CONVERGENCE ANALYSIS FOR ASYMMETRIC DEFFUANT–WEISBUCH MODEL

Jiangbo Zhang

In this paper, we investigate the convergence behavior of the asymmetric Deffuant–Weisbuch (DW) models during the opinion evolution. Based on the convergence of the asymmetric DW model that generalizes the conventional DW model, we first propose a new concept, the separation time, to study the transient behavior during the DW model's opinion evolution. Then we provide an upper bound of the expected separation time with the help of stochastic analysis. Finally, we show relations of the separation time with model parameters by simulations.

Keywords: opinion dynamics, asymmetric Deffuant–Weisbuch model, convergence, separation time

Classification: 91C99, 91D30, 60G40

1. INTRODUCTION

Opinion dynamics, which was discussed several decades ago [6, 13], had attracted much attention as a challenging research topic because of its many potential applications in various disciplines. Particularly, the bounded confidence opinion models became popular based on simulations studies in recent years. These models, such as the Hegselmann–Krause (HK) model [7] and the Deffuant–Weisbuch (DW) model [3, 17], were proposed in order to understand the evolution of opinions in a group. Both the HK model and the DW model are successful in describing opinion aggregation or evolution as revealed in numerical simulations and physical analysis (see [3, 5, 11]).

In recent years, mathematical analysis has been paid much attention to opinion dynamics, and related collective behavior of multi-agent systems have been widely applied in [1, 8, 9, 14, 15]. However, because the inter-agent topology may keep changing and is dependent of opinion states, many analysis methods (for example, in [1, 12]) cannot be applied to these opinion models. Recently, with the help of stochastic analysis and Lyapunov methods, [16] provided a new method for the convergence analysis of the homogeneous HK model. In addition, the convergence of generalized DW models was discussed in [18].

DOI: 10.14736/kyb-2014-1-0032

The objective of our research in this paper is to study the convergence behavior of the asymmetric DW dynamics, proposed in [18]. The main contributions include

- To study the transient behavior during the opinion evolution, we give a new concept, the separation time, and show that it is a stopping time;
- We estimate the upper bound of the expected separation time for the asymmetric DW model.

The rest of this paper is organized as follows. Section 2 formulates the asymmetric DW opinion protocol and then introduces preliminary notations on probability and graph theories. Section 3 introduces the separation time and then estimates its expected upper bound. Section 4 presents simulation results. Finally, Section 5 provides concluding remarks.

2. THE ASYMMETRIC DW DYNAMICS

In this section, we introduce the DW models, and compare the asymmetric DW model with the conventional DW model.

2.1. Model Description

The conventional DW model can be found in [3], formulated as follows. Agent i has an opinion value $x_i(t) \in \mathbb{R}$ at time $t \geq 0$, $1 \leq i \leq n$. Without loss of generality, initial opinions, $x_i(0)$, $1 \leq i \leq n$, are limited in $[0,1]$ (noting that this can be easily extended in any set in $\mathbb{R}$). Denote $\varepsilon_0 \in (0,1)$ as the confidence radius and $\gamma_0 \in (0,1)$ as the trust weight. Then the conventional DW protocol in [3] is described as

$$\begin{cases} x_i(t+1) = x_i(t) + \gamma_0 \mathbb{1}_{\{|x_j(t)-x_i(t)| \leq \varepsilon_0\}} \cdot (x_j(t) - x_i(t)); \\ x_j(t+1) = x_j(t) + \gamma_0 \mathbb{1}_{\{|x_j(t)-x_i(t)| \leq \varepsilon_0\}} \cdot (x_i(t) - x_j(t)), \end{cases} \tag{1}$$

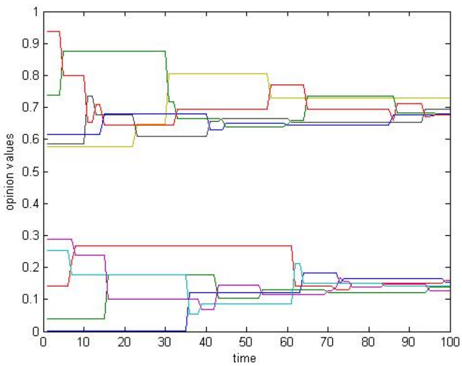
where i, j are selected randomly with a uniform distribution in $\mathcal{V} = \{1, 2, \ldots, n\}$ at time t. $\mathbb{1}$ is the indicator function, that is, $\mathbb{1}_{\{\omega\}} = 1$ if ω holds and $\mathbb{1}_{\{\omega\}} = 0$ otherwise.

