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BIVARIATE COPULAS: TRANSFORMATIONS, ASYMMETRY AND MEASURES OF CONCORDANCE

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The present paper introduces a group of transformations on the collection of all bivariate copulas. This group contains an involution which is particularly useful since it provides (1) a criterion under which a given symmetric copula can be transformed into an asymmetric one and (2) a condition under which for a given copula the value of every measure of concordance is equal to zero. The group also contains a subgroup which is of particular interest since its four elements preserve symmetry, the order between two copulas and the value of every measure of concordance.

Keywords: bivariate copulas, transformations, asymmetry, measures of concordance

Classification: 60E05, 62E10, 62H20, 20C99

1. INTRODUCTION

The present paper introduces a group of transformations on the collection of all bivariate copulas. This group has eight elements and is generated by two of its involutions. One of these involutions turns out to be particularly useful since it provides

– a criterion under which a given symmetric copula can be transformed into an asymmetric one and
– a condition under which for a given copula the value of every measure of concordance is equal to zero.

The group also contains a subgroup which is of particular interest since its four elements preserve symmetry, the order between two copulas and the value of every measure of concordance, whereas the other four elements of the group reverse the order and change the sign of the value of a measure of concordance. Besides, using the transformations of this group provides a very straight method for establishing the properties of survival copulas and the Fréchet–Hoeffding bounds and may even be helpful for proving that a function on the unit square is indeed a copula.

The group of transformations of copulas considered here is a realization of the dihedral group with eight elements. It is thus isomorphic to the well–known group of symmetries on the unit square considered by Edwards et al. [4] and by Taylor [11] in connection with measures of concordance; see also Klement et al. [6] and Nelsen [8] (Exercise 2.6),

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who used the group of symmetries on the unit square to define certain transformations of copulas, including the transformation of a copula into its survival copula. Using the group of transformations of copulas is slightly more abstract than using the group of symmetries on the unit square, but it is also much simpler since it avoids the use of the volume measure related to a copula.

As noted before, the present paper contains a contribution to the construction of asymmetric (or nonexchangeable) copulas from a given copula and it will be shown that even certain symmetric copulas can be used to produce asymmetric ones. While most copulas considered in the literature are symmetric, several authors recognized the need for asymmetric copulas and studied the construction of asymmetric copulas and the measurement of asymmetry; see e. g. Nelsen [9], De Baets et al. [1], Liebscher [7] and Durante and Papini [2], to mention only some of the early papers on this topic. According to Durante and Sempi [3], asymmetry should be a principal challenge in the future development of copula theory.

This paper is organized as follows: In Section 2 we introduce and study a group of transformations which, for the sake of convenience, are defined not only for copulas but for real functions on the unit square. In Section 3 we give a detailed discussion of the number of new functions generated by the transformations of the group, with particular attention to symmetry or asymmetry of these new functions. In Section 4 we consider applications of these transformations and their general properties to copulas, and in Section 5 we study the relation between the group of transformations and measures of concordance.

2. A GROUP OF TRANSFORMATIONS OF BIVARIATE REAL FUNCTIONS

Let $\mathcal{M}$ denote the collection of all functions $[0,1]^2 \to \mathbb{R}$. Then $\mathcal{M}$ is an ordered vector space under the coordinatewise defined linear operations and order relation. A function $C \in \mathcal{M}$ is said to be

- *symmetric* if it satisfies $C(u,v) = C(v,u)$ for all $u, v \in [0,1]$ and it is said to be
- *asymmetric* if it is not symmetric.

A map $\varphi : \mathcal{M} \to \mathcal{M}$ is said to be a *transformation* on $\mathcal{M}$.

Let Φ denote the collection of all transformations on $\mathcal{M}$ and define the *composition* $\circ : \Phi \times \Phi \to \Phi$ by letting $(\varphi \circ \psi)(C) := \varphi(\psi(C))$. The composition is associative and the transformation $\iota \in \Phi$ given by $\iota(C) := C$ is called the *identity* on $\mathcal{M}$ and satisfies $\iota \circ \varphi = \varphi = \varphi \circ \iota$ for every $\varphi \in \Phi$. We thus obtain the following result:

Lemma 2.1. $(\Phi, \circ)$ is a semigroup with neutral element ι.

Define now $\pi, \nu_1 : \mathcal{M} \to \mathcal{M}$ by letting

$$\begin{aligned} (\pi(C))(u,v) &:= C(v,u) \\ (\nu_1(C))(u,v) &:= v - C(1-u,v) \end{aligned}$$

and define $\nu_2, \nu, \sigma_1, \sigma_2, \sigma : \mathcal{M} \to \mathcal{M}$ by letting

$$\begin{aligned} \nu_2 &:= \pi \circ \nu_1 \circ \pi \\ \nu &:= \nu_1 \circ \nu_2 \\ \sigma_1 &:= \pi \circ \nu_1 \\ \sigma_2 &:= \pi \circ \nu_2 \\ \sigma &:= \pi \circ \nu. \end{aligned}$$

We are interested in the properties of the subset

$$\Gamma \;:=\; \{\iota, \nu_1, \nu_2, \nu, \pi, \sigma_1, \sigma_2, \sigma\}$$

of Φ. The following result provides representations of the functions $\gamma(C)$ with $C \in \mathcal{M}$ and $\gamma \in \Gamma$ (including the trivial cases for the sake of completeness):

Lemma 2.2. The following identities hold for every function $C \in \mathcal{M}$:

$$\begin{aligned} (\iota(C))(u,v) &= C(u,v) \\ (\nu_1(C))(u,v) &= v - C(1-u,v) \\ (\nu_2(C))(u,v) &= u - C(u,1-v) \\ (\nu(C))(u,v) &= u + v - 1 + C(1-u,1-v) \\ (\pi(C))(u,v) &= C(v,u) \\ (\sigma_1(C))(u,v) &= u - C(1-v,u) \\ (\sigma_2(C))(u,v) &= v - C(v,1-u) \\ (\sigma(C))(u,v) &= u + v - 1 + C(1-v,1-u). \end{aligned}$$

In particular, C is symmetric if and only if $C = \pi(C)$, and in this case $\nu(C)$ and $\sigma(C)$ are symmetric as well.

Proof. The identities for ι, π, ν_1 are immediate from the definitions. Furthermore, we have

$$\begin{aligned} (\nu_2(C))(u,v) &= ((\pi \circ \nu_1 \circ \pi)(C))(u,v) \\ &= (\pi((\nu_1 \circ \pi)(C)))(u,v) \\ &= ((\nu_1 \circ \pi)(C))(v,u) \\ &= (\nu_1(\pi(C)))(v,u) \\ &= u - (\pi(C))(1-v,u) \\ &= u - C(u,1-v). \end{aligned}$$

The proofs of the remaining identities are similar and hence omitted. □

A transformation $\varphi \in \Phi$ is said to be an *involution* if $\varphi \circ \varphi = \iota$.

