, $M \geq M_\epsilon$, there exists an optimal solution $(\beta_M^{opt}(\cdot), z_M^{opt}(\cdot))$ associated with the approximating OCP (9). Moreover, this solution possesses the following consistency property:

$$|\tilde{J}(\nu^{opt}(\cdot)) - J(\beta_M^{opt}(\cdot))| \leq \epsilon_s, \tag{10}$$

where

$$\nu^{opt}(\cdot)) \in \mathcal{U}^{(n+1)} \times \Lambda(n+1)$$

is a solution of the corresponding Gamkrelidze-relaxed OCP.

Proof. Since $\nu^{opt}(\cdot))$ is an admissible control to the Gamkrelidze-relaxed OCP, Theorem 1 guarantees that there is a sequence $\{\beta_M(\cdot)\}$ of generalized controls in (7) such that

$$\lim_{M\to\infty} ||z_M^\beta(\cdot) - \eta^{opt}(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0,$$

where $\eta^{opt}(\cdot)$ is an optimal trajectory of the Gamkrelidze system associated with $\nu^{opt}(\cdot))$. Let t_{sl} be a exact "sliding time" defined from the original sliding condition

$$\sigma(t, x(t)) = 0.$$

The continuity of the function $\sigma(\cdot,\cdot)$ and the absolute continuity of the trajectory of a Gamkrelidze system imply existence of numbers $\epsilon_s^1 > 0$, $M_1 \in \mathbb{N}$ such that for a "quasi sliding time" $\tilde{t}_{sl}$ and an associated sequence of trajectories $\{z_M^{\beta}(\cdot)\}$ the approximate sliding condition

$$-\epsilon_s^1 \leq \sigma(t, z(t)) \leq \epsilon_s^1, \ \forall t \in [\tilde{t}_{sl}, t_f],$$

is satisfied. This last condition holds for all $M \geq M_1$. The affine structure of the original differential inclusion (4) and the approximating quasi Gamkrelidze system (7) implies the convexity of the extended velocity vector (orientor) field. Moreover, the control set $\aleph(M)$ is compact [10]. From [40], we deduce that the solutions of (7) are uniformly bounded. By definition, the sliding manifold is a closed set. Since the solution set associated with (9) is non-empty, the existence of an optimal solution to this auxiliary OCP follows from the extended Filippov existence result [10] for optimal control problems with constraints (refer to [22, 41]).

Under the basic assumptions, the functions $\tilde{J}(\cdot)$ and $\sigma(\cdot,\cdot)$ are continuous. Then, for any $\epsilon_s^2 > 0$, there exists a number $M_2 \in \mathbb{N}$ such that

$$|\tilde{J}(\nu^{opt}(\cdot)) - J(\beta_M^{opt}(\cdot))| \leq \epsilon_s^2$$

for all $M > M_2$ (see [20]). The trajectory $z_M^{opt}(\cdot)$ is a solution to the differential inclusion (7) and also satisfies the relaxed dynamic system

$$\dot{\eta}(t) = \int_U \big[a(t,\eta(t)) + b(t,\eta(t))u\big]\mu(t)(\mathrm{d}u), \tag{11}$$

where $\eta(0) = x_0$, $t \in [0, t_f]$, and $\mu(\cdot)$ is so-called relaxed control. An (11) admissible relaxed control (see [13, 22]) in that case can be determined as

$$\mu := \sum_{k=1}^{M} \beta_M^{k,opt}(t)\delta_{\nu_k},$$

where $\beta_M^{k,opt}(t)$ are components of $\beta_M^{opt}(t)$ and δ_{ν_k}, $k = 1, \ldots, M$, are Dirac delta-functions. Based on [22], we deduce that $z_M^{opt}(\cdot)$ is also a solution to the original Gamkrelidze system (6) with the corresponding admissible $\nu(\cdot)$ from $\mathcal{U}^{(n+1)} \times \Lambda(n+1)$.