Note that in the conventional DW model (1) only two agents are selected at each time, and both learn each other if their distance is not larger than ε_0. However, in many cases [10], some person i may choose another person j and learn his/her opinion who may not choose and learn the person i's opinion at the same time, or maybe even when they collect opinions from each other, trust weights are different when they update their opinions. Based on the practical observation, we consider an asymmetric DW model. The update rule of the opinions in the asymmetric DW model can be formulated as

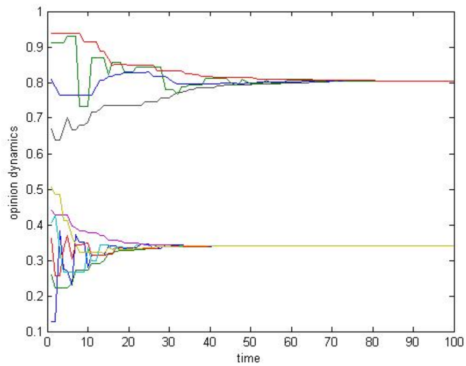
$$x_i(t+1) = x_i(t) + \gamma_i \mathbb{1}_{\{|x_{r_i(t)}(t)-x_i(t)| \leq \varepsilon_0\}} \cdot (x_{r_i(t)}(t) - x_i(t)), \forall 1 \leq i \leq n, t \geq 0, \tag{2}$$

where $\gamma_i \in (0,1)$ is the trust combination weight of the agent i, and $r_i(t)$ denotes the agent index chosen by agent i at time $t \in \mathbb{N}$, which is a random variable uniformly and independently distributed in the agent set $\mathcal{V} = \{1, 2, \ldots, n\}$. Denote the agent neighbor of agent i in $\mathcal{V}$ at time t as $\mathcal{N}_i(t) = \{j : |x_j(t) - x_i(t)| \leq \varepsilon_0\}$.

For illustration, a simulation is given for a comparison between the conventional DW model and the asymmetric DW model. We take 10 agents whose initial opinions are randomly distributed in the interval $[0, 1]$ and the confidence radius $\varepsilon_0 = 0.3$. The curves in Figure 1 show that the convergence of the agents' opinions in the asymmetric DW dynamics is faster than that in the conventional DW dynamics. Moreover, opinion values in the conventional DW model changes more sharply than those in the asymmetric DW model as shown in Figure 1.



(a) The conventional DW dynamics



(b) The asymmetric DW dynamics

Fig. 1. Conventional DW dynamics (1) vs. asymmetric DW dynamics (2).

2.2. Graph and Probability Space

To study the DW model (2), some concepts in graph theory and stochastic analysis are required.

We first present some basic definitions about graph theory [4]. An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, 2, \ldots, n\}$ of vertexes (or nodes) and an edge set $\mathcal{E}$, in which an edge is an pair of distinct nodes of $\mathcal{V}$. A subgraph of $\mathcal{G}$ is a graph whose vertex set is a subset of $\mathcal{V}$, and whose adjacency relation is a subset of $\mathcal{E}$ restricted to this subset. We provide following concepts.

- A *path* is an edge sequence that all edges are connected one by one, such as a path from i_1 to i_k $\{(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)\}$. i and j are *connected* if there exists at least a path from i to j.

- A *connected component* of an undirected graph $\mathcal{G}$ is a subgraph in which any two vertices are connected to each other by paths, and in which any vertex is not connected to the vertex outside this subgraph. A *complete connected component* of $\mathcal{G}$ is a connected component in which any two vertices are connected.

- $|\mathcal{V}|$ is the agent number of the set $\mathcal{V}$.

A directed graph is defined by $\overrightarrow{\mathcal{G}} = (\mathcal{V}, \overrightarrow{\mathcal{E}})$ in which any edge $(i,j) \in \overrightarrow{\mathcal{E}}$ is directed and denotes the information flow from j to i. We can define the path, the union of two graphs, connected component and complete subgraph similarly as above.

To describe the randomness of the DW model (2), we introduce some notations of probability theory (referring to [2]).

A probability space is usually defined as $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is the state space and $\mathbf{P}$ is a normalized measure on a σ-algebra $\mathcal{F}$ composed by subsets of Ω. A set $Q \in \mathcal{F}$ is called an event. The nonnegative real number $\mathbf{P}(Q)$ is the probability of the event Q. In the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, the event Q is established almost surely (a.s.) if $\mathbf{P}(Q) = 1$, and is a null event if $\mathbf{P}(Q) = 0$. If $A = (a_{ij})_{n\times n} \in [0,1]^{n\times n}$, $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1$ for all $i,j \in \mathcal{V}$, then A is a *row-stochastic matrix* (We simply call it *stochastic matrix* in this paper).

The probability space that we consider for the DW dynamics is constructed by n-agent infinite time opinions trajectories. For the conventional DW model (1) and the asymmetric DW model (2), $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))'$ is, in fact, the abbreviation of $\mathbf{x}(t,\omega) = (x_1(t,\omega), \ldots, x_n(t,\omega))'$. Denote $\omega_t = \{x_1(t), \ldots, x_n(t)\}$. Let $\{\mathcal{F}_t\}$ be a filtration on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_{t+1} = \sigma(\mathbf{x}(0), \ldots, \mathbf{x}(t)), t \geq 0$. $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_\infty = \mathcal{F}$. We use $(\Omega, \mathcal{F}, \mathbf{P})$ to denote the probability space generalized by the DW model (2).

Based on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ of the model (2), for any event ω, we obtain a directed graph $\overrightarrow{\mathcal{G}}_t(\omega)$ in which $(i,j) \in \overrightarrow{\mathcal{E}}_t(\omega)$ if and only if $r_i(t,\omega) = j$ and $|x_j(t,\omega) - x_i(t,\omega)| \leq \varepsilon_0$ at time $t \in \mathbb{N}$. Generally, we get a directed graph sequence $\{\overrightarrow{\mathcal{G}}_t(\omega)\}_{t\geq 0}$ for any event $\omega \in \Omega$. Then we denote an undirected graph sequence on (1) and (2) as $\{\mathcal{G}_t\}_{t\geq 0}$ where $(i,j) \in \mathcal{E}_t$ if and only if $|x_i(t) - x_j(t)| \leq \varepsilon_0$.

