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ON TRANSIENT QUEUE-SIZE DISTRIBUTION IN THE BATCH-ARRIVALS SYSTEM WITH A SINGLE VACATION POLICY

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A queueing system with batch Poisson arrivals and single vacations with the exhaustive service discipline is investigated. As the main result the representation for the Laplace transform of the transient queue-size distribution in the system which is empty before the opening is obtained. The approach consists of few stages. Firstly, some results for a "usual" system without vacations corresponding to the original one are derived. Next, applying the formula of total probability, the analysis of the original system on a single vacation cycle is brought to the study of the "usual" system. Finally, the renewal theory is used to derive the general result. Moreover, a numerical approach to analytical results is discussed and some illustrative numerical examples are given.

Keywords: batch Poisson arrivals, queue-size distribution, renewal theory, single vacation, transient state

Classification: 90B22, 60K25

1. INTRODUCTION

Evidently, queueing systems with server vacations are good mathematical models for different real-life situations occurring e. g. in telecommunications, computer networks, manufacturing, logistics and transport. As a vacation we can treat a period of server's unavailability or maintenance, a repair time after the failure of the machine or e. g. temporary suspension of the ferry shipping due to severe weather conditions. The server vacation can also be a tool for reducing costs of system's operation.

Most of the results for vacation queueing models concerns the system in the equilibrium (stationary state). However, in general, the investigation of system's characteristics in the transient state is of great importance because of (at least) two main reasons:

- "input" parameters of the system (e. g. the arrival and service rates) may change in short periods of time;

- even when the parameters are "stable", the convergence rate of the transient distributions to the stationary ones is often slow.

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In the paper we deal with a batch-arrivals system of the $M^X/G/1$ type with a single compulsory vacation after each busy period (i. e. with exhaustive service). Such a system was analyzed in [6] where, inter alia, the representation for the stationary queue-size distribution was obtained. In [7] the case of the system with additional setup time before the first service after the vacation was considered. Besides, in [16] the queue-size distributions in the transient and stationary states were obtained, but explicitly only for the probability of one packet present in the system. The overview of results for different-type vacation systems can be found e. g. in [2] and in monographs [15] and [17].

The results for transient characteristics of batch-arrivals queueing systems with single vacations can also be found in [8, 9, 10] and [11]. In [8] the explicit representation for the joint transform of the first busy period, first vacation period and the number of customers served during the first busy period was derived for exponential vacations. Generalization of these results for the case of arbitrarily distributed single vacations can be found in [11]. The departure process in the system with single vacations was investigated in [9] and [10]. Transient analysis of the virtual waiting time in a finite-buffer system working under single vacation policy can be found in [13]. Moreover, the queue-size distribution in the finite-buffer system was studied in [12], where another restrictions in the service process was introduced: an AQM-type scheme with a dropping function.

In the article we propose a non-standard approach to the transient analysis of the queue-size distribution in the original system. The method is, in fact, a certain step-by-step procedure and can be described as follows:

- firstly, we consider the $M^X/G/1$ system without vacations corresponding to the original one (we call it a "usual" $M^X/G/1$-type system) and derive the formula for the Laplace transform of the queue-size distribution in such a system;
- next we investigate the queue-size distribution in a certain modified system with vacations, on the first vacation cycle. Using the formula of total probability we bring the analysis to the case of the corresponding "usual" system;
- in the last step we use a delayed renewal process of successive vacation cycles to obtain general results.

So, the paper is organized as follows. In the next Section 2 we give precise descriptions of considered queueing models. In Section 3 we investigate a "usual" system and find a representation for the Laplace transform of the transient queue-size distribution. Section 4 contains results obtained for the modified system with vacations. In Section 5 we state the main theorem that gives the formula for the Laplace transform of the queue-size distribution in the original system. The last Section 6 is devoted to numerical approach to theoretical results and contains computational examples.

2. QUEUEING MODELS

The original system is supposed to be of the $M^X/G/1$ type in which batches of packets arrive according to a Poisson process with constant rate λ and are being served individually with a distribution function $F(\cdot)$, according to the FIFO service discipline. The size of an arriving batch equals k with probability p_k, $\sum_{k=1}^{\infty} p_k = 1$. The system starts working in the "standard" way i. e. the first batch of packets joins at time $t = 0$ the empty system, and the service process begins immediately. After each busy period (when the system becomes empty) the server takes a compulsory vacation which is generally distributed with a distribution function $V(\cdot)$. If at least one batch of packets arrives during the vacation then, at the end of the vacation, the service process begins immediately. In the case of no arrivals during the vacation, at the vacation completion epoch the server is being activated (is on standby) and waits for the first group of packets to start the service process. Thus, we can observe the operation of the original system during successive vacation cycles C_i, $i = 0, 1, ...$ defined as follows:

$$C_0 = \tau_0, \quad C_i = v_i + \delta_i + \tau_i, \;\; i = 1, 2, ..., \tag{1}$$

where v_i denotes the ith vacation duration, δ_i is the ith standby time and τ_i stands for the ith busy period of the system. As above, we will often identify a particular period of system's operation (busy period, vacation etc.) with its duration. According to the order of summands in the formula (1), we define successive cycles in a non-standard way: they begin with a vacation which is followed by a standby time (maybe zero) and a busy period. Thanks to this, start moments of successive cycles are renewal moments due to the memoryless property of the distribution of interarrival times. Of course, the "zero" vacation cycle, that begins at $t = 0$, consists only of the busy period τ_0.

In Section 4 we consider a modified original system which starts working with the first vacation, thus $C_0 = 0$. We denote all probabilities for such a system by $\mathbf{P}_M\{\cdot\}$.