Lemma 2.3. Each of the transformations $\iota, \nu_1, \nu_2, \nu, \pi, \sigma$ is an involution.

Proof. It is obvious from the definitions that ι and π are involutions. Moreover, we have

$$\begin{aligned}((\nu_1 \circ \nu_1)(C))(u,v) &= (\nu_1(\nu_1(C)))(u,v)\\ &= v - (\nu_1(C))(1-u,v)\\ &= v - (v - C(u,v))\\ &= C(u,v)\end{aligned}$$

which shows that ν_1 is an involution as well. This implies that also $\nu_2 = \pi \circ \nu_1 \circ \pi$ is an involution. Furthermore, Lemma 2.2 yields $\nu_1 \circ \nu_2 = \nu = \nu_2 \circ \nu_1$ and hence $\nu \circ \nu = \nu_1 \circ \nu_2 \circ \nu_2 \circ \nu_1 = \nu_1 \circ \nu_1 = \iota$, which shows that ν is an involution. Finally, we have $\sigma = \pi \circ \nu = \pi \circ \nu_1 \circ \nu_2 = \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi$, which implies that σ is an involution as well. □

The following result provides representations of the transformations in Γ in terms of alternating compositions of π and ν_1 starting with π (including the trivial case for the sake of completeness):

Lemma 2.4. The following identities hold:

$$\begin{aligned}\pi &= \pi\\ \nu_1 \circ \pi &= \sigma_2\\ \pi \circ \nu_1 \circ \pi &= \nu_2\\ \nu_1 \circ \pi \circ \nu_1 \circ \pi &= \nu\\ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi &= \sigma\\ \nu_1 \circ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi &= \sigma_1\\ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi &= \nu_1\\ \nu_1 \circ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi &= \iota.\end{aligned}$$

Proof. The identities for π, ν_2, ν, σ are evident from the definitions. Furthermore, we have

$$\begin{aligned}\nu_1 \circ \pi &= \pi \circ \nu_2\\ \nu_1 \circ \nu_2 &= \nu_2 \circ \nu_1.\end{aligned}$$

This yields

$$\nu_1 \circ \pi = \pi \circ \nu_2 = \sigma_2$$

and

$$\nu_1 \circ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi = \pi \circ \nu_2 \circ \nu_1 \circ \nu_2 = \pi \circ \nu_1 \circ \nu_2 \circ \nu_2 = \pi \circ \nu_1 = \sigma_1$$

and hence

$$\pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi = \pi \circ \sigma_1 = \pi \circ \pi \circ \nu_1 = \nu_1$$

and

$$\nu_1 \circ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi \circ \nu_1 \circ \pi = \nu_1 \circ \nu_1 = \iota$$

which completes the proof. □

Composing the alternating compositions of Lemma 2.4 with π yields representations of the transformations in Γ in terms of alternating compositions of π and ν_1 starting with ν_1 instead of π. The following result is immediate from Lemma 2.4:

Theorem 2.5. $(\Gamma, \circ)$ is a group with neutral element ι and the composition $\circ$ satisfies

$\circ$	ι	ν_1	ν_2	ν	π	σ_1	σ_2	σ
ι	ι	ν_1	ν_2	ν	π	σ_1	σ_2	σ
ν_1	ν_1	ι	ν	ν_2	σ_2	σ	π	σ_1
ν_2	ν_2	ν	ι	ν_1	σ_1	π	σ	σ_2
ν	ν	ν_2	ν_1	ι	σ	σ_2	σ_1	π
π	π	σ_1	σ_2	σ	ι	ν_1	ν_2	ν
σ_1	σ_1	π	σ	σ_2	ν_2	ν	ι	ν_1
σ_2	σ_2	σ	π	σ_1	ν_1	ι	ν	ν_2
σ	σ	σ_2	σ_1	π	ν	ν_2	ν_1	ι

(such that, for example, $\sigma_1 \circ \nu_2 = \sigma$). In particular,

– the group $(\Gamma, \circ)$ is non–commutative and the transformations ι and ν are the only elements which commute with every other transformation in the group,
– the group $(\Gamma, \circ)$ has the non–trivial subgroups

$$\{\iota, \nu_1, \nu_2, \nu\}, \quad \{\iota, \sigma_1, \sigma_2, \nu\}, \quad \{\iota, \pi, \sigma, \nu\}$$

and

$$\{\iota, \nu_1\}, \quad \{\iota, \nu_2\}, \quad \{\iota, \nu\}, \quad \{\iota, \pi\}, \quad \{\iota, \sigma\}$$

which are all commutative, and
– the group $(\Gamma, \circ)$ is generated by each of the sets $\{\pi, \nu_1\}, \{\pi, \nu_2\}, \{\pi, \sigma_1\}, \{\pi, \sigma_2\}, \{\nu_1, \sigma_1\}, \{\nu_1, \sigma_2\}, \{\nu_2, \sigma_1\}, \{\nu_2, \sigma_2\}$.

In the sequel, we denote by γ^{-1} the inverse of $\gamma \in \Gamma$ with respect to the composition.

The table shows that the group Γ is a representation of the *dihedral group* D_4 containing eight elements. The group D_4 may also be represented in a different way which is more popular: Define the transformations $e, h, r : [0,1]^2 \to [0,1]^2$ by letting

$$\begin{aligned} e(u,v) &:= (u,v) \\ h(u,v) &:= (1-u,v) \\ r(u,v) &:= (1-v,u). \end{aligned}$$

Then h is the *reflection* at the line $\{\left(\frac{1}{2}, v\right) \mid v \in [0,1]\}$ and r is the counterclockwise *rotation* by 90°. With regard to the composition $\diamond$ of transformations on $[0,1]^2$, e is the identity and the smallest group $(G, \diamond)$ containing h and r is another representation of the dihedral group D_4.

Corollary 2.6. The groups $(G, \diamond)$ and $(\Gamma, \circ)$ are isomorphic under the isomorphism $g : G \to \Gamma$ satisfying $g(h) = \nu_1$ and $g(r) = \sigma_1$.