In the following, we focus on the analysis of the DW model (2) based on the graph sequence $\{\mathcal{G}_t\}$, though similar conclusions can be obtained for the conventional DW model (1).

3. SEPARATION TIME

In this section, we introduce a new concept, the separation time, and give a related analysis for the convergence behavior.

Define $\mathbf{x}(t) \in \mathbb{R}^n$ convergent to $\mathbf{x}^* \in [0,1]^n$ a.s. if $\mathbf{P}(\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*) = 1$. The following result was obtained in [18].

Lemma 1. For the asymmetric DW model (2), one of the following two results holds a.s. for any $i, j \in \mathcal{V}$:
(i) $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$,
(ii) $\lim_{t\to\infty} |x_i(t) - x_j(t)| > \varepsilon_0$.

By Lemma 1, we obtain that, for limit states, $\mathcal{G}_\infty(\omega)$ is composed by finite complete connected components a.s. This induces us to think about whether there exists a finite time $\tau(\omega)$ in which all graphs in $\{\mathcal{G}_\tau(\omega), \omega \in \Omega$ is a convergence event$\}$ are composed by finite complete connected components. The following lemma explains the existence of this finite time $\tau(\omega)$ for the event ω that agents converge.

Lemma 2. If the opinion convergence holds for a given event $\omega \in \Omega$, then $\tau(\omega) < \infty$.

Proof. Consider an event ω in which the opinion convergence holds. By Lemma 1, for the confidence radius $\varepsilon_0 > 0$, there must exist a finite time T such that when $t > T$,

[I] $|x_i(t) - x_j(t)| \le \varepsilon_0$ if $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$;

[II] $|x_i(t) - x_j(t)| > \varepsilon_0$ if $\lim_{t\to\infty} |x_i(t) - x_j(t)| > \varepsilon_0$.

Otherwise if there exists an infinite time sequence $\{t_k\}$ such that $[I], [II]$ do not hold, then we obtain that, correspondingly,

[I'] $\overline{\lim}_{t\to\infty}|x_i(t) - x_j(t)| \ge \varepsilon_0 > 0$ if $\{t_k\}$ satisfies that $|x_i(t_k) - x_j(t_k)| > \varepsilon_0$;

[II'] $\underline{\lim}_{t\to\infty}|x_i(t) - x_j(t)| \le \varepsilon_0$ if $\{s_k\}$ satisfies that $x_i(s_k) - x_j(s_k)| \le \varepsilon_0$.

Both $[I']$ and $[II']$ contradict with Lemma 1 because

$$\lim_{t\to\infty} |x_i(t) - x_j(t)| = \overline{\lim}_{t\to\infty}|x_i(t) - x_j(t)| = \underline{\lim}_{t\to\infty}|x_i(t) - x_j(t)|.$$

Hence, our result is obtained. □

With the existence of τ for the event in which opinions converge, we can further show that after time τ, all graphs $\{\mathcal{G}_t, t \ge \tau\}$ are the same. Obviously, they are composed by complete connected components.

Lemma 3. For any event $\omega \in \Omega$ and $t \in \mathbb{N}$, if $\mathcal{G}_t(\omega)$ is composed by complete connected components, then $\mathcal{G}_{t+1}(\omega) = \mathcal{G}_t(\omega)$.

Proof. If $\mathcal{G}_t$ is composed by complete connected components, then obviously if $(i, j) \in \mathcal{E}_t$ and $(j, k) \in \mathcal{E}_t$ then $(i, k) \in \mathcal{E}_t$ for any $i, j, k \in \mathcal{V}$.

Suppose $\mathcal{G}_{t+1}$ is not composed by complete connected components. For the event ω, we can assume that there exist $i, j, k \in \mathcal{V}$ such that

- $(i, j) \in \mathcal{E}_t$, $(j, k) \in \mathcal{E}_t$ and $(i, k) \in \mathcal{E}_t$;

- $(i, j) \in \mathcal{E}_{t+1}$, $(j, k) \in \mathcal{E}_{t+1}$ and $(i, k) \notin \mathcal{E}_{t+1}$.