Beyond the original model we consider the corresponding model of the $M^X/G/1$ type without vacations which is observed on its initial "zero" busy period τ_0 (in this system we have $v_i = 0$ and δ_i is the usual idle period, where $i = 1, 2, ...$). All essential "input" distributions (of interarrival times, service times and batch sizes) in the corresponding system without vacations are the same as in the original one. We call such a system a "usual" one. For the "usual" system we also define the following initial conditions:

- by $\mathbf{P}^U_{n,x}\{\cdot\}$ we denote the probability on condition that the system begins its operation with n packets present (at time $t = +0$), successive arriving batches of packets occur according to a Poisson process with intensity λ, and the first service is being completed exactly at time $x > 0$;
- by $\mathbf{P}^U_n\{\cdot\}$ and $\mathbf{E}^U_n\{\cdot\}$ we denote, respectively, the probability and the mean on condition that the system starts the operation containing (at time $t = +0$) exactly n packets;
- by $\mathbf{P}^U_{std}\{\cdot\}$ and $\mathbf{E}^U_{std}\{\cdot\}$ we denote, respectively, the probability and the mean on condition that the system works in the "standard" regime i. e. it is empty before the opening and the first group of packets occurs at time $t = 0$.

Let us observe the fact that the first two initial conditions shall not preclude the situation in which the system contains a number of packets before the opening.

We end this section with some important notations which will be used in the paper. So, let us denote by

- $X(t)$ – the number of packets present in the system at time t;
- $F^{j*}(\cdot)$ – the j-fold Stieltjes convolution of the distribution function $F(\cdot)$ with itself;

- $f(s) = \int_0^\infty e^{-st}\,\mathrm{d}F(t)$, $\mathrm{Re}(s) > 0$ – the Laplace-Stieltjes transform of the distribution function $F(\cdot)$;
- p_k^{j*} – the kth term of the j-fold convolution of the sequence (p_k) with itself;
- $p(z) = \sum_{k=1}^\infty p_k z^k$, $|z| \le 1$ – the probability generating function of the sequence (p_k);
- $V(\cdot)$ – the distribution function of a single vacation;
- $I\{\mathbb{A}\}$ – the indicator of the random event $\mathbb{A}$;
- I_+ – the positive projection of the Laplace transform, defined for any real function $k(\cdot)$ as

$$I_+\Big[\int_{-\infty}^{\infty} e^{-sx}k(x)\,\mathrm{d}x\Big] = \int_0^\infty e^{-sx}k(x)\,\mathrm{d}x,$$

if only $\int_{-\infty}^{\infty} e^{-\mathrm{Re}(s)x}|k(x)|\,\mathrm{d}x < \infty$.

3. QUEUE-SIZE DISTRIBUTION IN THE “USUAL” SYSTEM

Let us consider the $M^X/G/1$-type “usual” system on its initial “zero” busy period τ_0. The main goal of this section is to find a representation for the Laplace transform of the queue-size distribution conditioned by the number of packets present in the system at the opening, i. e. for the expression

$$\int_0^\infty e^{-\mu t}\mathbf{P}_n^U\{X(t) = m,\, t \in \tau_0\}\,\mathrm{d}t, \quad \mu > 0. \tag{2}$$

In [5] the following theorem was proved:

Theorem 3.1. For arbitrary $0 < \mathrm{Re}(s) < \mu$, $n \ge 1$ and $m \ge 1$ the following formula is true:

$$\begin{aligned}
&\int_{x=0}^\infty e^{-(s-\mu)x}\int_{t=0}^\infty e^{-\mu t}\mathbf{P}_{n+1,x}^U\{X(t) = m,\, t \in \tau_0\}\,\mathrm{d}t\mathrm{d}x \\
&= \frac{1}{\mu(\lambda+\mu)(s-\mu)}\sum_{k=0}^\infty p_{m-(n+1)}^{k*}\Big(\frac{\lambda}{\lambda+\mu}\Big)^k + \sum_{i=0}^n I_+\Big[\frac{\big(f(\mu-s)\big)^{n-i}G(s,m-i)}{s-\mu}\Big] \\
&+ I_+\Bigg[\frac{\lambda\big(f(\mu-s)\big)^n}{(\lambda+s)\varphi_+(s,\mu)}\sum_{k=1}^\infty p_k\sum_{i=0}^{k-1} I_+\Big[\frac{\big(f(\mu-s)\big)^{k-i}G(s,m-i)}{(s-\mu)\varphi_-(s,\mu)}\Big]\Bigg] \\
&- \frac{\varphi_+(\mu,\mu)}{\mu}I_+\Big[\frac{\lambda\big(f(\mu-s)\big)^n}{(\lambda+s)(s-\mu)\varphi_+(s,\mu)}\Big]\sum_{i=0}^m\Big(\frac{\lambda}{\lambda+\mu}\Big)^i\big(p_m^{i*} - p_m^{(i+1)*}\big),
\end{aligned} \tag{3}$$

where

$$G(s,n) = \begin{cases} \frac{1}{\lambda+s}\sum_{k=0}^n\big(p_{n-1}^{k*} - p_n^{k*}\big)\big(\frac{\lambda}{\lambda+s}\big)^k, & n \ge 0, \\ 0, & n < 0. \end{cases}$$

The functions $\varphi_+(\cdot,\cdot)$ and $\varphi_-(\cdot,\cdot)$ on the right side of (3) are connected with the following Wiener–Hopf-type factorization identity (see [4]):

$$1-\frac{\lambda}{\lambda+s}p\big(f(\mu-s)\big)=\varphi_+(s,\mu)\cdot\varphi_-(s,\mu),\quad \mathrm{Re}(s)\in[0,\mu]. \tag{4}$$

