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LEFT AND RIGHT SEMI-UNINORMS ON A COMPLETE LATTICE

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Uninorms are important generalizations of triangular norms and conorms, with a neutral element lying anywhere in the unit interval, and left (right) semi-uninorms are non-commutative and non-associative extensions of uninorms. In this paper, we firstly introduce the concepts of left and right semi-uninorms on a complete lattice and illustrate these notions by means of some examples. Then, we lay bare the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation. Finally, we discuss the relations between the upper approximation left (right) semi-uninorms of a given binary operation and the lower approximation left (right) semi-uninorms of its dual operation.

Keywords: fuzzy connective, uninorm, left (right) semi-uninorm, upper (lower) approximation

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1. INTRODUCTION

Uninorms, introduced by Yager and Rybalov [30], and studied by Fodor et al. [9], are special aggregation operators that have proven useful in many fields like fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modeling [10, 22, 27, 28, 29]. Uninorms are interesting because their structure is a special combination of t-norms and t-conorms [9]. It is well known that a uninorm U can be conjunctive or disjunctive whenever $U(0,1) = 0$ or 1, respectively. This fact allows to use uninorms in defining fuzzy implications and coimplications [3, 19, 20].

There are real-life situations when truth functions can not be associative or commutative. By throwing away the commutativity from the axioms of uninorms, Mas et al. [17, 18] introduced the concepts of left and right uninorms on $[0,1]$, Wang and Fang [25, 26] studied the residual operators and the residual coimplicators of left (right) uninorms on a complete lattice. By removing the associativity and commutativity from the axioms of uninorms, Liu [15] introduced the concept of semi-uninorms on a complete lattice. In this paper, motivated by these generalizations, we will generalize the concepts of both left (right) uninorms and semi-uninorms, introduce a new concept, called the left (right) semi-uninorm, illustrate these notions by means of some examples and lay bare the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a given binary operation on a complete lattice.

This paper is organized as follows. In section 2, we introduce the concepts of left and right semi-uninorms on a complete lattice and illustrate these concepts by means of some examples. In section 3, we give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation. In section 4, we discuss the relations between the upper approximation left (right) semi-uninorms of a given binary operation and the lower approximation left (right) semi-uninorms of its dual operation.

The knowledge about lattices required in this paper can be found in [5].

Throughout this paper, unless otherwise stated, L always represents any given complete lattice with maximal element 1 and minimal element 0; J stands for any index set.

2. LEFT AND RIGHT SEMI-UNINORMS

Noting that the commutativity and associativity are not desired for aggregation operators in a lot of cases. In this section, based on [15, 17, 25, 26], we introduce the concepts of left and right semi-uninorms on a complete lattice and illustrate these notions by means of some examples.

Definition 2.1. A binary operation U on L is called a left (right) semi-uninorm if it satisfies the following two conditions:

(U1) there exists a left (right) neutral element, i. e., an element $e_L \in L$ ($e_R \in L$) satisfying $U(e_L, x) = x$ ($U(x, e_R) = x$) for all $x \in L$,

(U2) U is non-decreasing in each variable.

For any left (right) semi-uninorm U on L, U is said to be left-conjunctive (right-conjunctive) if $U(0, 1) = 0$ ($U(1, 0) = 0$). U is said to be conjunctive if both $U(0, 1) = 0$ and $U(1, 0) = 0$ since it satisfies the classical boundary conditions of AND. If $U(1, 0) = 1$ ($U(0, 1) = 1$), then we call U left-disjunctive (right-disjunctive). We call U disjunctive if both $U(1, 0) = 1$ and $U(0, 1) = 1$ by a similar reason.

If a left (right) semi-uninorm U is associative, then U is the left (right) uninorm (see [25, 26]).

If a left (right) semi-uninorm U with left (right) neutral element e_L (e_R) has a right (left) neutral element e_R (e_L), then $e_L = U(e_L, e_R) = e_R$. Let $e = e_L = e_R$. Here, U is the semi-uninorm (see [15]). In particular, if the neutral element $e = 1$, then the semi-uninorm U becomes a t-seminorm (see [21]) or a semi-copula (see [4, 8]); if the neutral element $e = 0$, then the semi-uninorm U becomes a t-semiconorm (see [7]).