For the connected component G containing i, j, k, by the definition of $\{\mathcal{G}_t\}$ we can see that

$$\max_{s,l\in G} |x_s(t) - x_l(t)| \le \varepsilon_0, \min_{s\in G^c,l\in G} |x_s(t) - x_l(t)| > \varepsilon_0.$$

Moreover, it is obtained obviously that both inequalities also hold at time $t+1$ because $\min_{s\in G,l\in G^c} |x_{r_s(t)}(t) - x_l(t)| > \varepsilon_0$ and $\min_{s\in G,l\in G^c} |x_{r_l(t)}(t) - x_s(t)| > \varepsilon_0$. However, $(i, k) \notin \mathcal{E}_{t+1}$ will contradict with $\min_{s\in G^c,l\in G} |x_s(t) - x_l(t)| > \varepsilon_0$ because:

- If $r_k(t), r_i(t) \in G$, then

$$|x_i(t+1) - x_k(t+1)| \le \max_{s,l\in\{i,k,r_k(t+1),r_i(t+1)\}} |x_s(t+1) - x_l(t+1)| \le \varepsilon_0;$$

- If $r_k(t) \in G, r_i(t) \in G^c$ or $r_k(t) \in G^c, r_i(t) \in G$, then $x_i(t+1) = x_i(t)$ or $x_k(t+1) = x_k(t)$. Similar with above, $|x_i(t+1) - x_k(t+1)| \le \varepsilon_0$;

- If $r_k(t) \in G^c, r_i(t) \in G^c$, then $|x_i(t+1) - x_k(t+1)| = |x_i(t) - x_k(t)| \le \varepsilon_0$.

Thus, we obtain the conclusion. □

By Lemmas 2 and 3, we have proved that there exists such a time $\tau(\omega)$ that all graphs $\{\mathcal{G}_t(\omega), t \ge \tau(\omega)\}$ are composed by the same complete connected components for the event ω that opinions converge. This induces us to think about the low bound of $\tau(\omega)$. Hence, we introduce the definition of the separation time to describe this time as follows.

Definition 1. For the DW models (1) and (2), the *separation time* $T^*(\omega)$ is a mapping $T^*(\omega) : \Omega \to \bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$ such that

$$T^*(\omega) = \inf\{\tau(\omega) : \mathcal{G}_{\tau(\omega)} \text{ is composed by complete connected components.}\} \quad (3)$$

Moreover, by Lemma 3 we can see that all agent subgroups are not changed after the separation time T^*, and hence we say that agent subgroups are *steady* after the separation time $T^*(\omega)$ for the event ω that agent opinions converge. Specially, when opinions do not converge for the event ω that agent opinions diverge, we only need to take $T^*(\omega) = \infty$. By Lemma 2, it is obvious that the mapping $T^*(\omega)$ can be well defined for any event ω.

It is also not difficult to prove that $T^*(\omega)$ is a stopping time for the event ω that agent opinions converge. In fact, recalling the equation (3), T^* is clearly the hitting time of $\{\mathcal{G}_t(\omega)$ is composed by complete connected components$\}$. When $t < T^*(\omega)$, all the graphs $\mathcal{G}_1(\omega), \dots, \mathcal{G}_{t-1}(\omega)$ are not composed by complete connected components. Hence, $T^*(\omega)$ is a stopping time.

Actually, the separation time of (2) is the first moment after which all opinion subgroups will not be changed. The estimation of the separation time is important because we can previously estimate when the steady opinion separation will appear. We will give the following example to show the difference between the separation time and the time when separation (maybe unsteady) occurs. With $n = 40$, $\varepsilon_0 = 0.2$, two subgroups are formed at time $t = 25$ in Figure 2. But they are divided with the distance less than $\varepsilon_0 = 0.2$ at this time and there is still some possibility for them to merge again though subgroups look separated. In fact, T^* is about 45 in this example.

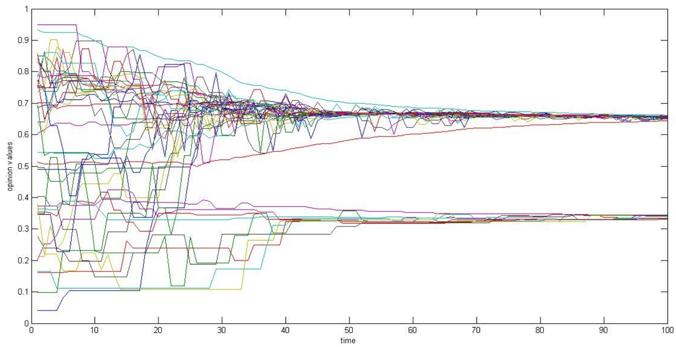


Fig. 2. A simulation for (2).

In the following we will estimate an expected upper bound of the separation time for the DW dynamics (2).

Rewrite (2) as a matrix form $\mathbf{x}(t+1) = W(t)\mathbf{x}(t)$. The system matrix $W(t)$ is a stochastic matrix and

$$W_{sk}(t) = \begin{cases} 1 - \omega_s \mathbb{1}_{\{|x_k(t)-x_s(t)|\le\varepsilon_0\}} & \text{if } r_s(t) \neq s, k = s; \\ \omega_s \mathbb{1}_{\{|x_k(t)-x_s(t)|\le\varepsilon_0\}} & \text{if } r_s(t) = k, k \neq s; \\ 1 & \text{if } r_s(t) = s, k = s; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $W(t)$ is measurable with respect to $\mathcal{F}_{t+1}$, i. e., $W_{ij}(t)$ is measurable with respect to $\mathcal{F}_{t+1}$ for any $i, j \in \mathcal{V}$ and $t \ge 0$, we can see that $\{W(t)\}$ is a random chain adapted to $\{\mathcal{F}_t\}$, or simply $\{W(t)\}$ is an adapted random chain.