The components $\varphi_+(s,\mu)$ and $\varphi_-(s,\mu)$ of the factorization (4) are regular and non-zero functions in half-planes $\mathrm{Re}(s)>0$ and $\mathrm{Re}(s)<\mu$, respectively. Moreover, the following properties hold true (see [3, 4, 14]):

$$\begin{aligned}\varphi_+(s,\mu)&=1+\int_0^\infty e^{-sx}\,\mathrm{d}P_+(x,\mu),\quad \mathrm{Re}(s)\geq 0,\\ \frac{1}{\varphi_+(s,\mu)}&=1+\int_0^\infty e^{-sx}\,\mathrm{d}Q_+(x,\mu),\quad \mathrm{Re}(s)\geq 0\end{aligned}\tag{5}$$

and, similarly,

$$\begin{aligned}\varphi_-(s,\mu)&=1+\int_{-\infty}^0 e^{-sx}\,\mathrm{d}P_-(x,\mu),\quad \mathrm{Re}(s)\leq \mu,\\ \frac{1}{\varphi_-(s,\mu)}&=1+\int_{-\infty}^0 e^{-sx}\,\mathrm{d}Q_-(x,\mu),\quad \mathrm{Re}(s)\leq \mu.\end{aligned}\tag{6}$$

The functions $P_\pm(x,\mu)$ and $Q_\pm(x,\mu)$ have bounded variations for any $\mu>0$ and besides $Q_\pm(0,\mu)=P_\pm(0,\mu)=0$.

In theorem below, basing on the conclusion of Theorem 3.1, we find the explicit representation for (2).

Theorem 3.2. The Laplace transform of the queue-size distribution in the "usual" $M^X/G/1$-type system without vacations, starting with fixed number $n\geq 1$ of packets present just after the opening, on its initial busy period τ_0, is given by the formula

$$\begin{aligned}Q_n^U(m,\mu)&=\int_0^\infty e^{-\mu t}\mathbf{P}_n^U\{X(t)=m,\ t\in\tau_0\}\,\mathrm{d}t\\ &=\frac{1}{\mu(\lambda+\mu)}\sum_{k=0}^\infty p_{m-n}^{k*}\Big(\frac{\lambda}{\lambda+\mu}\Big)^k+W_1(m,n,\mu)\\ &\quad-\frac{\varphi_+(\mu,\mu)}{\mu}\sum_{i=0}^m\Big(\frac{\lambda}{\lambda+\mu}\Big)^i\big(p_m^{i*}-p_m^{(i+1)*}\big)W_2(n,\mu)+W_3(m,n,\mu),\end{aligned}$$

where $\mu>0$, $m\geq 1$, and the functions $W_1(m,n,\mu)$, $W_2(n,\mu)$ and $W_3(m,n,\mu)$ are defined as follows:

$$\begin{aligned}&W_1(m,n,\mu)\\ &=\sum_{i=0}^m\int_{u=-\infty}^0\int_{y=-\infty}^{+0}\int_{t=0}^\infty e^{-\mu t}\Theta(t-y-u,m-i,\mu)\,\mathrm{d}H(t,i)\,\mathrm{d}Q_-^{(0)}(y,\mu)\\ &\times d_u\Big[\int_{z=-\min(0,u)}^\infty e^{-\mu z}\int_{v=-0}^{u+z}\big(1-e^{-\lambda(u+z-v)}\big)\,\mathrm{d}Q_+^{(0)}(v,\mu)\,\mathrm{d}F^{n*}(z)\Big],\end{aligned}\tag{7}$$

$$W_2(n,\mu) = \int_{v=-\infty}^{0} e^{\mu v} \int_{y=-0}^{-v} e^{-\mu y}\,\mathrm{d}Q_+^{(0)}(y,\mu) d_v\Big(\int_{z=-\min(0,v)}^{\infty} (1-e^{-\lambda(v+z)})e^{-\mu z}\,\mathrm{d}F^{n*}(z)\Big) \quad (8)$$

and

$$W_3(m,n,\mu) = \sum_{i=0}^{n} \int_0^{\infty} e^{-\mu y}\Theta(y,m-i,\mu)\,\mathrm{d}F^{(n-i)*}(y), \quad (9)$$

where

$$\begin{aligned} Q_+^{(0)}(x,\mu) &= I\{x>0\} + Q_+(x,\mu),\\ Q_-^{(0)}(x,\mu) &= -I\{x<0\} + Q_-(x,\mu), \end{aligned} \quad (10)$$

$$\Theta(x,k,\mu) = e^{-\lambda x}\sum_{i=0}^{k}\big(p_{k-1}^{i*} - p_k^{i*}\big)\int_0^{\infty} e^{-(\lambda+\mu)y}\frac{\big(\lambda(x+y)\big)^i}{i!}\,\mathrm{d}y \quad (11)$$

and

$$H(t,k) = \sum_{i=k+1}^{\infty} p_i F^{(i-k)*}(t). \quad (12)$$

Here and further the notation $d_x(\ldots)$ indicates that the Stieltjes integration is executed with respect to the argument x.

P r o o f. Let us invert the expression on the right side of (3) on the argument s. Note that the following identity holds true:

$$\sum_{i=0}^{m}\sum_{k=i+1}^{\infty} p_k I_+\Big[\frac{G(s,m-i)\big(f(\mu-s)\big)^{k-i}}{(s-\mu)\varphi_-(s,\mu)}\Big] = \sum_{i=0}^{m}\int_{x=0}^{\infty} e^{-sx}\int_{y=-\infty}^{+0}\int_{t=0}^{\infty} e^{-\mu t}\Theta(x+t-y,m-i,\mu)\,\mathrm{d}H(t,i)\,\mathrm{d}Q_-^{(0)}(y,\mu)\,\mathrm{d}x, \quad (13)$$

where $\Theta(x,k,\mu)$ and $H(t,k)$ were defined in (11) and (12), respectively.