The group $(G, \diamond)$ has been used by Edwards et al. [4] and by Taylor [11] in connection with measures of concordance; see Section 5 below. Also, Klement et al. [6] and subsequently Nelsen [8] (Exercise 2.6) used transformations of copulas which can be identified with those of the isomorphic and commutative subgroups $\{e, h, h \diamond r \diamond r, r \diamond r\}$ of $(G, \diamond)$ resp. $\{\iota, \nu_1, \nu_2, \nu\}$ of $(\Gamma, \circ)$.

To complete the discussion of the properties of Γ, let us consider the properties of the transformations in Γ with regard to convexity and the order relation in the ordered vector space $\mathcal{M}$.

Lemma 2.7. Consider $C, D \in \mathcal{M}$ and $a \in (0,1)$. Then the identity

$$\gamma(aC + (1-a)D) = a\gamma(C) + (1-a)\gamma(D)$$

holds for every $\gamma \in \Gamma$.

Proof. Because of Lemma 2.4 it is sufficient to prove the identity for $\gamma \in \{\pi, \nu_1\}$. The identity is obvious for $\gamma = \pi$, and its proof for $\gamma = \nu_1$ is straightforward. □

With regard to the order relation on $\mathcal{M}$, a transformation $\gamma \in \Gamma$ is said to be

- *order preserving* if, for any $C, D \in \mathcal{M}$, $C \leq D$ implies $\gamma(C) \leq \gamma(D)$;
- *order reversing* if, for any $C, D \in \mathcal{M}$, $C \leq D$ implies $\gamma(C) \geq \gamma(D)$.

The following result is evident from Lemma 2.2:

Lemma 2.8. (1) Each of the transformations ι, π, σ, ν is order preserving.

(2) Each of the transformations $\nu_1, \nu_2, \sigma_1, \sigma_2$ is order reversing.

Summarizing Lemmas 2.2, 2.3 and 2.8, we see that the transformations in the subgroup $\{\iota, \pi, \sigma, \nu\}$ of Γ are symmetry and order preserving involutions; for another such property of the subgroup $\{\iota, \pi, \sigma, \nu\}$, see Theorem 5.1 below.

3. ORBITS AND ASYMMETRIC FUNCTIONS

In the present section, we study the group Γ with regard to the construction of asymmetric functions from a symmetric one.

By Lemma 2.2, the application of the transformations in Γ to a function $C \in \mathcal{M}$ produces eight functions in $\mathcal{M}$ which, however, need not be distinct, as will be seen later. To determine the number of distinct functions which are generated by the application of the transformations in Γ to a function $C \in \mathcal{M}$, we have to determine the possible values of the cardinality $|\Gamma(C)|$ of the set

$$\Gamma(C) \quad := \quad \{D \in \mathcal{M} \mid D = \gamma(C) \text{ for some } \gamma \in \Gamma\}$$

which is called the *orbit* of C. More generally, for a subgroup Ψ of Γ, we define

$$\Psi(C) \quad := \quad \{D \in \mathcal{M} \mid D = \psi(C) \text{ for some } \psi \in \Psi\}.$$

The following result is evident:

Lemma 3.1. Consider $C, D \in \mathcal{M}$ and a subgroup Ψ of Γ. Then the following are equivalent:

(a) $D \in \Psi(C)$
(b) $C \in \Psi(D)$.

To study the cardinality of the orbit of a function $C \in \mathcal{M}$, we define the relation $\sim_C$ on Γ by letting

$$\gamma \sim_C \delta$$

for $\gamma, \delta \in \Gamma$ satisfying $\gamma(C) = \delta(C)$. Then $\sim_C$ is an equivalence relation on Γ. For $\gamma \in \Gamma$, we denote by

$$C(\gamma)$$

the equivalence class containing $\gamma \in \Gamma$.

Lemma 3.2. For every $C \in \mathcal{M}$, the equivalence class $C(\iota)$ is a subgroup of Γ and every $\gamma \in \Gamma$ satisfies $|C(\gamma)| = |C(\iota)|$.

P r o o f. It is evident that $C(\iota)$ is a subgroup of Γ. Moreover, for every $\delta \in \Gamma$, the fact that Γ is a group implies that $\delta(C) = \gamma(C)$ if and only if $(\gamma^{-1} \circ \delta)(C) = \iota(C)$. Therefore, the map $T_\gamma : C(\gamma) \to C(\iota)$ given by $T_\gamma(\delta) := \gamma^{-1} \circ \delta$ is a bijection with inverse $(T_\gamma)^{-1} = T_{\gamma^{-1}}$ and this yields $|C(\gamma)| = |C(\iota)|$. □

We thus obtain the following result on the cardinality of the orbit of a function $C \in \mathcal{M}$:

Theorem 3.3. For every $C \in \mathcal{M}$, the cardinality of the orbit of C is equal to the number of equivalence classes of the equivalence relation $\sim_C$ and satisfies

$$|\Gamma(C)| \in \{1, 2, 4, 8\}.$$

P r o o f. It is obvious that the cardinality of the orbit of C is equal to the number of equivalence classes of the equivalence relation $\sim_C$. By Lemma 3.2, all equivalence classes have the same cardinality, and this implies that the number of equivalence classes is $|\Gamma|/|C(\iota)|$. We have $|\Gamma| = 8$ and Theorem 2.5 yields $|C(\iota)| \in \{1, 2, 4, 8\}$. Therefore, we obtain $|\Gamma|/|C(\iota)| \in \{1, 2, 4, 8\}$. □

The following result provides some more precise information on the orbit of a function $C \in \mathcal{M}$ in the cases where one or both of the generators π and ν_1 of the group Γ satisfy $\pi(C) = C$ resp. $\nu_1(C) = C$:

Lemma 3.4. Consider $C \in \mathcal{M}$.

(1) If $\pi(C) = C$, then $\Gamma(C) = \{\iota, \nu_1, \nu_2, \nu\}(C)$ and $|\Gamma(C)| \in \{1, 2, 4\}$.
(2) If $\nu_1(C) = C$, then $\Gamma(C) = \{\iota, \pi, \sigma, \nu\}(C)$ and $|\Gamma(C)| \in \{1, 2, 4\}$.
(3) $\pi(C) = C$ and $\nu_1(C) = C$ if and only if $|\Gamma(C)| = 1$.

Proof. Assertions (1) and (2) follow from Theorem 2.5 and assertion (3) is evident from Lemma 2.4. □

We thus obtain the following result on the orbit of a symmetric or asymmetric function $C \in \mathcal{M}$:

Theorem 3.5. Consider $C \in \mathcal{M}$.