Upon defining

$$\epsilon_s := \max(\epsilon_s^1, \epsilon_s^2), \ M_\epsilon := \max(M_1, M_2),$$

the preceding inequalities yield

$$-\epsilon_s \leq \sigma(t, z_M(t)) \leq \epsilon_s, \ \forall t \in [\tilde{t}_{sl}, t_f], \text{ and } |\tilde{J}(\nu^{opt}(\cdot)) - J(\beta_M^{opt}(\cdot))| \leq \epsilon_s$$

for a given ϵ_s. The proof is completed. □

Theorem 2 establishes the existence of a minimizing sequence (see e. g., [32]) for the Gamkrelidze-relaxed OCP (9). Recall that a sequence $\{v_s(\cdot)\}$ from the set of admissible control functions is called a minimizing sequence if

$$\lim_{s\to\infty} J(v_s(\cdot)) = \min J(\cdot).$$

The minimizing sequence from Theorem 2 provides a theoretical basis for possible numerically consistent approximations of the relaxed OCP (9). On the other hand, we are mostly interested in consistent approximations associated with the original (non-relaxed) OCP (5). This fact motivates the following partial result.

Theorem 3. Let the original system (1) – (3) satisfy all the assumptions of Section 1. Consider a control

$$\nu(\cdot) \in \mathcal{U}^{(n+1)} \times \Lambda(n+1)$$

and the corresponding solution $\eta(\cdot)$ of the Gamkrelidze system (6). Then, there exists a piecewise constant control function $v(\cdot) \in \mathcal{U}$ such that the solution $x^v(\cdot)$ of the original system (1) exists on a given time interval $[0, t_f]$ and

$$\lim_{M \to \infty} ||x^v(\cdot) - \eta(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0.$$

P r o o f. As mentioned, $\eta(\cdot)$ is a solution of the relaxed system (11) with a generalized control

$$\mu := \sum_{k=1}^{M} \beta_M^{k,opt}(t)\delta_{\nu_k}$$

(see the proof of Theorem 2). The affine structure of the original control system (1) and the convexity of the set U imply convexity of the right-hand side of the differential inclusion (7). The existence of a piecewise constant control function $v(\cdot) \in \mathcal{U}$ such that

$$\lim_{M \to \infty} ||x^v(\cdot) - \eta(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0.$$

now follows from the results of [22]. The proof is completed. □

Theorem 3 establishes existence of a conventional (non-relaxed) control function that possesses a strong approximability property for trajectories of the original and corresponding Gamkrelidze control systems. Assuming existence of an optimal solution $(\beta_M^{opt}(\cdot), z_M^{opt}(\cdot))$ to the auxiliary OCP (9) implies, in view of Theorem 1 and 3, the following approximation

$$\lim_{M \to \infty} ||x^v(\cdot) - \eta^{opt}(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0. \tag{12}$$

Summing up, one can say that for any $\epsilon_s > 0$ and the associated M_ϵ there always exists a piecewise control function $v(\cdot) \in \mathcal{U}$ from Theorem 3 such that the trajectory $x^v(\cdot)$ of (1) generated by $v(\cdot)$ approximates an optimal trajectory η^{opt} of the Gamkrelidze-relaxed OCP. This qualitative observation can be formalized and extended as follows.

Theorem 4. Let the original system (1) – (3) satisfy all the assumptions of Section 1. Consider a given accuracy $\epsilon_s > 0$ and the corresponding number $M_\epsilon \in \mathbb{N}$ that guarantee existence of an optimal solution $(\beta_M^{opt}(\cdot), z_M^{opt}(\cdot))$ for every (9) with $M \geq M_\epsilon$, which possesses the consistency property (10) from Theorem 2. Then, there exists a piecewise constant control function $v(\cdot) \in \mathcal{U}$ such that the solution $x^v(\cdot)$ of the original system (1) exists on a given time interval $[0, t_f]$ and