A random vector process $\{\pi(t)\}$ is an absolute probability process for $\{W(t)\}$ if

$$E[\pi^T(t+1)W(t)|\mathcal{F}_t] = \pi^T(t) \qquad \text{for all } k \ge 0,$$

and $\pi(t)$ is a stochastic vector ($\sum_{i=1}^n \pi_i(t) = 1$, $\pi_i(t) \ge 0$, $i \in \mathcal{V}$) a.s. for any $t \ge 0$ ([16]).

Based on these notations, we present and prove the main result of this paper as follows.

Theorem 1. For the model (2),

$$E[T^*(\omega)] \leq 1 + \frac{n^n \overline{\gamma}^{n-1}}{\varepsilon_0^2 \underline{\gamma}^n (1-\overline{\gamma})(\min\{\underline{\gamma}, 1-\overline{\gamma}\})^{n-1}} \tag{4}$$

where $\underline{\gamma} = \min_{1\leq i\leq n} \gamma_i$ and $\overline{\gamma} = \max_{1\leq i \leq n} \gamma_i$.

Proof. For graphs $\{\mathcal{G}_t(\omega)\}_{t\geq 0}$ and $\omega \in \Omega$, if the initial graph $\mathcal{G}_0(\omega)$ owns more than one connected component, then we only need to analyze components respectively. Simply we assume that $\mathcal{G}_0(\omega)$ owns one connected component for the event $\omega \in \Omega$ in the following analysis.

By the definition of $T^*(\omega)$, for $t < T^*(\omega)$, we can find at least one mapping $i_t(\omega) \in \mathcal{V} : \Omega \to \mathcal{V}$, such that there exist $j_t, k_t \in \mathcal{N}_{i_t}(t)$, $|x_{j_t}(t) - x_{k_t}(t)| \geq \varepsilon_0$.

The following steps are carried out for the proof.

(i) We have proved that there exists a sequence $\{\pi(t)\}$ which is the absolute probability sequence of $\{W(t)\}$ (in [18]). For

$$V_\pi(\mathbf{x}(t), t) = \sum_{i=1}^n \pi_i(t)(x_i(t) - \sum_{j=1}^n \pi_j(t) x_j(t))^2$$

and any $t > 0$,

$$E[\sum_{t=0}^{\infty} \sum_{i<j} L_{ij}(t)(x_i(t) - x_j(t))^2] \leq E[V_\pi(\mathbf{x}(0), 0)] < \infty \tag{5}$$

where $L(t) = W^T(t) diag(\pi(t+1)) W(t)$ (Corollary 4.3 in [16]).

Moreover, for $V_\pi(\mathbf{x}(t), t)$, we can get $V_\pi(\mathbf{x}(t), t) \leq \sum_{i=1}^n \pi_i(t) = 1$.

In fact, $\{W(t)\}$ has an absolute probability sequence $\{\pi(t)\}$ that is uniformly bounded below by $\kappa = (\frac{\delta}{n})^{n-1}$, $\delta = \frac{\underline{\gamma}\min\{1-\overline{\gamma}, \underline{\gamma}\}}{\overline{\gamma}}$ (Lemma 2 in [18]). Hence,

$$L_{ij}(t) \geq \kappa M_{ij}(t),$$

where $M(t) = W^T(t) W(t)$.

(ii) By $\mathbf{P}(r_i(t) = j) = \frac{1}{n}$ for any $i, j \in \mathcal{V}$, $t \geq 0$,

$$\begin{aligned} E[W_{ij}(t) W_{ii}(t)] &= \gamma_i(1-\gamma_i)\mathbf{P}(W_{ij}(t) = \gamma_i, W_{ii}(t) = 1-\gamma_i) \\ &+ 0\mathbf{P}(W_{ij}(t) = 0, W_{ii}(t) = 1) + 0\mathbf{P}(W_{ij}(t) = 0, W_{ii}(t) = 1-\gamma_i) = \frac{\gamma_i(1-\gamma_i)}{n}. \end{aligned}$$

Moreover,

$$M_{ij}(t) = \sum_{k=1}^n (W^T)_{ik}(t) W_{kj}(t) = \sum_{k=1}^n W_{ki}(t) W_{kj}(t).$$

Therefore, for $j_t \in \mathcal{N}_{i_t}(t)$, by

$$E[M_{j_t i_t}(t)] = \sum_{s=1}^{n} E[W_{s j_t}(t) W_{s i_t}(t)] \geq E[W_{i_t j_t}(t) W_{i_t i_t}(t)] \geq \frac{\underline{\gamma}(1-\overline{\gamma})}{n},$$

$$\min_{t<T^*} E[M_{j_t,i_t}(t)] \geq \frac{\underline{\gamma}(1-\overline{\gamma})}{n}. \tag{6}$$

In the same way, $\min_{t<T^*} E[M_{k_t,i_t}(t)] \geq \frac{\underline{\gamma}(1-\overline{\gamma})}{n}$.

(iii) $\{r_i(t), 1 \leq i \leq n\}$ are independent of agent locations at time t. Therefore, $\mathbf{P}(|x_i(t) - x_j(t)| = a, r_s(t) = l) = \mathbf{P}(|x_i(t) - x_j(t)| = a)\mathbf{P}(r_s(t) = l)$ for any $i, j, s, l \in \mathcal{V}$, $a \in [0, 1]$.