Moreover, we have

$$\begin{aligned} &\frac{\lambda\big(f(\mu-s)\big)^n}{(\lambda+s)\varphi_+(s,\mu)}\\ &= \int_{x=-\infty}^{\infty} e^{-sx}\,\mathrm{d}_x\Big[\int_{y=-\min(0,x)}^{\infty}(1-e^{-\lambda(x+y)})e^{-\mu y}\,\mathrm{d}F^{n*}(y)\Big]\cdot\int_{-0}^{\infty} e^{-sv}\,\mathrm{d}Q_+^{(0)}(v,\mu)\\ &= \int_{x=-\infty}^{\infty} e^{-sx}\,\mathrm{d}_x\Big[\int_{y=-\min(0,x)}^{\infty} e^{-\mu y}\int_{v=-0}^{x+y}(1-e^{-\lambda(x+y-v)})\,\mathrm{d}Q_+^{(0)}(v,\mu)\,\mathrm{d}F^{n*}(y)\Big]. \end{aligned} \quad (14)$$

Now, the formulae (13) and (14) lead to

$$
\begin{aligned}
&I_+\Bigg[\frac{\lambda\big(f(\mu-s)\big)^n}{(\lambda+s)\varphi_+(s,\mu)}\sum_{k=1}^{\infty}p_k\sum_{i=0}^{k-1}I_+\Big[\frac{\big(f(\mu-s)\big)^{k-i}G(s,m-i)}{(s-\mu)\varphi_-(s,\mu)}\Big]\Bigg]\\
&=\sum_{i=0}^{m}\int_{x=0}^{\infty}e^{-sx}\int_{u=-\infty}^{x}\int_{y=-\infty}^{+0}\int_{t=0}^{\infty}e^{-\mu t}\Theta(x+t-y-u,m-i,\mu)\,\mathrm{d}H(t,i)dQ_-^{(0)}(y,\mu)\\
&\times d_u\Big[\int_{z=-\min(0,u)}^{\infty}e^{-\mu z}\int_{v=-0}^{u+z}(1-e^{-\lambda(u+z-v)})\,\mathrm{d}Q_+^{(0)}(v,\mu)\,\mathrm{d}F^{n*}(z)\Big]\,\mathrm{d}x\\
&=\int_0^{\infty}e^{-sx}W_1(x,m,n,\mu)\,\mathrm{d}x. \qquad (15)
\end{aligned}
$$

Next, from the obvious representation

$$
\frac{\lambda}{\lambda+s}\big(f(\mu-s)\big)^n=\int_{x=-\infty}^{\infty}e^{-sx}\,\mathrm{d}_x\Big[\int_{y=-\min(0,x)}^{\infty}(1-e^{-\lambda(x+y)})e^{-\mu y}\,\mathrm{d}F^{n*}(y)\Big],
$$

we derive the formula

$$
\begin{aligned}
&I_+\Big[\frac{\lambda\big(f(\mu-s)\big)^n}{(\lambda+s)(s-\mu)\varphi_+(s,\mu)}\Big]=\int_{x=0}^{\infty}e^{-(s-\mu)x}\int_{v=-\infty}^{x}e^{\mu v}\int_{y=-0}^{x-v}e^{-\mu y}\,\mathrm{d}Q_+^{(0)}(y,\mu)\\
&\times d_v\Big(\int_{z=-\min(0,v)}^{\infty}(1-e^{-\lambda(v+z)})e^{-\mu z}\,\mathrm{d}F^{n*}(z)\Big)\,\mathrm{d}x=\int_0^{\infty}e^{-sx}W_2(x,n,\mu)\,\mathrm{d}x. \qquad (16)
\end{aligned}
$$

It is easy to verify that

$$
\frac{G(s,k)-G(\mu,k)}{\mu-s}=\int_0^{\infty}e^{-sx}\Theta(x,k,\mu)\,\mathrm{d}x,
$$

where $\Theta(x,k,\mu)$ was introduced in (11).

From the definition of the projection I_+ follows that for any j

$$
I_+\Big[\frac{(f(\mu-s))^j}{\mu-s}\Big]=0.
$$

Hence we obtain

$$
\begin{aligned}
&\sum_{i=0}^{n}I_+\left[\frac{\big(f(\mu-s)\big)^{n-i}G(s,m-i)}{\mu-s}\right]\\
&=\sum_{i=0}^{n}I_+\left[\frac{\big(f(\mu-s)\big)^{n-i}\big(G(s,m-i)-G(\mu,m-i)\big)}{\mu-s}\right]\\
&=\sum_{i=0}^{n}\int_{x=0}^{\infty}e^{-sx}\int_{y=0}^{\infty}e^{-\mu y}\Theta(x+y,m-i,\mu)F^{(n-i)*}(y)\,\mathrm{d}x\\
&=\int_0^{\infty}e^{-sx}W_3(x,m,n,\mu)\,\mathrm{d}x. \qquad (17)
\end{aligned}
$$

Let us note that for any random event $\mathbb{A}$ the following relationship is true:

$$\lim_{x\downarrow 0}\mathbf{P}^U_{n+1,x}\{\mathbb{A}\} = \mathbf{P}^U_n\{\mathbb{A}\}, \quad n \geq 1. \tag{18}$$

As a consequence, we can observe that

$$\lim_{x\downarrow 0} W_1(x,m,n,\mu) = W_1(m,n,\mu),$$
$$\lim_{x\downarrow 0} W_2(x,n,\mu) = W_2(n,\mu)$$

and

$$\lim_{x\downarrow 0} W_3(x,m,n,\mu) = W_3(m,n,\mu),$$

where $W_1(m,n,\mu)$, $W_2(n,\mu)$ and $W_3(m,n,\mu)$ were defined in (7), (8) and (9), respectively. □