Clearly, $U(0, 0) = 0$ and $U(1, 1) = 1$ hold for any left (right) semi-uninorm U on L. Moreover, the left (right) neutral elements need not to be unique. In fact, the projection operator given by $U(x, y) = x$ for all $x, y \in L$ is such that any element in L is a right neutral element. But, left (right) neutral elements are all idempotent (see [2]) because $U(e_L, e_L) = e_L$ ($U(e_R, e_R) = e_R$) for any left (right) neutral element e_L (e_R) of U.

Definition 2.2. (Wang and Fang [26]) A binary operation U on L is called left (right) infinitely $\vee$-distributive if

$$U\big(\bigvee_{j\in J} x_j, y\big) = \bigvee_{j\in J} U(x_j, y)\ \Big(U\big(x, \bigvee_{j\in J} y_j\big) = \bigvee_{j\in J} U(x, y_j)\Big) \quad \forall x, y, x_j, y_j \in L;$$

left (right) infinitely $\wedge$-distributive if

$$U\big(\bigwedge_{j\in J} x_j, y\big) = \bigwedge_{j\in J} U(x_j, y)\ \Big(U\big(x, \bigwedge_{j\in J} y_j\big) = \bigwedge_{j\in J} U(x, y_j)\Big) \quad \forall x, y, x_j, y_j \in L.$$

If a binary operation U is left infinitely $\vee$-distributive ($\wedge$-distributive) and also right infinitely $\vee$-distributive ($\wedge$-distributive), then U is said to be infinitely $\vee$-distributive ($\wedge$-distributive).

Noting that the least upper bound of the empty set is 0 and the greatest lower bound of the empty set is 1 (see [6]), we have that

$$U(0, y) = U\big(\bigvee_{j\in\emptyset} x_j, y\big) = \bigvee_{j\in\emptyset} U(x_j, y) = 0\ \Big(U(x, 0) = U\big(x, \bigvee_{j\in\emptyset} y_j\big) = \bigvee_{j\in\emptyset} U(x, y_j) = 0\Big)$$

for any $x, y \in L$ when U is left (right) infinitely $\vee$-distributive and

$$U(1, y) = U\big(\bigwedge_{j\in\emptyset} x_j, y\big) = \bigwedge_{j\in\emptyset} U(x_j, y) = 1\ \Big(U(x, 1) = U\big(x, \bigwedge_{j\in\emptyset} y_j\big) = \bigwedge_{j\in\emptyset} U(x, y_j) = 1\Big)$$

for any $x, y \in L$ when U is left (right) infinitely $\wedge$-distributive.

When $L = [0, 1]$, a binary function f on $[0, 1]^2$ is infinitely sup-distributive if and only if, for any $x_0, y_0 \in [0, 1]$, $f(x, y_0)$ and $f(x_0, y)$ are left-continuous and increasing and $f(x, 0) = f(0, y) = 0$ for any $x, y \in [0, 1]$; and f is infinitely inf-distributive if and only if, for any $x_0, y_0 \in [0, 1]$, $f(x, y_0)$ and $f(x_0, y)$ are right-continuous and increasing and $f(x, 1) = f(1, y) = 1$ for any $x, y \in [0, 1]$ (see [11]).

For the sake of convenience, we introduce the following symbols:

$\mathcal{U}_s^{e_L}(L)$: the set of all left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_s^{e_R}(L)$: the set of all right semi-uninorms with right neutral element e_R on L;

$\mathcal{U}_{s\vee}^{e_L}(L)$: the set of all right infinitely $\vee$-distributive left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{\vee s}^{e_R}(L)$: the set of all left infinitely $\vee$-distributive right semi-uninorms with right neutral element e_R on L;

$\mathcal{U}_{s\wedge}^{e_L}(L)$: the set of all right infinitely $\wedge$-distributive left semi-uninorms with left neutral element e_L on L;

$\mathcal{U}_{\wedge s}^{e_R}(L)$: the set of all left infinitely $\wedge$-distributive right semi-uninorms with right neutral element e_R on L.

Now, we illustrate the notions of left (right) semi-uninorms by means of some examples.