(1) If C is symmetric, then $\nu(C)$ is symmetric as well and $|\Gamma(C)| \in \{1, 2, 4\}$. Moreover,
  - $|\Gamma(C)| = 4$ if and only if $\nu_1(C)$ and $\nu_2(C)$ are asymmetric;
  - $|\Gamma(C)| = 2$ if and only if $\nu_1(C)$ is symmetric and distinct from C.

(2) If C is asymmetric, then $\pi(C)$ is asymmetric as well and distinct from C and $|\Gamma(C)| \in \{2, 4, 8\}$.

Proof. Assume first that C is symmetric. Then we have $\pi(C) = C$. Therefore, we have $(\pi \circ \nu)(C) = (\nu \circ \pi)(C) = \nu(C)$, which means that $\nu(C)$ is symmetric, and Lemma 3.4(1) yields $|\Gamma(C)| \leq 4$.

- In the case $|\Gamma(C)| = 4$ Lemma 3.4(1) yields $\nu_1(C) \neq \nu_2(C)$. Therefore, it then follows from $(\pi \circ \nu_1)(C) = (\nu_2 \circ \pi)(C) = \nu_2(C)$ and $(\pi \circ \nu_2)(C) = (\nu_1 \circ \pi)(C) = \nu_1(C)$ that $\nu_1(C)$ and $\nu_2(C)$ are asymmetric. Conversely, if $\nu_1(C)$ and $\nu_2(C)$ are asymmetric, then $\nu_1(C)$ and $\nu_2(C)$ are both distinct from C, and we also have $\nu_2(C) = (\pi \circ \nu_1 \circ \pi)(C) = (\pi \circ \nu_1)(C) \neq \nu_1(C)$. This yields $3 \leq |\Gamma(C)| \leq 4$ and hence, by Theorem 3.3, $|\Gamma(C)| = 4$.
- In the case $|\Gamma(C)| = 2$, Lemma 3.4(3) yields $\nu_1(C) \neq C$. Therefore, we have $\Gamma(C) = \{C, \nu_1(C)\}$. We also have $\nu_2(C) = \nu_1(C)$, since $\nu_2(C) = C$ implies $\nu_1(C) = (\nu_1 \circ \pi)(C) = (\pi \circ \nu_2)(C) = \pi(C) = C$, which is impossible. This yields $(\pi \circ \nu_1)(C) = (\nu_2 \circ \pi)(C) = \nu_2(C) = \nu_1(C)$, which means that $\nu_1(C)$ is symmetric. Conversely, if $\nu_1(C)$ is symmetric and distinct from C, then we have $\nu_2(C) = (\nu_2 \circ \pi)(C) = (\pi \circ \nu_1)(C) = \nu_1(C)$ and hence $\nu(C) = (\nu_2 \circ \nu_1)(C) = (\nu_2 \circ \nu_2)(C) = C$, and Lemma 3.4(1) yields $|\Gamma(C)| = 2$.

Assume now that C is asymmetric. Then we have $(\pi \circ \pi)(C) = C \neq \pi(C)$. Therefore, $\pi(C)$ is asymmetric and $|\Gamma(C)| \geq 2$. □

Theorem 3.5 implies that for a symmetric function $C \in \mathcal{M}$ it is sufficient to check $\nu_1(C)$ for symmetry in order to decide whether the orbit of C contains an asymmetric function or not:

Corollary 3.6. For every symmetric function $C \in \mathcal{M}$, the following are equivalent:

(a) The orbit of C contains an asymmetric function.
(b) The function $\nu_1(C)$ is asymmetric.

Theorem 3.5 also provides some information on the number of asymmetric functions in the orbit of an arbitrary function $C \in \mathcal{M}$:

Corollary 3.7. Consider $C \in \mathcal{M}$.

(1) If $|\Gamma(C)| = 8$, then every function in the orbit of C is asymmetric.
(2) If $|\Gamma(C)| = 4$, then either two or all of the functions in the orbit of C are asymmetric.
(3) If $|\Gamma(C)| = 2$, then either none or all of the functions in the orbit of C are asymmetric.
(4) If $|\Gamma(C)| = 1$, then C is symmetric.

Proof. Assertion (1) follows from Lemma 3.4. Assume now that $|\Gamma(C)| = 4$.

– If C is symmetric, then, by Theorem 3.5, $\nu(C)$ is symmetric as well and the functions $\nu_1(C)$ and $\nu_2(C)$ are asymmetric. This means that the orbit of C contains exactly two asymmetric functions.
– If C is asymmetric, then, by Theorem 3.5, $\pi(C)$ is asymmetric as well and this implies that the orbit of C contains at least two asymmetric functions. If the orbit of C contains a symmetric function D, then Lemma 3.1 yields $\Gamma(D) = \Gamma(C)$ and it follows from the preceding argument that $\Gamma(D)$ and hence $\Gamma(C)$ contains exactly two asymmetric functions.

Assertion (3) follows from Theorem 3.5 and assertion (4) is evident. □

The assertions of Corollary 3.7 will be illustrated by the examples given in the following section. As can be seen from these examples, there exists

– an asymmetric function $C \in \mathcal{M}$ with $|\Gamma(C)| = 8$ (Example 4.3(1)),
– an asymmetric function $C \in \mathcal{M}$ with $|\Gamma(C)| = 4$ and the property that every function in the orbit of C is asymmetric as well (Example 4.3(2)),
– an asymmetric function $C \in \mathcal{M}$ with $|\Gamma(C)| = 4$ and the property that the orbit of C contains a symmetric function (Example 4.3(3)),
– a symmetric function $C \in \mathcal{M}$ with $|\Gamma(C)| = 4$ and the property that the orbit of C contains an asymmetric function (Example 4.3(4)),
– an asymmetric function $C \in \mathcal{M}$ with $|\Gamma(C)| = 2$ (Example 4.4),
– a symmetric function $C \in \mathcal{M}$ with $|\Gamma(C)| = 2$ (Example 4.2(2)), and
– a (symmetric) function $C \in \mathcal{M}$ with $|\Gamma(C)| = 1$ (Example 4.2(3); see also Lemma 3.8 below).

The analysis of the orbit of a function $C \in \mathcal{M}$ with respect to the cardinality of the orbit and the existence of symmetric or asymmetric functions in the orbit is thus complete.

A function $C \in \mathcal{M}$ is said to be *invariant* under Γ if it satisfies $|\Gamma(C)| = 1$. The following characterization of invariant functions is evident from Lemmas 3.4(3) and 2.7 but deserves an explicit statement:

Lemma 3.8. For every $C \in \mathcal{M}$, the following are equivalent:

(a) C is invariant.
(b) C satisfies $C = \pi(C)$ and $C = \nu_1(C)$.
(c) There exists some $D \in \mathcal{M}$ such that

$$C = \frac{1}{8} \sum_{\gamma \in \Gamma} \gamma(D).$$

In particular, every invariant function is symmetric.