In [5] one can find the following formula (in a slightly another form) for the Laplace transform of the queue-size distribution in the "usual" $M^X/G/1$-type system working under the "standard" initial condition, on its first "zero" busy period:

Theorem 3.3. For any $\mu > 0$ and $m \geq 1$ the following representation holds true:

$$Q^U_{std}(m,\mu) = \int_0^\infty e^{-\mu t}\mathbf{P}^U_{std}\{X(t) = m,\, t \in \tau_0\}\,\mathrm{d}t = \frac{\varphi_+(\mu,\mu)}{\lambda+\mu}\sum_{k=0}^{m-1} p_m^{(k+1)*}\Big(\frac{\lambda}{\lambda+\mu}\Big)^k$$
$$+\sum_{k=0}^{m}\int_{y=-\infty}^{+0}\int_{t=0}^{\infty} e^{-\mu t}\Theta(t-y, m-k,\mu)\,\mathrm{d}H(t,k)dQ^{(0)}_-(y,\mu), \tag{19}$$

where the formulae for functions $Q^{(0)}_-(\cdot,\cdot)$, $\Theta(\cdot,\cdot,\cdot)$ and $H(\cdot,\cdot)$ are given in (10), (11) and (12), respectively.

4. QUEUE-SIZE DISTRIBUTION IN THE MODIFIED SYSTEM

In this section we derive the formula for the Laplace transform of the queue-size distribution in the modified system with vacations, on its first vacation cycle C_1. In fact, we prove the following theorem:

Theorem 4.1. The Laplace transform of the queue-size distribution in the modified $M^X/G/1$-type system on its first vacation cycle C_1 is following:

$$Q_M(m,\mu) = \int_0^\infty e^{-\mu t}\mathbf{P}_M\{X(t) = m, t \in C_1\}\,\mathrm{d}t$$
$$= I\{m \geq 1\}\Big(\sum_{i=1}^{m} p_m^{i*}\beta_i(\mu) + \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} p_j^{i*}\alpha_i(\mu)Q^U_j(m,\mu)$$
$$+\frac{\lambda}{\lambda+\mu}\alpha_0(\mu)Q^U_{std}(m,\mu)\Big) + I\{m=0\}\frac{1}{\lambda+\mu}, \tag{20}$$

where $\mu > 0$, $m \geq 0$, and

$$\alpha_i(\mu) = \int_0^\infty e^{-(\lambda+\mu)x} \frac{(\lambda x)^i}{i!} \, \mathrm{d}V(x), \tag{21}$$

$$\beta_i(\mu) = \int_0^\infty e^{-(\lambda+\mu)x} \frac{(\lambda x)^i}{i!} \big(1 - V(x)\big) \, \mathrm{d}x. \tag{22}$$

P r o o f. Recall that the modified system begins its operation with the vacation time of the first vacation cycle C_1. Let us note that the following formula is true:

$$\mathbf{P}_M\{X(t) = m,\, t \in C_1\} = \sum_{i=1}^{4} \mathbf{P}_M\{(X(t) = m,\, t \in C_1) \cap A_i\},$$

where A_i, $i = 1, 2, 3, 4$, are the following random events:

- A_1 — t is "inside" C_1 and the first arrival occurs before t, but the first vacation v_1 ends after t;
- A_2 — t is "inside" C_1, the first arrival occurs before t during the first vacation v_1, and the vacation also ends before t (so, t is in the first busy period τ_1 of the modified system);
- A_3 — t is "inside" C_1 and the first arrival occurs before t but after the completion epoch of the first vacation v_1 (the first arrival occurs when the system is on standby and waits for packets);
- A_4 — t is "inside" C_1 and the first arrival occurs after t.

It is intuitively clear that in the case of A_2 the analysis of the modified system can be reduced to the case of the vacationless "usual" system with fixed number of packets just after the opening, and in the case of A_3 we can use the corresponding "usual" system working in the "standard" regime. Indeed, we have

$$\mathbf{P}_M\{(X(t) = m,\, t \in C_1) \cap A_1\} = I\{m \geq 1\} \sum_{i=1}^{m} p_m^{i*} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \big(1 - V(t)\big), \tag{23}$$

$$\begin{aligned} \mathbf{P}_M\{(X(t) = m,\, t \in C_1) \cap A_2\} &= I\{m \geq 1\} \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} p_j^{i*} \\ &\times \int_0^t \frac{(\lambda x)^i}{i!} e^{-\lambda x} \mathbf{P}_j^U\{X(t-x) = m,\, t - x \in \tau_0\} \, \mathrm{d}V(x), \end{aligned} \tag{24}$$

$$\begin{aligned} &\mathbf{P}_M\{(X(t) = m,\, t \in C_1) \cap A_3\} \\ &\quad = I\{m \geq 1\} \lambda \int_0^t e^{-\lambda y} V(y) \mathbf{P}_{std}^U\{X(t-y) = m,\, t - y \in \tau_0\} \, \mathrm{d}y, \end{aligned} \tag{25}$$

$$\mathbf{P}_M\{(X(t) = m,\, t \in C_1) \cap A_4\} = I\{m = 0\} e^{-\lambda t}. \tag{26}$$

Now, by introducing sequences (21)–(22), the representations (23)–(26) lead to (20). □

For the case of the system with individual arrivals, in which $p_1 = 1$ and $p_k = 0$ for $k \geq 2$, we obtain the following corollary:

Corollary 4.2. For $\mu > 0$ and $m \geq 0$ in the case of single arrivals the following formula is true:

$$\begin{aligned} Q_M(m,\mu) =& I\{m \geq 1\}\Big(\beta_m(\mu) + \sum_{i=1}^{\infty} \alpha_i(\mu) Q_i^U(m,\mu) \\ &+ \frac{\lambda}{\lambda+\mu}\alpha_0(\mu) Q_1^U(m,\mu)\Big) + I\{m=0\}\frac{1}{\lambda+\mu}, \end{aligned}$$

where, of course, the proper "usual" system without vacations is also characterizing by single arrivals and, besides, $Q_{std}^U(m,\mu) = Q_1^U(m,\mu)$.