Denote $S(\mathbf{x}(0), |x_i(t) - x_j(t)|)$ as the set of all possible values of $|x_i(t) - x_j(t)|$ at time t given the initial values $\mathbf{x}(0)$ for (2), $i, j \in \mathcal{V}$. Obviously, $|S(\mathbf{x}(0), |x_i(t) - x_j(t)|)| < \infty$ for the finite t and $\mathbf{x}(0) \in [0, 1]^n$.

We use $\{f_k(\gamma_i, \gamma_j), k = 1, 2, 3, 4\}$ to denote all possible values of $M_{ij}(t)$ and obtain that:

$$f_k(\gamma_i, \gamma_j) = \begin{cases} \gamma_i(1-\gamma_i) & \text{if } r_i(t) = j, r_j(t) \neq i; \\ \gamma_j(1-\gamma_j) & \text{if } r_j(t) = i, r_i(t) \neq j; \\ \gamma_i(1-\gamma_i) + \gamma_j(1-\gamma_j) & \text{if } r_j(t) = i, r_i(t) = j; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for any $i, j \in \mathcal{V}$, we denote $f_k(\gamma_i, \gamma_j)$ as f_k simply and can get that

$$\begin{aligned} &E[M_{ij}(t)(x_i(t) - x_j(t))^2] = E[M_{ij}(t)(x_i(t) - x_j(t))^2 | \mathcal{F}_{t+1}] \\ &= \int_{[0,1]^n} \sum_{a \in S(\mathbf{x}(0), |x_i(t) - x_j(t)|)} \sum_k a^2 f_k \mathbf{P}(|x_i(t) - x_j(t)| = a, M_{ij}(t) = f_k) d\mathbf{x}(0) \\ &= \sum_k \left(\int_{[0,1]^n} \sum_{a \in S(\mathbf{x}(0), |x_i(t) - x_j(t)|)} a^2 \mathbf{P}(|x_i(t) - x_j(t)| = a) d\mathbf{x}(0) \right) f_k \mathbf{P}(M_{ij}(t) = f_k) \\ &= E[M_{ij}(t)] E[(x_i(t) - x_j(t))^2]. \end{aligned} \tag{7}$$

Therefore, for $t < T^*$, by (7) we have

$$\begin{aligned} &\sum_{i<j} E[M_{ij}(t)(x_i(t) - x_j(t))^2] \geq \sum_{i<j, i,j \in \mathcal{N}_{i_t}(t)} E[M_{ij}(t)(x_i(t) - x_j(t))^2] \\ &\geq E[M_{j_t i_t}(t)(x_{j_t}(t) - x_{i_t}(t))^2] + E[M_{k_t i_t}(t)(x_{k_t}(t) - x_{i_t}(t))^2] \\ &= E[M_{j_t i_t}(t)] E[(x_{j_t}(t) - x_{i_t}(t))^2] + E[M_{k_t i_t}(t)] E[(x_{k_t}(t) - x_{i_t}(t))^2] \\ &\geq \min\{E[M_{j_t,i_t}(t)], E[M_{k_t,i_t}(t)]\} E[(x_{j_t}(t) - x_{i_t}(t))^2 + (x_{k_t}(t) - x_{i_t}(t))^2] \\ &\geq \min\{E[M_{j_t,i_t}(t)], E[M_{k_t,i_t}(t)]\} E[(x_{j_t}(t) - x_{k_t}(t))^2] \\ &\geq \varepsilon_0^2 \min\{E[M_{j_t,i_t}(t)], E[M_{k_t,i_t}(t)]\}. \end{aligned} \tag{8}$$

Also, because T^* is a stopping time, by (6), we obtain

$$\begin{aligned} E[\sum_{t=0}^{T^*-1} M_{j_t i_t}(t)] &= E[\sum_{t=0}^{\infty} M_{j_t i_t}(t)\mathbb{1}_{\{T^*\geq t+1\}}] \geq \min_{t<T^*} E[M_{j_t i_t}(t)]E[\sum_{t=0}^{\infty} \mathbb{1}_{\{T^*\geq t+1\}}] \\ &\geq \frac{\underline{\gamma}(1-\overline{\gamma})}{n} E[\sum_{t=0}^{\infty} \mathbb{1}_{\{T^*\geq t+1\}}] = \frac{\underline{\gamma}(1-\overline{\gamma})}{n}(E[T^*]-1). \end{aligned} \quad (9)$$

Similarly, $E[\sum_{t=0}^{T^*-1} M_{k_t i_t}(t)] \geq \frac{\underline{\gamma}(1-\overline{\gamma})}{n}(E[T^*]-1)$.

In a sum, by inequalities (5), (6), (8) and (9), we have

$$\begin{aligned} 1 &\geq E[V_\pi(\mathbf{x}(0),0)] \geq E[\sum_{t=0}^{T^*-1} \sum_{i<j} L_{ij}(t)(x_i(t)-x_j(t))^2] \\ &\geq \kappa\varepsilon_0^2 \min\{E[\sum_{t=0}^{T^*-1} M_{j_t i_t}(t)], E[\sum_{t=0}^{T^*-1} M_{k_t i_t}(t)]\} \\ &\geq \kappa\varepsilon_0^2 \min_{t<T^*}\{E[M_{j_t i_t}(t)], E[M_{k_t i_t}(t)]\}E[T^*-1] \\ &\geq \kappa\frac{\underline{\gamma}(1-\overline{\gamma})}{n}\varepsilon_0^2(E[T^*]-1). \end{aligned} \quad (10)$$

Then, by (10), $E[T^*(\omega)] \leq 1 + \frac{n}{\kappa\varepsilon_0^2\underline{\gamma}(1-\overline{\gamma})}$.