$$|\tilde{J}(v(\cdot)) - J(\beta_M^{opt}(\cdot))| \leq \epsilon_s. \tag{13}$$

Proof. Theorem 2 yields the existence of a number $\epsilon_s/2$ such that

$$|\tilde{J}(\nu^{opt}(\cdot)) - J(\beta_M^{opt}(\cdot))| \leq \epsilon_s/2,$$

where $\nu^{opt}(\cdot)) \in \mathcal{U}^{(n+1)} \times \Lambda(n+1)$ is an optimal solution of the corresponding Gamkrelidze-relaxed OCP. Using the continuity argument for the objective function $\tilde{J}(\cdot)$ and the result of Theorem 3, we obtain

$$|\tilde{J}(v(\cdot)) - \tilde{J}(\nu^{opt}(\cdot))| \leq \epsilon_s/2.$$

Finally, we have

$$\begin{aligned}
&|\tilde{J}(v(\cdot)) - J(\beta_M^{opt}(\cdot))| \\
&\leq |\tilde{J}(\nu^{opt}(\cdot)) - J(\beta_M^{opt}(\cdot))| + |\tilde{J}(v(\cdot)) - \tilde{J}(\nu^{opt}(\cdot))| \leq \epsilon_s.
\end{aligned}$$

The proof is completed. □

We now make some remarks related to the obtained result and a possible numerical interpretation of Theorem 4. The structure of the original sliding mode-type and auxiliary quasi Gamkrelidze dynamic systems makes it possible to apply various numerical algorithms to the OCPs (1) and (9). Theorem 4 guarantees existence of an approximating optimal control strategy from a simple class of admissible piecewise constant control strategies $v(\cdot)$. Those simple controls are of the open-loop type. Specific design of such a piecewise-constant control function $v(\cdot)$ can be realized using the Tikhomirov approach developed in [43] for a general class of OCPs. A Tikhomirov control sequence with a finite number of switchings is in fact a possible realization of the control strategy $v(\cdot)$ from Theorem 4. The conventional sliding mode-motivated strategy for variable structure systems also has a piecewise structure, which is represented by the single- or multi-valued *sign* functions. Apparently, a piecewise-constant control strategy $v(\cdot)$ generalizes this traditional sliding mode *sign*-based synthesis. The reconstruction of a suitable feedback-type function $w(\cdot, \cdot)$ can be obtained from the relation $u_w(t) \equiv v(t)$ for every $t \in [0, t_f]$. Therefore, the resulting feedback control $w(\cdot, \cdot)$ has a piecewise structure as well. This designed feedback control strategy does not necessarily satisfy the ideal sliding condition $\sigma(t, x(t)) = 0$. Otherwise, the trajectory of the closed-loop sliding mode system (1) corresponding to the control $w(\cdot, \cdot)$ possesses the quasi sliding properties in the sense of the inequality

$$-\epsilon_s \leq \sigma(t, z(t)) \leq \epsilon_s \ \forall t \in [\tilde{t}_{sl}, t_f],$$

pertinent to the auxiliary OCP (9).

4. NUMERICAL ASPECTS

Let us now illustrate the developed approximation-based numerical approach to OCPs (5) with an example. Consider a nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1^2 + x_2 + x_1 u_1 \\ \dot{x}_2 &= x_1^2 + x_2 u_2\end{aligned} \tag{14}$$

and the associated OCP (5) with $U = [-1,1]\times[-1,1]$, $t_f = 1$, and $f^0(t,x,u) = \frac{1}{2}\|\sigma(x)\|^2$, $\sigma(x) = x_1 + x_2$. The initial conditions for (14) satisfy the relation $x_1(0) + x_2(0) = 0$ and are assumed to be nontrivial.