Invariant functions are of particular interest with regard to measures of concordance; see Section 5 below.

4. COPULAS

In this section, we study the group Γ with respect to copulas.

A (*bivariate*) *copula* is a function $C : [0,1]^2 \to [0,1]$ satisfying the following conditions:

(i) The inequality $0 \leq C(x,y) - C(x,v) - C(u,y) + C(u,v)$ holds for all $(u,v),(x,y) \in [0,1]^2$ such that $(u,v) \leq (x,y)$.
(ii) The identity $C(u,0) = 0 = C(0,v)$ holds for all $u,v \in [0,1]$.
(iii) The identities $C(u,1) = u$ and $C(1,v) = v$ hold for all $u,v \in [0,1]$.

We denote by $\mathcal{C}$ the collection of all copulas. Then $\mathcal{C}$ is a convex subset of $\mathcal{M}$ and the following result shows that $\mathcal{C}$ is invariant under Γ:

Theorem 4.1. $\Gamma(\mathcal{C}) = \mathcal{C}$.

Proof. Since $\iota \in \Gamma$, we have $\mathcal{C} \subseteq \Gamma(\mathcal{C})$. To prove the converse inclusion, it is sufficient to prove that, for every copula $C \in \mathcal{C}$, the functions $\pi(C)$ and $\nu_1(C)$ are copulas as well since, by Lemma 2.4, every element of Γ is an alternating composition of π and ν_1. For a copula $C \in \mathcal{C}$, it is evident that $\pi(C)$ is a copula as well, and it is thus sufficient to prove that also $\nu_1(C)$ is a copula.
To this end, consider first $(u,v),(x,y) \in [0,1]^2$ such that $(u,v) \leq (x,y)$. Then we have $(1-x,v) \leq (1-u,y)$ and hence

$$\begin{aligned}
&(\nu_1(C))(x,y) - (\nu_1(C))(x,v) - (\nu_1(C))(u,y) + (\nu_1(C))(u,v) \\
&\quad= (y - C(1-x,y)) - (v - C(1-x,v)) - (y - C(1-u,y)) + (v - C(1-u,v)) \\
&\quad= C(1-u,y) - C(1-u,v) - C(1-x,y) + C(1-x,v) \\
&\quad\geq 0
\end{aligned}$$

which proves (i). Also, for $u,v \in [0,1]$ we have $(\nu_1(C))(u,0) = 0 - C(1-u,0) = 0$ and $(\nu_1(C))(0,v) = v$ as well as $(\nu_1(C))(u,1) = 1 - C(1-u,1) = 1 - (1-u) = u$ and $(\nu_1(C))(1,v) = v$, which proves (ii) and (iii). □

Theorem 4.1 provides an efficient tool for establishing the basic properties of the standard copulas:

Example 4.2. (Standard copulas)

(1) *Survival copulas:* For a copula C, the function $\widehat{C} : [0,1]^2 \to \mathbb{R}$ given by

$$\widehat{C}(u,v) := u + v - 1 + C(1-u,1-v)$$

is called the *survival copula* of C. By Lemma 2.2 we have $\widehat{C} = \nu(C)$, and it now follows from Theorem 4.1 that the survival copula of C is a copula.

(2) *Fréchet–Hoeffding bounds:* Each of the functions $W, M : [0,1]^2 \to \mathbb{R}$ given by

$$\begin{aligned} W(u,v) &:= \max\{u+v-1, 0\} \\ M(u,v) &:= \min\{u,v\} \end{aligned}$$

is a symmetric copula with $\Gamma(W) = \{W, M\} = \Gamma(M)$, and every copula $C \in \mathcal{C}$ satisfies

$$W \leq C \leq M.$$

Indeed, the functions W and M are symmetric, and it is easy to see that M is a copula and that every copula C satisfies $C \leq M$. Moreover, we have $W(u,v) = \max\{u+v-1, 0\} = v - \min\{1-u, v\} = v - M(1-u, v) = (\nu_1(M))(u,v)$ and hence

$$W = \nu_1(M).$$

It now follows from Theorem 4.1 that W is a copula. Moreover, since W and M are symmetric, it follows that $\nu_2(W) = (\pi \circ \nu_1 \circ \pi)(W) = M$ and hence $\nu(W) = (\nu_1 \circ \nu_2)(W) = W$. This yields $\Gamma(W) = \{W, M\} = \Gamma(M)$.
Furthermore, for every copula C, Theorem 4.1 asserts that $\nu_1(C)$ is a copula as well and hence satisfies $\nu_1(C) \leq M$, and it now follows from Lemma 2.8 and the identity established before that $C = (\nu_1 \circ \nu_1)(C) = \nu_1(\nu_1(C)) \geq \nu_1(M) = W$.

(3) *Product copula:* The function $\Pi \in \mathcal{M}$ given by

$$\Pi(u,v) := uv$$

is a symmetric copula with $|\Gamma(C)| = 1$; in particular, Π is invariant.

Theorem 3.3 asserts that the orbit of a copula C satisfies $|\Gamma(C)| \in \{1, 2, 4, 8\}$. The product copula satisfies $|\Gamma(\Pi)| = 1$, and the following examples show that also the other possible values of the cardinality of the orbit of a copula can be attained:

Example 4.3. (Distortions of the product copula)

(1) The function $C \in \mathcal{M}$ given by $C(u,v) := uv - u^2v(1-u)(1-v)$ is a copula and satisfies

$$\begin{aligned} (\iota(C))(u,v) &= uv - u^2v(1-u)(1-v) \\ (\nu_1(C))(u,v) &= uv + uv(1-u)^2(1-v) \\ (\nu_2(C))(u,v) &= uv + u^2v(1-u)(1-v) \\ (\nu(C))(u,v) &= uv - uv(1-u)^2(1-v) \\ (\pi(C))(u,v) &= uv - uv^2(1-u)(1-v) \\ (\sigma_1(C))(u,v) &= uv + uv(1-u)(1-v)^2 \\ (\sigma_2(C))(u,v) &= uv + uv^2(1-u)(1-v) \\ (\sigma(C))(u,v) &= uv - uv(1-u)(1-v)^2 \end{aligned}$$

and hence $|\Gamma(C)| = 8$.