Example 2.3. Let $L = \{0, a, b, c, d, 1\}$ be a lattice, where $0 < a < b < d < 1,\ 0 < a < c < d < 1$, $b \wedge c = a$ and $b \vee c = d$. Define two binary operations U_1, U_2 on L as follows:

U_1	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	c	c	1
b	0	a	b	c	d	1
c	0	a	c	d	d	1
d	0	a	d	d	d	1
1	0	1	1	1	1	1

U_2	0	a	b	c	d	1
0	0	0	0	0	0	1
a	0	0	a	0	c	1
b	0	a	b	c	d	1
c	0	0	c	0	c	1
d	0	d	d	d	d	1
1	1	1	1	1	1	1

Obviously, U_1 and U_2 are neither commutative nor associative. It is easy to verify that U_1 is a conjunctive infinitely $\vee$-distributive semi-uninorm with the neutral element b and U_2 is a disjunctive infinitely $\wedge$-distributive semi-uninorm with the neutral element b.

Example 2.4. Let $L = \{0, a, b, c, 1\}$ be a lattice, where $0 < a < b < 1,\ 0 < a < c < 1$, $b \wedge c = a$ and $b \vee c = 1$. Define a binary operation U on L as follows:

U	0	a	b	c	1
0	0	0	0	0	0
a	0	0	a	c	1
b	0	a	b	c	1
c	0	a	b	c	1
1	0	1	1	1	1

Clearly, U is a conjunctive left semi-uninorm with two left neutral elements b and c. But, U has no right neutral element. It is easy to see that U is neither commutative nor associative. Moreover, U is neither left infinitely $\vee$-distributive ($\wedge$-distributive) nor right infinitely $\vee$-distributive ($\wedge$-distributive).

Example 2.5. Let $e_L \in L$,

$$U_{sW}^{e_L}(x,y) = \begin{cases} y & \text{if } x \geq e_L, \\ 0 & \text{otherwise,} \end{cases} \qquad U_{sM}^{e_L}(x,y) = \begin{cases} y & \text{if } x \leq e_L, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{sW}^{e_L *}(x,y) = \begin{cases} 1 & \text{if } y = 1, \\ y & \text{if } x \geq e_L,\ y \neq 1, \\ 0 & \text{otherwise,} \end{cases} \quad U_{sM}^{e_L *}(x,y) = \begin{cases} 0 & \text{if } y = 0, \\ y & \text{if } x \leq e_L,\ y \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

where x and y are elements of L. Then $U_{sW}^{e_L}$ and $U_{sM}^{e_L}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_s^{e_L}(L)$; $U_{sW}^{e_L}$ and $U_{sM}^{e_L *}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_{s\vee}^{e_L}(L)$; $U_{sW}^{e_L *}$ and $U_{sM}^{e_L}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_{s\wedge}^{e_L}(L)$.

Example 2.6. Let $e_R \in L$,

$$U_{sW}^{e_R}(x,y) = \begin{cases} x & \text{if } y \geq e_R, \\ 0 & \text{otherwise,} \end{cases} \qquad U_{sM}^{e_R}(x,y) = \begin{cases} x & \text{if } y \leq e_R, \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{sW}^{e_R\,*}(x,y) = \begin{cases} 1 & \text{if } x = 1, \\ x & \text{if } y \geq e_R,\ x \neq 1, \\ 0 & \text{otherwise,} \end{cases} \quad U_{sM}^{e_R\,*}(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ x & \text{if } y \leq e_R,\ x \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

where x and y are elements of L. Then $U_{sW}^{e_R}$ and $U_{sM}^{e_R}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_s^{e_R}(L)$; $U_{sW}^{e_R}$ and $U_{sM}^{e_R\,*}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_{\vee s}^{e_R}(L)$; $U_{sW}^{e_R\,*}$ and $U_{sM}^{e_R}$ are, respectively, the smallest and greatest elements of $\mathcal{U}_{\wedge s}^{e_R}(L)$.

3. THE UPPER AND LOWER APPROXIMATION LEFT (RIGHT) SEMI-UNINORMS OF A BINARY OPERATION

Constructing logic operators is an interesting work. Recently, Jenei and Montagna [12, 13, 14] introduced several new types of constructions of left-continuous t-norms and Wang [24] laid bare the formulas for calculating the smallest pseudo-t-norm that is stronger than a binary operation. In this section, we continue the work in [24] and give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation.