Hence, the conclusion is obtained. □

Remark 1. Since the conventional DW model (1) is a special case of the asymmetric DW model (2), it is not difficult to apply the estimation method of Theorem 1 in the conventional DW model (1) to obtain a similar result. Additionally, our estimation is much larger than the actual separation time and the following section will show the difference by simulations.

4. SIMULATIONS

In this section, we present several numerical simulations to further illustrate phenomena about the separation time. All figures are done after the averaging of 100 simulations.

First, we present an experiment about relations of the separation time with the agent number. The confidence radius ε_0 is fixed at 0.3 and initial opinions are randomly distributed in the interval $[0,1]$. To demonstrate the influence of the agent number n on the separation time, we let n vary from 10 to 100, and the result is shown in Figure 3. We can see that the separation time is a roughly increasing function of the parameter n. Clearly, the more agents are, the larger the separation time is.

To show that our main result Theorem 1 is a conservative estimation one, we can present following simulations on the separation time. Our estimated upper bound of the expected separation time in Theorem 1 is larger than n^n. However, we can see from

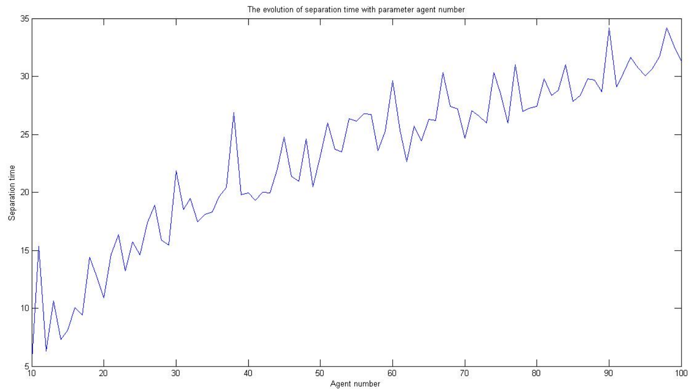


Fig. 3. The agent number vs. the separation time for (2).

Figure 3 that separation times are always less than n, which is much less than the result and shows the necessary to further estimate the separation time in future.

When $t > T^*$, there exist some steady opinion subgroups. By Lemma 1, the steady distances of different subgroups are lager than ε_0. We use average subgroup distance to describe the average of all adjacent subgroup distances. The following simulations show phenomena after the separation time T^*. The confidence radius ε_0 is 0.15 and the terminal time is 10000. By Figure 4, we can see that the average subgroup distance is almost not sensitive to the agent number n.

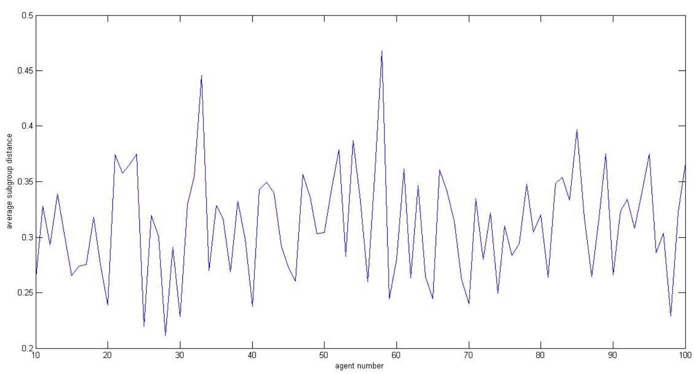


Fig. 4. The agent number vs. average subgroup distance for (2).

Second, we illustrate the influence of the confidence radius on the separation time.

We keep $n = 100$, and let ε_0 vary from 0.05 to 1 with the step 0.025. Figure 5 shows that the separation time increase first and then decrease as the parameter ε_0 increases from 0.05 to 1. There exists a value ε_0^* in which T^* gets the maximum value. When $\varepsilon_0 < \varepsilon_0^*$, separated steady subgroups are formed early. When $\varepsilon_0 > \varepsilon_0^*$, we can see that the separation time is a roughly decreasing function of parameter ε_0, which implies that the larger the confidence radius is, the smaller the separation time is.

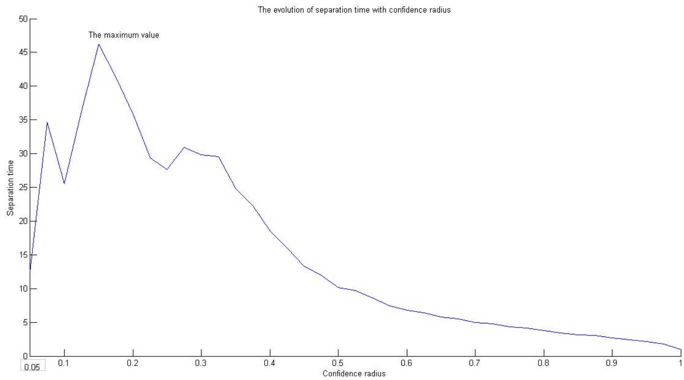


Fig. 5. The confidence radius vs. the separation time for (2).