(2) The function $C \in \mathcal{M}$ given by $C(u,v) := uv + uv(1-u)(1-v)(1-2u)/2$ is a copula and satisfies

$$\begin{array}{rcccl} (\iota(C))(u,v) &=& uv + uv(1-u)(1-v)(1-2u)/2 &=& (\nu_1(C))(u,v) \\ (\nu(C))(u,v) &=& uv - uv(1-u)(1-v)(1-2u)/2 &=& (\nu_2(C))(u,v) \\ (\pi(C))(u,v) &=& uv + uv(1-u)(1-v)(1-2v)/2 &=& (\sigma_1(C))(u,v) \\ (\sigma(C))(u,v) &=& uv - uv(1-u)(1-v)(1-2v)/2 &=& (\sigma_2(C))(u,v) \end{array}$$

and hence $|\Gamma(C)| = 4$.

(3) The function $C \in \mathcal{M}$ given by $C(u,v) := uv - u^2v(1-u)(1-v)^2$ is a copula and satisfies

$$\begin{array}{rcccl} (\iota(C))(u,v) &=& uv - u^2v(1-u)(1-v)^2 &=& (\sigma(C))(u,v) \\ (\nu_1(C))(u,v) &=& uv + uv(1-u)^2(1-v)^2 &=& (\sigma_1(C))(u,v) \\ (\nu_2(C))(u,v) &=& uv + u^2v^2(1-u)(1-v) &=& (\sigma_2(C))(u,v) \\ (\nu(C))(u,v) &=& uv - uv^2(1-u)^2(1-v) &=& (\pi(C))(u,v) \end{array}$$

and hence $|\Gamma(C)| = 4$.

(4) The function $C \in \mathcal{M}$ given by $C(u,v) := uv + u^2v^2(1-u)(1-v)$ is a symmetric copula and satisfies

$$\begin{array}{rcl} (\iota(C))(u,v) &=& uv + u^2v^2(1-u)(1-v) \\ (\nu_1(C))(u,v) &=& uv - uv^2(1-u)^2(1-v) \\ (\nu_2(C))(u,v) &=& uv - u^2v(1-u)(1-v)^2 \\ (\nu(C))(u,v) &=& uv + uv(1-u)^2(1-v)^2 \end{array}$$

and hence $|\Gamma(C)| = |\{\iota, \nu_1, \nu_2, \nu\}(C)| = 4$.

(5) The function $C \in \mathcal{M}$ given by $C(u,v) := uv - uv(1-u)(1-v)$ is a symmetric copula and satisfies

$$\begin{array}{rcccl} (\iota(C))(u,v) &=& uv - uv(1-u)(1-v) &=& (\nu(C))(u,v) \\ (\nu_1(C))(u,v) &=& uv + uv(1-u)(1-v) &=& (\nu_2(C))(u,v) \end{array}$$

and hence $|\Gamma(C)| = |\{\iota, \nu_1, \nu_2, \nu\}(C)| = 2$.

In (1) and (2), all copulas in the orbit of C are asymmetric. The copula C considered in (3) is asymmetric while that in (4) is symmetric, but both copulas have the same orbit. In (5), all copulas in the orbit of C are symmetric. These observations are in accordance with the results of Section 3.

These examples illustrate a benefit resulting from the use of transformations in Γ: For a function $C \in \mathcal{M}$, it is usually easy to check whether or not properties (ii) and (iii) of a copula are fulfilled. By contrast, is it not always evident that property (i) of a copula is fulfilled as well or even that $C(u,v) \in [0,1]$ holds for all $(u,v) \in [0,1]^2$; see

Example 4.3(4). In such a case, it may be helpful to show that the orbit of C contains a copula since, by Theorem 4.1, the function C is a copula if and only if some (and hence each) of the functions in its orbit is a copula.

To complete the examples with regard to the orbit of a copula, we present an example of an asymmetric copula $C \in \mathcal{C}$ with $|\Gamma(C)| = 2$:

Example 4.4. (Asymmetric copula with minimal orbit) We define a function $C \in \mathcal{M}$ as follows:

– For all $u, v \in [0,1]$ define

$$\begin{array}{llllllll} C(u,0) & := & 0 & C\big(u,\tfrac{1}{2}\big) & := & \tfrac{u}{2} & C(u,1) & := & u \\ C(0,v) & := & 0 & C\big(\tfrac{1}{2},v\big) & := & \tfrac{v}{2} & C(1,v) & := & v. \end{array}$$

– Define

$$\begin{array}{llll} C\big(\tfrac{1}{6},\tfrac{1}{6}\big) := \tfrac{1}{36} & C\big(\tfrac{1}{6},\tfrac{2}{6}\big) := \tfrac{1}{36} & C\big(\tfrac{1}{6},\tfrac{4}{6}\big) := \tfrac{5}{36} & C\big(\tfrac{1}{6},\tfrac{5}{6}\big) := \tfrac{5}{36} \\ C\big(\tfrac{2}{6},\tfrac{1}{6}\big) := \tfrac{3}{36} & C\big(\tfrac{2}{6},\tfrac{2}{6}\big) := \tfrac{4}{36} & C\big(\tfrac{2}{6},\tfrac{4}{6}\big) := \tfrac{8}{36} & C\big(\tfrac{2}{6},\tfrac{5}{6}\big) := \tfrac{9}{36} \\ C\big(\tfrac{4}{6},\tfrac{1}{6}\big) := \tfrac{3}{36} & C\big(\tfrac{4}{6},\tfrac{2}{6}\big) := \tfrac{8}{36} & C\big(\tfrac{4}{6},\tfrac{4}{6}\big) := \tfrac{16}{36} & C\big(\tfrac{4}{6},\tfrac{5}{6}\big) := \tfrac{21}{36} \\ C\big(\tfrac{5}{6},\tfrac{1}{6}\big) := \tfrac{5}{36} & C\big(\tfrac{5}{6},\tfrac{2}{6}\big) := \tfrac{11}{36} & C\big(\tfrac{5}{6},\tfrac{4}{6}\big) := \tfrac{19}{36} & C\big(\tfrac{5}{6},\tfrac{5}{6}\big) := \tfrac{25}{36}. \end{array}$$

– Define

$$S \quad := \quad \big\{0, \tfrac{1}{6}, \tfrac{2}{6}, \tfrac{3}{6}, \tfrac{4}{6}, \tfrac{5}{6}, 1\big\}^2.$$