5. GENERAL RESULTS FOR THE TRANSIENT QUEUE-SIZE DISTRIBUTION

Consider the original system with single vacations working in the "standard" regime. From the memoryless property of exponential distribution of interarrival times follows that the sequence C_k, $k \geq 0$, of successive vacation cycles forms a delayed renewal process. Let us denote by $B_0(\cdot)$ and $B_1(\cdot)$ distribution functions of random variables C_0 and C_1, respectively.

Theorem 5.1. The representation for the Laplace transform of the queue-size distribution in the $M^X/G/1$ system with single vacations, working in the "standard" regime is following:

$$\int_0^{\infty} e^{-\mu t}\mathbf{P}\{X(t)=m\}\,\mathrm{d}t = Q_{std}^U(m,\mu) + Q_M(m,\mu)\frac{b_0(\mu)}{1-b_1(\mu)}, \tag{27}$$

where $\mu > 0$, $m \geq 0$, $Q_{std}^U(m,\mu)$ and $Q_M(m,\mu)$ are taken from (19) and (20), respectively, and $b_0(\mu)$ and $b_1(\mu)$ are Laplace-Stieltjes transforms of distribution functions $B_0(\cdot)$ and $B_1(\cdot)$, respectively.

P r o o f. Note that the following formula is true:

$$\begin{aligned} &\int_0^{\infty} e^{-\mu t}\mathbf{P}\{X(t)=m\}\,\mathrm{d}t = \sum_{k=0}^{\infty}\int_0^{\infty} e^{-\mu t}\mathbf{P}\{X(t)=m,\, t \in C_k\}\,\mathrm{d}t \\ &= \int_0^{\infty} e^{-\mu t}\mathbf{P}_{std}^U\{X(t)=m,\, t\in\tau_0\}\,\mathrm{d}t \\ &+ \sum_{k=1}^{\infty}\int_0^{\infty} e^{-\mu t}\int_0^t \mathbf{P}_M\{X(t-y)=m,\, t-y\in C_1\}\,\mathrm{d}(B_0 * B_1^{(k-1)*})(y)\,\mathrm{d}t. \end{aligned} \tag{28}$$

In consequence, (28) leads to (27).

To complete the proof we need the formulae for $b_0(\mu)$ and $b_1(\mu)$. One can find in [9] the following result:

$$b_0(\mu) = \mathbf{E}_{std}^U\{e^{-\mu\tau_0}\} = 1 - \varphi_+(0,\mu).$$

Moreover, let us note that the formula of total probability gives

$$b_1(\mu) = \sum_{i=1}^{\infty}\sum_{n=i}^{\infty} p_n^{i*}\mathbf{E}_n^U\{e^{-\mu\tau_0}\}\int_0^{\infty} e^{-(\lambda+\mu)y}\frac{(\lambda y)^i}{i!}\,\mathrm{d}V(y)$$
$$+\,\lambda b_0(\mu)\int_0^{\infty} e^{-(\lambda+\mu)y}V(y)\,\mathrm{d}y, \tag{29}$$

where (see [9])

$$\mathbf{E}_n^U\{e^{-\mu\tau_0}\} = \big(f(\mu)\big)^n - \varphi_+(0,\mu)\int_{y=0}^{\infty} e^{-\mu y}\int_{v=-0}^{y}\big(1-e^{-\lambda(y-v)}\big)dQ_+^{(0)}(v,\mu)\,\mathrm{d}F^{n*}(y). \tag{30}$$

Indeed, the first summand on the right side of (29) relates to the situation in which there is at least one arrival during the vacation. If the number of packets at the end of the vacation equals n, then the evolution of the system in the following busy period corresponds to the evolution of the "usual" system, that contains n packets initially, on its first "zero" busy period. Similarly, the second summand on the right side of (29) concerns the situation in which the first packet arrives after the vacation. □

Theorem 5.1 gives the representation for the LT of the queue-size distribution using only transforms of crucial "input" distributions of the system and components of the factorization identity of Wiener–Hopf type connected with them, without introducing any additional random walk.

However, let us note that the formula (27), due to its form, can not be a base for obtaining the stationary queue-size distribution using the Tauberian theorem.

6. NUMERICAL RESULTS

It is easy to realize that the efficient numerical application of the results given in Theorems 3.2, 3.3, 4.1 and 5.1 requires finding the components $\varphi_+(s,\mu)$ and $\varphi_-(s,\mu)$ of the factorization identity (4), and the functions $Q_+(x,\mu)$ and $Q_-(x,\mu)$ defined in (5) and (6), respectively. Of course, it is impossible to find the universal formulae for these functions: they essentially depend on the arrival and service processes ("shapes" of distribution functions and their parameters).