The corresponding Hamiltonian system has the following form:

$$\begin{aligned}\dot{p}_1 &= -(-2p_1x_1 + p_1u_1 + 2p_2x_1) + (x_1 + x_2), \\ \dot{p}_2 &= -(p_1 + p_2u_2) + (x_1 + x_2),\end{aligned} \tag{15}$$

where $H(u,x,p) = p_1(-x_1^2 + x_2 + x_1u_1) + p_2(x_1^2 + x_2u_2) - \frac{1}{2}(x_1 + x_2)^2$ is the Hamiltonian of the problem. Using the phase restriction $\sigma(x(t)) = 0$ in (5) and the non-triviality condition for $x_1(0)$, $x_2(0)$ in (14) implies $u_1^{opt}(t) - u_2^{opt}(t) = 1$. The obtained condition makes it possible to calculate the optimal feedback-type control for the OCP under consideration. We use here the notation associated with OCP (5) (see Section 2).

Note that $\dot{x}_1^{opt}(t) + \dot{x}_2^{opt}(t) = 0$ (as shown in Figure 1). Therefore, we deal here with an optimal sliding mode motion.

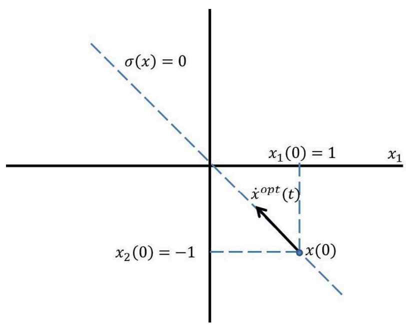

Fig. 1. Optimal sliding mode motion.

Our objective now is to represent the same optimal sliding type dynamics using the approximating technique discussed in Section 3. In this example, we select $x_1(0) = 1$, $x_2(0) = -1$. Based on the restrictions for the control function, we assign $U_M = U_M^{x_1} \times U_M^{x_2}$ with $U_M^{x_1} = [-1;1]_M$ and $U_M^{x_2} = [-1;1]$. Thus, the auxiliary dynamic system (7) is given as follows:

$$\begin{aligned}\dot{z}_1(t) &= -z_1^2(t) + z_2(t) + \sum_{k=1}^{M} \beta_1^k(t) z_1(t) \omega_k^1 \\ \dot{z}_2(t) &= z_1^2(t) + \sum_{k=1}^{M} \beta_2^k(t) z_2(t) \omega_k^2.\end{aligned} \tag{16}$$

Here, $\omega_k^1 \in U_M^{x_1} = [-1;1]_M$ and $\omega_k^2 \in U_M^{x_2} = [-1;1]_M$. We choose $M = 100$ to ensure that $|\omega - \omega_k| \leq 10^{-2}$ for $\omega_1, \omega_2 \in [-1;-1]$. Note that the corresponding cost function takes the form $J(\beta_M(\cdot)) = \int_0^2 (z_1(t) + z_2(t))^2 \, \mathrm{d}t$ and, moreover, the exact sliding mode condition $\sigma(x) = 0$ is replaced by the approximating conditions (inequalities) in (9). The new control vector in (16) is $(\beta_1(\cdot), \beta_2(\cdot))^T$, where $\sum_{k=1}^M \beta_j^k(t) = 1$ for $j = 1, 2$, and $\beta_j^k(t) \geq 0$ for $j = 1, 2$ and all $k = 1, \ldots, M$. We now solve the OCP (9) with $\beta_1^k = \{0.5 + 1/n, 0, \ldots, 0.5 - 1/n\}$ and $\beta_2^k = \{1 - 1/n, 0, \ldots, 1/n\}$, where $n \in \mathbb{N}$, and obtain the following results for different values of n (as shown in Figure 2).