Finally, we present Figure 6 to show the relation of average subgroup distance with the confidence radius. The confidence radius is the same as before. Figure 6 shows that the average steady subgroup distance is increasing first and then becomes zero after certain confidence radius.

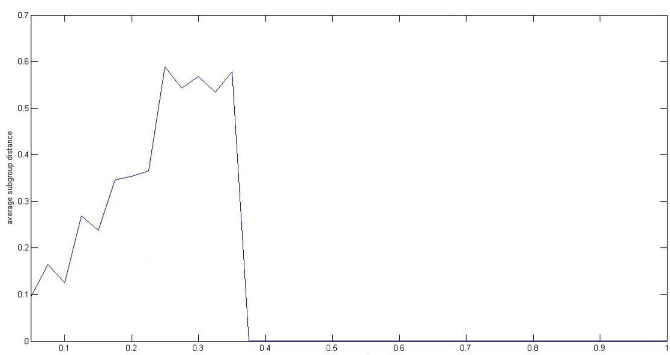


Fig. 6. Confidence radius vs. average subgroup distance for (2).

5. CONCLUDING REMARKS

In this paper, we proposed the separation time to investigate dynamical behaviors of the asymmetric DW opinion dynamics. After the introduction of the separation time, we estimated the upper bound of the expectation of the separation time using stochastic analysis. Then we gave simulations on the separation time of the DW dynamics.

These results helped us to understand how the separation time functions as an important definition in the DW opinion dynamics. However, many interesting opinion dynamics problems, such as further estimation of the expected separation time on the DW model and the estimation for the number of steady subgroups of the DW model, remain to be solved.

ACKNOWLEDGEMENT

This work was supported by the NNSF of China under Grant 61174071.

(Received May 15, 2013)

REFERENCES

[1] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah: Randomized gossip algorithms. Trans. Inform. Theory *52* (2006), 2508–2530.

[2] Y. Chow and H. Teicher: Probability Theory: Independence Interchangeability Martingales. Second edition. Springer-Verlag, Heidelberg – Berlin 1978.

[3] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch: Mixing beliefs among interacting agents. Adv. in Complex Systems *3* (2000), 87–98.

[4] C. Godsil and G. Royle: Algebraic Graph Theory. Springer-Verlag, New York 2001.

[5] P. Groeber, F. Schweitzer, and K. Press: How groups can foster consensus: The case of local cultures. J. Artificial Societies and Social Simulations *12* (2009), 2/4.

[6] F. Harary: A criterion for unanimity in French's theory of social power. In: Studies in Social Power, Ann Arbor 1959, pp. 168–182.

[7] R. Hegselmann and U. Krause: Opinion dynamics and bounded confidence models. Analysis and Simulation. J. Artificial Societies and Social Simulations *5* (2002), 3.

[8] Y. Hong, L. Gao, D. Cheng, and J. Hu: Lyapuov-based approach to multi-agent systems with switching jointly connected interconnection. IEEE Trans. Automat. Control *52* (2007), 943–948.

[9] J. Hu: On robust consensus of multi-agent systems with communication delays. Kybernetika *45* (2009), 768–784.

[10] L. Jiang, D. Hua, J. Zhu, B. Wang, and T. Zhou. Opinion dynamics on directed small-world networks. The European Physical Journal B *65* (2008), 2, 251–255.

[11] J. Lorenz: Continuous opinion dynamics under bounded confidence: A Survey. Internat. J. of Modern Physics C *18* (2007), 12, 1819–1838.

[12] Y. Lou and Y. Hong: Target containment control of multi-agent systems with random switching interconnection topologies. Automatica *48* (2012), 5, 879–885.

[13] M. Serge and Z. Marisa: The group as a polarizer of attitudes. J. Personal. Soc. Psychology *12* (1969), 1, 125–135.

[14] G. Shi and Y. Hong: Global target aggregation and state agreement of nonlinear multi-agent systems with switching topologies. Automatica *45* (2009), 1165–1175.

[15] J. Thunberg, W. Song, Y. Hong, and X. Hu: Distributed attitude synchronization using backstepping and sliding mode control. Journal of Control Theory and Applications *12* (2014), 1, 48–55.

[16] B. Touri: Product of Random Stochastic Matrices And Distributed Averaging. Ph.D. Thesis, Dept. of Industrial Engineering in the Graduate College of the University of Illinois at Urbana-Champaign, 2011.

[17] G. Weisbuch, G. Deffuant, F. Amblard, and J. Nadal: Meet, discuss and segregate! Complexity *7* (2002), 3, 55–63.

[18] J. Zhang and Y. Hong: Opinion evolution analysis for short-range and long-range Deffuant–Weisbuch models. Physica A *392* (2013), 5289–5297.

Jiangbo Zhang, No.55, Zhongguancun East Road, Academy of Mathematics and Systems Science, Beijing 100190. P. R. China.

e-mail: jbzhang@amss.ac.cn