Let us consider, as an example, the $M/M/1$-type queueing model with single Poisson arrivals with intensity λ, and with exponential service times with parameter σ. Moreover, let us assume that single vacations are also exponentially distributed with mean θ^{-1}. Thus, we have

$$F(t) = 1-e^{-\sigma t},\quad V(t) = 1-e^{-\theta t},\ \ t>0,$$

and

$$p_1 = 1,\quad p_k = 0,\ \ k\ge 2.$$

Applying the factorization identity (4) to the considered queueing model we get

$$\varphi_+(s,\mu) = \frac{s-s_-(\mu)}{\lambda+s},$$

where

$$s_-(\mu) = \frac{\sigma + \mu - \lambda - \sqrt{(\sigma + \mu - \lambda)^2 + 4\lambda\mu}}{2}.$$

From (5) we have

$$dQ_+(x, \mu) = \big(\lambda + s_-(\mu)\big)e^{s_-(\mu)x}.$$

After simplifications, from (29), we obtain

$$b_1(\mu) = \frac{\theta}{\lambda + \mu + \theta} \sum_{n=1}^{\infty} \mathbf{E}_n^U\{e^{-\mu\tau_0}\}\Big(\frac{\lambda}{\lambda + \mu + \theta}\Big)^n + \frac{\lambda\theta}{(\lambda + \mu)(\lambda + \mu + \theta)} b_0(\mu),$$

where (see (30))

$$\begin{aligned}\mathbf{E}_n^U\{e^{-\mu\tau_0}\} &= \Big(\frac{\sigma}{\sigma + \mu}\Big)^n \\ &- \frac{\varphi_+(0, \mu)\sigma^n}{(n-1)!}\Bigg[\int_0^{\infty} e^{-(\mu+\sigma)y}\big(1 - e^{-\lambda y}\big)y^{n-1}\,\mathrm{d}y \\ &+ \int_{y=0}^{\infty} e^{-(\mu+\sigma)y} \int_{v=0}^{y} \big(1 - e^{-\lambda(y-v)}\big)\,\mathrm{d}Q_+(v, \mu)\,\mathrm{d}y\Bigg].\end{aligned}$$

Finally, we have from (27)

$$\int_0^{\infty} e^{-\mu t}\mathbf{P}\{X(t) = 0\}\,\mathrm{d}t = \frac{b_0(\mu)}{(\lambda + \mu)\big(1 - b_1(\mu)\big)}.$$

Let us investigate the influence of key system parameters λ, σ and θ on the probability $\mathbf{P}\{X(t) = 0\}$ for three different time moments $t = 1$, $t = 10$ and $t = 30$. All computations we execute using the *Mathematica* environment. Besides, to obtain the value of $\mathbf{P}\{X(t) = 0\}$ for the fixed t, we use the algorithm of numerical Laplace transform inversion based on the Bromwich integral, introduced and described in details in [1], taking the values of the operating parameters suggested in [1] i. e. $L = 1$, $A = 19$, $m = 11$ and $n = 38$.

Firstly, let us investigate the case of the fixed $\lambda = \sigma = 1$ (so, the case of the critically loaded system, $\rho = \frac{\lambda}{\sigma} = 1$) and the changing parameter θ of the exponentially distributed vacation time. Probabilities $\mathbf{P}\{X(t) = 0\}$ for three different values of t are given in Table 1 (with a precision of 6 significant digits) and visualized in Figure 1. As one can observe, in general, the higher the value of the parameter θ, the greater the probability $\mathbf{P}\{X(t) = 0\}$. It is intuitively clear: with the increase of the parameter θ, the average length $\frac{1}{\theta}$ of the single vacation decreases and, in consequence, the "chances" that the system will be empty increase, because the service of packets for only a short time is suspended. Of course, at $\rho = 1$, the probabilities $\mathbf{P}\{X(t) = 0\}$ decrease with the passage of time.

Let us take into consideration the dependence of the probability $\mathbf{P}\{X(t) = 0\}$ on the value of the service rate σ. The results for fixed values $\lambda = \theta = 1$ are given in Table 2

No.	Parameter θ	$\mathbf{P}\{X(1)=0\}$	$\mathbf{P}\{X(10)=0\}$	$\mathbf{P}\{X(30)=0\}$
1	0.001	0.264960	0.0109274	1.92741×10^{-3}
2	0.01	0.265162	0.0136904	3.20427×10^{-3}
3	0.1	0.267129	0.0384402	0.0219771
4	0.5	0.274654	0.104000	0.0764955
5	1	0.281872	0.136514	0.0910024
6	1.5	0.287290	0.149944	0.0952199
7	2	0.291399	0.156486	0.0971539
8	5	0.302586	0.165981	0.100139
9	10	0.306633	0.167824	0.100805
10	25	0.308173	0.168414	0.101035

Tab. 1. Probabilities $\mathbf{P}\{X(t)=0\}$ for $\lambda=\sigma=1$ and different θ's.

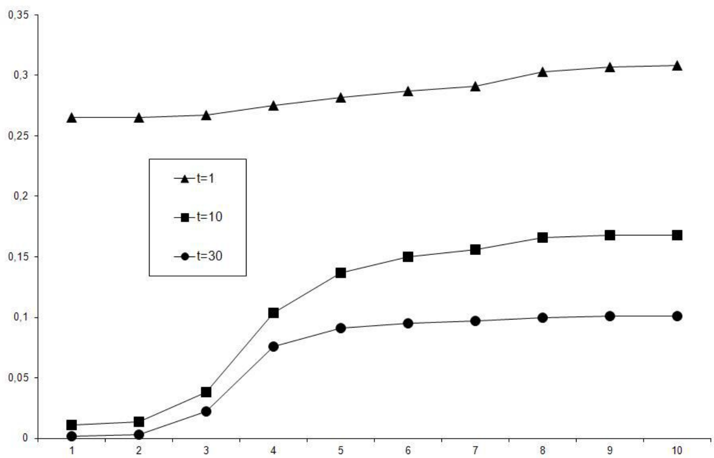

Fig. 1. Probabilities $\mathbf{P}\{X(t)=0\}$ for $\lambda=\sigma=1$ and different θ's.

and presented geometrically in Figure 2. Let us note that for $\rho \geq 1$ the probabilities $\mathbf{P}\{X(t)=0\}$ increase with the increase of the parameter σ, since the mean service time $\frac{1}{\sigma}$ decreases. Besides, obviously, due to the critically load or overload of the system, the probabilities essentially decrease with the passage of time. For $\rho < 1$ the first phenomenon also occurs but the probabilities for different values of t are similar. Indeed, for small values of the traffic load the system relatively quickly stabilizes.