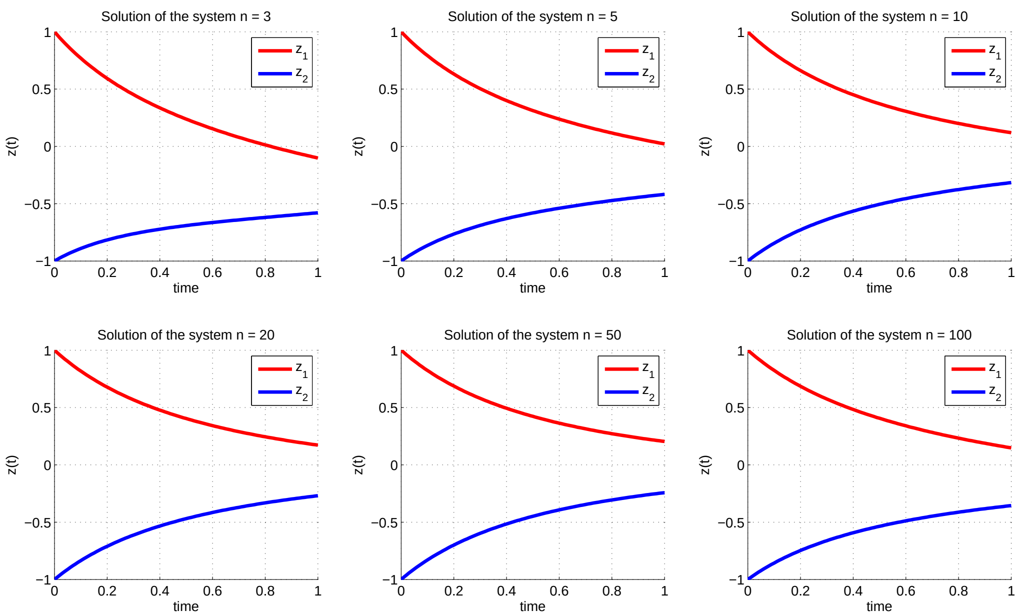

Fig. 2. Solutions of the equations (16) for $n = 3, 5, 10, 20, 50, 100$.

Evidently, the obtained simulation results demonstrate reliable convergence of the designed approximations to the optimal sliding mode trajectories on the manifold $\sigma(x) = x_1 + x_2 = 0$.

5. CONCLUDING REMARKS

In this paper, we have developed a new approach to solving OCPs corresponding to control systems with sliding modes, which generalizes specific approximation techniques from [10]. The proposed approach possesses general theoretic nature and can be applied to a large class of dynamic systems with discontinuous right-hand sides. For affine systems with sliding mode-type feedback control, we have obtained some specific approximating differential inclusions. These new mathematical objects constitute an analytic extension of the conventional Filippov-like differential inclusions associated with classic control systems. The consistency of the proposed theoretical approach is also established.

The methodology discussed in this paper can be applied not only to the classical sliding mode dynamics but also to high-order realizations of the sliding mode-type optimal control problems. In that case, the generic OCP needs to be extended by additional state/derivatives constraints related to an admissible feedback control law. These restrictions are usually given in the following form

$$w(t,x) := \tilde{w}(\sigma(t,x), \dot{\sigma}(t,x), \dots, \sigma^{(l-1)}(t,x)).$$

Here, $l \in N$ is a relative degree of the system under consideration. The resulting high-order OCP constitutes an optimization problem, which satisfies not only the conventional sliding mode constraint $\sigma(t,x) = 0$ but also a set of additional constrains

$$\dot{\sigma}(t,x) = 0, \dots \sigma^{(l-1)}(t,x) = 0.$$

The results obtained in our contribution need to be extended to effective numerically tractable approximation schemes and computational algorithms. Finally, note that the mathematical tools and analytic techniques used in this paper lead to a new methodology for solving OCPs corresponding to switched and hybrid systems.

ACKNOWLEDGEMENT

This work was partially supported by the Mexican National Science and Technology Council (CONACyT) under Grant no. 129081.

(Received May 20, 2013)

REFERENCES

[1] C. D. Aliprantis and K. C. Border: Infinite Dimensional Analysis. Springer, Berlin 1999.

[2] K. Atkinson and W. Han: Theoretical Numerical Analysis. Springer, New York 2005.