For any nonempty subfamily $\{T_j \mid j \in J\}$ of $L^{L\times L}$, the least upper bound $\vee_{j\in J}T_j$ and the greatest lower bound $\wedge_{j\in J}T_j$ of T_j's, respectively, define by

$$\big(\bigvee_{j\in J} T_j\big)(x,y) = \bigvee_{j\in J} T_j(x,y) \quad \text{and} \quad \big(\bigwedge_{j\in J} T_j\big)(x,y) = \bigwedge_{j\in J} T_j(x,y) \quad \forall x,y \in L.$$

It is easy to verify that $(L^{L\times L}, \leq, \vee, \wedge)$ is a complete lattice. Moreover, we have the following two theorems.

Theorem 3.1.

1. $\mathcal{U}_s^{e_L}(L)$ is a complete sublattice of $L^{L\times L}$ with $U_{sW}^{e_L}$ and $U_{sM}^{e_L}$ as its minimal and maximal elements, respectively.
2. $\mathcal{U}_s^{e_R}(L)$ is a complete sublattice of $L^{L\times L}$ with $U_{sW}^{e_R}$ and $U_{sM}^{e_R}$ as its minimal and maximal elements, respectively.

Theorem 3.2.

1. $\mathcal{U}_{s\wedge}^{e_L}(L)$ is a complete sublattice of $L^{L\times L}$ with $U_{sW}^{e_L\,*}$ and $U_{sM}^{e_L}$ as its minimal and maximal elements, respectively.

2. $\mathcal{U}^{e_R}_{\wedge s}(L)$ is a complete sublattice of $L^{L\times L}$ with ${U^{e_R}_{sW}}^*$ and $U^{e_R}_{sM}$ as its minimal and maximal elements, respectively.

3. $\mathcal{U}^{e_L}_{s\vee}(L)$ is a complete sublattice of $L^{L\times L}$ with $U^{e_L}_{sW}$ and ${U^{e_L}_{sM}}^*$ as its minimal and maximal elements, respectively.

4. $\mathcal{U}^{e_R}_{\vee s}(L)$ is a complete sublattice of $L^{L\times L}$ with $U^{e_R}_{sW}$ and ${U^{e_R}_{sM}}^*$ as its minimal and maximal elements, respectively.

Proof. We only prove that statement (1) holds.

Suppose that $U_j \in \mathcal{U}^{e_L}_{s\wedge}(L)$ $(j \in J)$ and $J \neq \emptyset$. Then it follows from Theorem 3.1 that $\wedge_{j\in J} U_j \in \mathcal{U}^{e_L}_{s}(L)$. Moreover, we have

$$\begin{aligned}&\big(\bigwedge_{j\in J} U_j\big)\big(x, \bigwedge_{k\in K} y_k\big) = \bigwedge_{j\in J} U_j\big(x, \bigwedge_{k\in K} y_k\big) = \bigwedge_{j\in J}\bigwedge_{k\in K} U_j(x,y_k)\\ =\ &\bigwedge_{k\in K}\bigwedge_{j\in J} U_j(x,y_k) = \bigwedge_{k\in K}\big(\bigwedge_{j\in J} U_j(x,y_k)\big) = \bigwedge_{k\in K}\Big(\big(\bigwedge_{j\in J} U_j\big)(x,y_k)\Big),\end{aligned}$$

where K is any index set, and x and y_k $(k \in K)$ are any elements of L. Hence, $\wedge_{j\in J} U_j \in \mathcal{U}^{e_L}_{s\wedge}(L)$. Noting that fact $U^{e_L}_{sM} \in \{U \in \mathcal{U}^{e_L}_{s\wedge}(L) \mid U_j \leq U\ \forall j \in J\}$, let $U^* = \wedge\{U \in \mathcal{U}^{e_L}_{s\wedge}(L) \mid U_j \leq U\ \forall j \in J\}$, then $U^* \in \mathcal{U}^{e_L}_{s\wedge}(L)$ and $U^* = \vee_{j\in J} U_j$. Thus, $\mathcal{U}^{e_L}_{s\wedge}(L)$ is a complete sublattice of $L^{L\times L}$ with $U^{e_L}_{sM}$ and ${U^{e_L}_{sW}}^*$ as its maximal and minimal elements, respectively. □