Lastly, let us investigate the dependence of the probabilities $\mathbf{P}\{X(t)=0\}$, for different t's, on the values of the intensity of arrivals λ. The results of computations are presented in Table 3 and in Figure 3.

No.	Parameter σ	ρ	$\mathbf{P}\{X(1)=0\}$	$\mathbf{P}\{X(10)=0\}$	$\mathbf{P}\{X(30)=0\}$
1	0.5	2	0.160800	0.0260793	3.49230×10^{-3}
2	0.8	1.25	0.237612	0.0851513	0.0364590
3	0.9	1.11	0.260390	0.110238	0.0608582
4	1	1	0.281872	0.136514	0.0910024
5	1.5	0.67	0.372072	0.261207	0.249883
6	2	0.5	0.438770	0.350276	0.348107
7	2.5	0.4	0.487863	0.408518	0.406858
8	5	0.2	0.595778	0.529860	0.527123
9	10	0.1	0.636336	0.587221	0.583858
10	20	0.05	0.656812	0.527518	0.524331

Tab. 2. Probabilities $\mathbf{P}\{X(t)=0\}$ for $\lambda=\theta=1$ and different σ's.

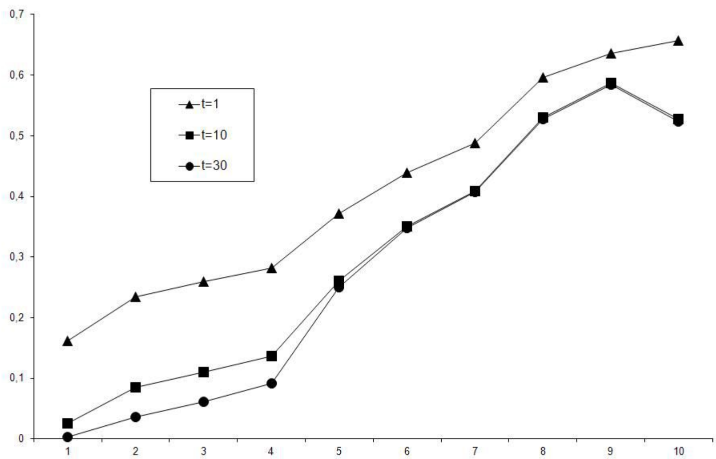

Fig. 2. Probabilities $\mathbf{P}\{X(t)=0\}$ for $\lambda=\theta=1$ and different σ's.

Let us observe that as λ increases, since then the traffic load increases, the probabilities that the system is empty decrease essentially for $t=1$, $t=10$ and $t=30$. Note besides that e. g. the values of $\mathbf{P}\{X(10)=0\}$ and $\mathbf{P}\{X(30)=0\}$ for $\lambda=0.1$ and $\lambda=0.2$ are much smaller than the corresponding probabilities for $t=1$. Indeed, $t=10$ and $t=30$ are then multiples of the mean interarrival times, i. e. $\frac{1}{\lambda}=10$ and $\frac{1}{\lambda}=5$, and hence are close to the "expected" moments of arrivals.

No.	Parameter λ	ρ	$\mathbf{P}\{X(1)=0\}$	$\mathbf{P}\{X(10)=0\}$	$\mathbf{P}\{X(30)=0\}$
1	0.1	0.1	0.583161	0.890725	0.891907
2	0.2	0.2	0.537976	0.772760	0.774534
3	0.5	0.5	0.422281	0.446559	0.439493
4	0.8	0.8	0.331371	0.226070	0.192710
5	0.9	0.9	0.305626	0.176388	0.135365
6	1	1	0.281872	0.136514	0.0910024
7	1.5	1.5	0.187897	0.0356308	7.32826×10^{-3}
8	2	2	0.125282	0.0100606	5.69436×10^{-4}
9	2.5	2.5	0.0834153	3.72751×10^{-3}	1.00379×10^{-4}
10	4	4	0.0244725	6.42696×10^{-4}	5.97557×10^{-6}

Tab. 3. Probabilities $\mathbf{P}\{X(t)=0\}$ for $\sigma=\theta=1$ and different λ's.

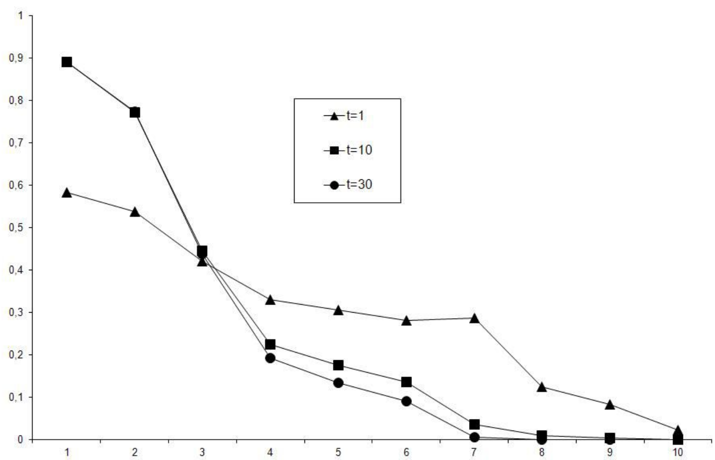

Fig. 3. Probabilities $\mathbf{P}\{X(t)=0\}$ for $\sigma=\theta=1$ and different λ's.

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