By the previous definitions, $C(u,v)$ is, in particular, defined for every $(u,v) \in S$. Moreover, for every $(u,v) \in [0,1]^2$ for which $C(u,v)$ has not yet been defined, there exist unique $(s,t),(x,y) \in S$ satisfying $(s,t) \le (u,v) \le (x,y)$ and $x - s = \frac{1}{6} = y - t$; this implies that there exist $a, b, c, d \in [0,1]$ satisfying $a + b + c + d = 1$ and $(u,v) = a(s,t) + b(s,y) + c(x,t) + d(x,y)$ and we thus define

$$C(u,v) \quad := \quad a\,C(s,t) + b\,C(s,y) + c\,C(x,t) + d\,C(x,y).$$

It now follows from Nelsen [8] (Proof of Lemma 2.3.5) that C is a copula, and it follows from the definition of C that this copula is asymmetric and hence satisfies $C \neq \pi(C)$. Moreover, it is straightforward to show that, for a convex combination $(u,v) = a(s,t) + b(s,y) + c(x,t) + d(x,y)$, the identity

$$(\gamma(C))(u,v) \quad = \quad a\,(\gamma(C))(s,t) + b\,(\gamma(C))(s,y) + c\,(\gamma(C))(x,t) + d\,(\gamma(C))(x,y)$$

holds for every $\gamma \in \{\iota, \nu_1, \nu_2, \nu\}$. This yields $C(\iota) = \{\iota, \nu_1, \nu_2, \nu\}$ and hence $C(\pi) = \{\pi, \sigma_1, \sigma_2, \sigma\}$. Since $\pi(C) \neq C$, it now follows from Theorem 3.3 that $|\Gamma(C)| = 2$.

On certain parts of the interior of the unit square, the copula considered in the previous example coincides with the product copula; it is thus another distortion of the product copula. Moreover, the copula C of Example 4.4 satisfies neither $C \le \Pi$ nor $C \ge \Pi$, whereas each of the copulas C considered in Examples 4.3 has the remarkable property that every copula D in the orbit of C satisfies $D \le \Pi$ or $D \ge \Pi$. This observation suggests the following result:

Theorem 4.5. Consider $C \in \mathcal{C}$ satisfying $C \leq \Pi$ or $C \geq \Pi$. Then C has the following properties:

(1) The number of copulas $D \in \Gamma(C)$ satisfying $D \leq \Pi$ is equal to the number of copulas $D \in \Gamma(C)$ satisfying $D \geq \Pi$.
(2) If $\gamma \in C(\iota)$ holds for some $\gamma \in \{\nu_1, \nu_2, \sigma_1, \sigma_2\}$, then $C = \Pi$.

Proof. Assume first that there exists some $\gamma \in C(\iota)$ satisfying $\gamma \in \{\nu_1, \nu_2, \sigma_1, \sigma_2\}$. In the case $C \leq \Pi$ it follows from Example 4.2(3) and Lemma 2.8 that

$$\Pi = \gamma(\Pi) \leq \gamma(C) = \iota(C) = C \leq \Pi$$

and in the case $C \geq \Pi$ it follows that

$$\Pi = \gamma(\Pi) \geq \gamma(C) = \iota(C) = C \geq \Pi.$$

In either case we obtain $C = \Pi$, which proves (2).
Assume now that $C(\iota) \cap \{\nu_1, \nu_2, \sigma_1, \sigma_2\} = \emptyset$. Since $C(\iota)$ is a subgroup of Γ, Theorem 2.5 implies that $C(\iota)$ is one of the sets $\{\iota, \pi, \sigma, \nu\}, \{\iota, \nu\}, \{\iota, \pi\}, \{\iota, \sigma\}, \{\iota\}$. For each of these cases we determine the equivalence classes of Γ with respect to $\sim_C$:

– In the case $C(\iota) = \{\iota, \pi, \sigma, \nu\}$, the equivalence classes of $\sim_C$ are $\{\iota, \pi, \sigma, \nu\}$ and $\{\nu_1, \nu_2, \sigma_1, \sigma_2\}$.
– In the case $C(\iota) = \{\iota, \nu\}$, the equivalence classes of $\sim_C$ are $\{\iota, \nu\}, \{\pi, \sigma\}$, $\{\nu_1, \nu_2\}$ and $\{\sigma_1, \sigma_2\}$.
– In the case $C(\iota) = \{\iota, \pi\}$, the equivalence classes of $\sim_C$ are $\{\iota, \pi\}, \{\nu, \sigma\}$, $\{\nu_1, \sigma_2\}$ and $\{\nu_2, \sigma_1\}$.
– In the case $C(\iota) = \{\iota, \sigma\}$, the equivalence classes of $\sim_C$ are $\{\iota, \sigma\}, \{\nu, \pi\}$, $\{\nu_1, \sigma_1\}$ and $\{\nu_2, \sigma_2\}$.
– In the case $C(\iota) = \{\iota\}$, the equivalence classes of $\sim_C$ are $\{\iota\}, \{\nu_1\}, \{\nu_2\}, \{\nu\}$, $\{\pi\}, \{\sigma_1\}, \{\sigma_2\}$ and $\{\sigma\}$.

In either case, the elements of an equivalence class are either all order preserving or all order reversing, and the number of equivalence classes containing order preserving transformations is equal to the number of equivalence classes containing order reversing transformations. Because of (2), this proves (1). □

Corollary 4.6. Consider $C \in \mathcal{C}$ satisfying $C \leq \Pi$ or $C \geq \Pi$. If C is invariant, then $C = \Pi$.

This means that a copula which is invariant and distinct from the product copula cannot be compared with the product copula.

5. MEASURES OF CONCORDANCE

In this final section, we study the group Γ with respect to measures of concordance for copulas.

There is a rich literature on measures of concordance. For example, Scarsini [10] introduced the notion of a measure of concordance for bivariate random vectors with continuous marginal distribution functions, using the unique copula determined by the distribution function of such a random vector; more recently, Taylor [11] proposed a definition of a measure of concordance for copulas, which is closely related to that of Scarsini but avoids the use of random vectors. Taylor's definition of a measure of concordance involves the group $(G, \diamond)$ of transformations on the unit square, which was also considered by Edwards et al. [4].

In the spirit of Taylor [11] we propose a quite general definition of a measure of concordance for copulas, using the group $(\Gamma, \circ)$ instead of $(G, \diamond)$:

A function $\kappa : \mathcal{C} \to [-1, 1]$ is said to be a *measure of concordance* if it has the following properties:

(i) $\kappa(M) = 1$ and $\kappa(\Pi) = 0$.
(ii) The identities $\kappa(\pi(C)) = \kappa(C)$ and $\kappa(\nu_1(C)) = -\kappa(C)$ hold for every $C \in \mathcal{C}$.