[3] D. Gómez-Gutiérrez, S. Čelikovský, A. Ramírez-Trevino, J. Ruiz-León, and S. Di Gennaro: Robust regulation via sliding modes of a rotary inverted pendulum. In: Preprints 3rd IFAC Symposium on Robust Control Design, ÚTIA AV ČR, Praha 2000.

[4] S. A. Attia, V. Azhmyakov, and J. Raisch: On an optimization problem for a class of impulsive hybrid systems. In: Discrete Event Dynamical Systems, 2009.

[5] V. Azhmyakov and J. Raisch: Convex control systems and convex optimal control problems with constraints. IEEE Trans. Automat. Control *53* (2008), 993–998.

[6] V. Azhmyakov, V. G. Boltyanski, and A. Poznyak: Optimal control of impulsive hybrid systems. Nonlinear Anal.: Hybrid Systems *2* (2008), 1089–1097.

[7] V. Azhmyakov, R. Galvan-Guerra, and M. Egerstedt: Hybrid LQ-optimization using dynamic programming. In: Proc. 2009 American Control Conference, St. Louis 2009, pp. 3617–3623.

[8] V. Azhmyakov, M. Egerstedt, L. Fridman, and A. Poznyak: Continuity properties of nonlinear affine control systems: applications to hybrid and sliding mode dynamics. In: Proc. 2009 IFAC Conference on Analysis and Design of Hybrid Systems, Zaragoza 2009, pp. 204–209.

[9] V. Azhmyakov, V. G. Boltyanski, and A. Poznyak. The dynamic programming approach to multi-model robust optimization. Nonlinear Anal.: Theory, Methods Applications *72* (2010), 1110–1119.

[10] V. Azhmyakov: Optimal control of sliding mode processes: A general approach. In: Proc. 11th International Workshop on Variable Structure Systems, Mexico City 2010, pp. 504–509.

[11] V. Azhmyakov M. Basin, and A. E. Gil García: A general approach to optimal control processes associated with a class of discontinuous control systems: Applications to the sliding mode dynamics. In: Proc. 2012 IEEE International Conference on Control Applications, Dubrovnik 2012, pp. 1154–1159.

[12] G. Bartolini, L. Fridman, A. Pisano, and E. Usai (eds.): Modern Sliding Mode Control Theory. Lecture Notes in Control and Inform. Sci. *375*, Springer, Berlin 2008.

[13] L. D. Berkovitz: Optimal Control Theory. Springer, New York 1974.

[14] I. Boiko: Discontinuous Control Systems Frequency-Domain Analysis and Design. Birkhauser, New York 2009.

[15] I. Boiko, L. Fridman, A. Pisano, and E. Usai: On the transfer properties of the "generalized sub-optimal" second-order sliding mode control algorithm. IEEE Trans. Automat. Control *54* (2009), 399–403.

[16] S. Čelikovský: Numerical algorithm for nonsmooth stabilization of triangular form systems. Kybernetika *32* (1996), 261–274.

[17] M. S. Branicky, V. S. Borkar, and S. K. Mitter: A unifed framework for hybrid control: model and optimal control theory. IEEE Trans. Automat. Control *43* (1998), 31–45.

[18] P. Caines, M. Egerstedt, R. Malhame, and A. Schoellig: A hybrid Bellman equation for bimodal systems. Lecture Notes in Computer Sci. *4416*, Springer, Berlin 2007, pp. 656–659.

[19] K. Deimling: Multivalued Differential Equations. de Gruyter, Berlin 1992.

[20] R. E. Edwards: Functional Analysis. Dover, New York 1995.

[21] M. Egerstedt, Y. Wardi, and H. Axelsson: Transition-time optimization for switched-mode dynamical systems. IEEE Trans. Automat. Control *51* (2006), 110–115.

[22] H. O. Fattorini: Infinite-Dimensional Optimization and Control Theory. Cambridge University Press, Cambridge 1999.

[23] A. Ferrara and M. Rubbagoti: A sub-optimal second order sliding mode controller for systems with saturating actuators. IEEE Trans. Automat. Control *54* (2009), 1082–1087.

[24] A. F. Filippov: Differential Equations with Discontinuous Right-Hand Sides. Kluwer, Dordrecht 1988.

[25] A. G. Gallardo-Hernández, L. Fridman, S. Islas-Andrade, and Y. Shtessel: Quasi-continuous high order sliding modes controllers applied to glucose-insulin regulatory system models. In: Proc. 47th IEEE Conference on Decision and Control, Cancun 2008, pp. 2208–2213.

[26] R. Gamkrelidze: Principles of Optimal Control Theory. Plenum Press, London 1978.

[27] D. Gómez-Gutiérrez, S. Čelikovský, A. Ramírez-Trevino, J. Ruiz-León, and S. Di Gennaro: Sliding mode observer for switched linear systems. In: Proc. 2011 IEEE Conference on Automation Science and Engineering, IEEE Conference on Automation Science and Engineering, Trieste 2011.

[28] W. M. Haddad and V. Chellaboina: Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach. Princeton University Press, New Jersey 2008.

[29] J. Hale: Ordinary Differential Equations. J. Wiley, New York 1969.

[30] C. J. Himmelberg: Measurable relations. Fund. Math. *87* (1975), 53–72.

[31] A. Isidori: Nonlinear Control Systems. Springer, New York 1989.

[32] J. Jahn: Introduction to the Theory of Nonlinear Optimization. Springer, Berlin 2007.

[33] H. K. Khalil: Nonlinear Systems. Prentice Hall, New Jersey 2001.

[34] A. Levant: Universal SISO sliding-mode controllers with finite-time convergence. IEEE Trans. Automat. Control *46* (2001), 1447–1451.

[35] D. Liberzon: Switching in Systems and Control. Birkhäuser, Boston 2003.

[36] Y. Orlov: Discontinuous Systems: Lyapunov Analysis and Robust Synthesis under Uncertainty Conditions. Springer, New York 2008.

[37] B. E. Paden and S. S. Sastry: Calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulator. IEEE Trans. Circuits Systems *34* (1987), 73–82.

[38] A. Poznyak: Advanced Mathematical Tools for Automatic Control Engineers. Elsevier, Amsterdam 2008.

[39] V. G. Boltyanski and A. Poznyak: The Robust Maximum Principle. Birkhäuser, Boston 2011.

[40] R. Pytlak: Numerical Methods for Optimal Control Problems with State Constraints. Springer-Verlag, Berlin 1999.

[41] R. T. Rockafellar: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. *14* (1976), 877–898.

[42] L. E. Ramos-Velasco, J. J. Ruiz-León, and S. Čelikovský: Rotary inverted pendulum: Trajectory tracking via nonlinear control techniques. Kybernetika *38* (2002), 217–232.

[43] T. Roubíček: Approximation theory for generalized young measures. Numer. Funct. Anal. Optim. *16* (1995), 1233–1253.

[44] M. S. Shaikh and P. E. Caines: On the hybrid optimal control problem: theory and algorithms. IEEE Trans. Automat. Control *52* (2007), 1587–1603.

[45] V. Utkin: Sliding Modes in Control and Optimization. Springer, Berlin 1992.

Arturo Enrique Gil García, Department of Control and Automation, CINVESTAV, A.P. 14-740, Av. I.P.N. no. 2508, C. P. 07360, Mexico D.F. Mexico.
e-mail: egil@ctrl.cinvestav.mx

Vadim Azhmyakov, Department of Control and Automation, CINVESTAV, A.P. 14-740, Av. I.P.N. no. 2508, C.P. 07360, Mexico D.F. Mexico.
e-mail: vazhmyakov@ctrl.cinvestav.mx

Michael V. Basin, Department of Physical and Mathematical Sciences, University of Nuevo Leon, San Nicolas de la Garza. Mexico.
e-mail: mbasin@fcfm.uanl.mx