A measure of concordance κ is said to be *order preserving* if $\kappa(C) \leq \kappa(D)$ holds for all $C, D \in \mathcal{C}$ such that $C \leq D$, and it is said to be *continuous* if $\lim_{n\to\infty} \kappa(C_n) = \kappa(C)$ holds for every sequence $\{C_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$ and every $C \in \mathcal{C}$ satisfying $\lim_{n\to\infty} C_n(u, v) = C(u, v)$ for all $(u, v) \in [0, 1]^2$. In Scarsini [10] and Taylor [11] and in many other papers, being order preserving and continuous is part of the definition of a measure of concordance; see also Nelsen [8] (Definition 5.1.7).

In terms of our definition of a measure of concordance, Nelsen [8] (Definition 5.1.7) and Taylor [11] require in addition that the identity $\kappa(\nu_2(C)) = -\kappa(C)$ holds for every $C \in \mathcal{C}$, and Nelsen requires that also $\kappa(W) = -1$. These identities turn out to be a consequence of the definition:

Theorem 5.1. Consider a measure of concordance κ and a copula C. Then

$$\kappa(\gamma(C)) = \kappa(C)$$

holds for every $\gamma \in \{\iota, \pi, \sigma, \nu\}$ and

$$\kappa(\gamma(C)) = -\kappa(C)$$

holds for every $\gamma \in \{\nu_1, \nu_2, \sigma_1, \sigma_2\}$. In particular, $\kappa(W) = -1$, and if $\kappa(C) = 0$, then $\kappa(D) = 0$ holds for every $D \in \Gamma(C)$.

P r o o f. The general assertion follows from Lemma 2.4 and the final assertion then follows from Example 4.2(2). □

Theorem 5.1 emphasizes once more the particular role of the transformations which belong to the subgroup $\{\iota, \pi, \sigma, \nu\}$ of Γ: They do not affect the value of a measure of concordance and, as noticed at the end of Section 2, they are symmetry preserving and order preserving involutions.

Theorem 5.1 implies that for certain copulas *every* measure of concordance is equal to zero:

Corollary 5.2. Consider a copula C. If $\nu_1(C) = C$, then every measure of concordance κ satisfies

$$\kappa(C) = 0.$$

In particular, if C is invariant, then every measure of concordance κ satisfies $\kappa(C) = 0$.

The converse of the second implication of Corollary 5.2 is not true; see Example 4.4 for a copula C which is not invariant but satisfies $\nu_1(C) = C$ and hence $\kappa(C) = 0$.

To complete the discussion of relations between measures of concordance and the group Γ, we finally present a slight extension of the main result of Edwards et al. [4].

Proposition 5.3. For a copula D, the following are equivalent:

(a) D is invariant under Γ.
(b) There exist $a, b \in \mathbb{R}$ such that the map $\kappa_D : \mathcal{C} \to \mathbb{R}$ given by

$$\kappa_D(C) := a + b \int_{[0,1]^2} C(u,v)\, \mathrm{d}D(u,v)$$

is a measure of concordance.

In this case, κ_D satisfies

$$\kappa_D(C) = \frac{4 \int_{[0,1]^2} C(u,v)\, \mathrm{d}D(u,v) - 1}{4 \int_{[0,1]^2} M(u,v)\, \mathrm{d}D(u,v) - 1}$$

and is order preserving and continuous.

The arguments needed for the proof of Proposition 5.3 are all contained in the original proof of Edwards et al. [4] (Theorem 3.1). However, our condition (b) is weaker than theirs since it does not require the verification that κ_D is order preserving and continuous.

Example 5.4. By Example 4.2(3), the product copula Π is invariant, and it follows from Example 4.2(2) and Lemma 3.8 that the mean $(W{+}M)/2$ of the Fréchet–Hoeffding bounds is invariant as well. Since

$$\begin{aligned} \kappa_\Pi(C) &= 3\left(4 \int_{[0,1]^2} C(u,v)\, \mathrm{d}\Pi(u,v) - 1\right) \\ \kappa_{(W+M)/2}(C) &= 2\left(4 \int_{[0,1]^2} C(u,v)\, \mathrm{d}((W{+}M)/2)(u,v) - 1\right) \end{aligned}$$

we see that κ_Π is *Spearman's rho* and $\kappa_{(W+M)/2}$ is *Gini's gamma*, and it now follows from Proposition 5.3 that Spearman's rho and Gini's gamma are order preserving and continuous measures of concordance.

Corollary 5.5. Assume that $C, D \in \mathcal{C}$ are invariant. Then

$$\int_{[0,1]^2} C(u,v)\, \mathrm{d}D(u,v) = \frac{1}{4}.$$

This follows from Corollary 5.2 and Proposition 5.3.

REMARK

The question arises whether some or all results of this paper can be extended to copulas on $[0,1]^d$ with $d \geq 3$. Of course, the first step for passing from the bivariate case to the general multivariate case consists in the construction of an appropriate group of transformations which map the collection of all copulas on $[0,1]^d$ into itself. This step has been done in a recent paper by Fuchs [5] who also studied the impact of these transformations on symmetry, order and measures of concordance of copulas on $[0,1]^d$.

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REFERENCES

[1] B. De Baets, H. De Meyer and R. Mesiar: Asymmetric semilinear copulas. Kybernetika *43* (2007), 221–233.

[2] F. Durante and P. L. Papini: Componentwise concave copulas and their asymmetry. Kybernetika *45* (2009), 1003–1011.

[3] F. Durante and C. Sempi: Copula theory – an introduction. In: Copula Theory and Its Applications (P. Jaworski, F. Durante, W. K. Härdle and T. Rychlik, eds.), Springer, Berlin, Heidelberg 2010, pp. 3–31.

[4] H. H. Edwards, P. Mikusiński, and M. D. Taylor: Measures of concordance determined by D_4–invariant copulas. Internat. J. Math. Sci. *70* (2004), 3867–3875.

[5] S. Fuchs: Multivariate copulas – Transformations, symmetry, order and measures of concordance. Submitted for publication (2013).

[6] E. P. Klement, R. Mesiar, and E. Pap: Invariant copulas. Kybernetika *38* (2002), 275–285.

[7] E. Liebscher: Construction of asymmetric multivariate copulas. J. Multivariate Anal. *99* (2008), 2234–2250.

[8] R. B. Nelsen: An Introduction to Copulas. Second edition. Springer, New York 2006.

[9] R. B. Nelsen: Extremes of nonexchangeability. Statist. Papers *48* (2007), 329–336.

[10] M. Scarsini: On measures of concordance. Stochastica *8* (1984), 201–218.

[11] M. D. Taylor: Multivariate measures of concordance. Ann. Inst. Statist. Math. *59* (2007), 789–806.

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