For a binary operation A on L, if there exists $U \in \mathcal{U}^{e_L}_{s}(L)$ such that $A \leq U$, then it follows from Theorem 3.1 that $\bigwedge\{U \mid A \leq U, U \in \mathcal{U}^{e_L}_{s}(L)\}$ is the smallest left semi-uninorm that is stronger than A on L, we call it the upper approximation left semi-uninorm of A and written as $[A)^{e_L}_{s}$; if there exists $U \in \mathcal{U}^{e_L}_{s}(L)$ such that $U \leq A$, then $\bigvee\{U \mid U \leq A, U \in \mathcal{U}^{e_L}_{s}(L)\}$ is the largest left semi-uninorm that is weaker than A on L, we call it the lower approximation left semi-uninorm of A and written as $(A]^{e_L}_{s}$.

Similarly, we introduce the following symbols:

$[A)^{e_R}_{s}$: the upper approximation right semi-uninorm of A;

$(A]^{e_R}_{s}$: the lower approximation right semi-uninorm of A;

$(A]^{e_L}_{s\wedge}$: the right infinitely $\wedge$-distributive lower approximation left semi-uninorm of A;

$(A]^{e_R}_{\wedge s}$: the left infinitely $\wedge$-distributive lower approximation right semi-uninorm of A;

$[A)^{e_L}_{s\vee}$: the right infinitely $\vee$-distributive upper approximation left semi-uninorm of A;

$[A)^{e_R}_{\vee s}$: the left infinitely $\vee$-distributive upper approximation right semi-uninorm of A.

Now we consider how to construct the upper and lower approximation left (right) semi-uninorms of a binary operation.

Definition 3.3. Let $A \in L^{L\times L}$. Define the upper approximation A_u and the lower approximation A_l of A as follows:

$$A_u(x,y) = \bigvee\{A(u,v) \mid u \le x, v \le y\},\ A_l(x,y) = \bigwedge\{A(u,v) \mid u \ge x, v \ge y\}\ \ \forall\, x,y \in L.$$

Theorem 3.4. Let $A, B \in L^{L\times L}$. Then the following statements hold:

1. $A_l \le A \le A_u$.
2. $(A \vee B)_u = A_u \vee B_u$ and $(A \wedge B)_l = A_l \wedge B_l$.
3. A_u and A_l are non-decreasing in its each variable.
4. If A is non-decreasing in its each variable, then $A_u = A_l = A$.

Proof. Clearly, statements (1) and (2) hold.

3. We only prove that A_l is non-decreasing in its first variable.
If $x_1 \le x_2$, then

$$\{A(u,v) \mid u \ge x_1, v \ge y\} \supseteq \{A(u,v) \mid u \ge x_2, v \ge y\}.$$

Thus $A_l(x_1,y) \le A_l(x_2,y)$ for any $y \in L$ by Definition 3.3, i. e., A_l is non-decreasing in its first variable.

4. If A is non-decreasing in its each variable, then

$$A_l(x,y) = \bigwedge\{A(u,v) \mid u \ge x, v \ge y\} \ge \bigwedge\{A(x,y) \mid u \ge x, v \ge y\} = A(x,y)\ \forall x,y \in L$$

and hence $A_l \ge A$. Thus, it follows from statement (1) that $A_l = A$.
Similarly, we can show that $A_u = A$. □

As usual, the upper or lower approximation of a binary operation is neither a left semi-uninorm nor a right semi-uninorm.

Example 3.5. Let

$$A(x,y) = \begin{cases} \frac{1}{4}y & \text{if } x \le \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $A \le U_{sM}^{(\frac{1}{2})_L}$ and $A_u = A$. Clearly, A_u is not a left semi-uninorm. Let

$$U(x,y) = \begin{cases} \frac{1}{4}y & \text{if } x < \frac{1}{2}, \\ y & \text{if } x = \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that U is the upper approximation left semi-uninorm with left neutral element $\frac{1}{2}$ of A.

The following two theorems give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation.

Theorem 3.6. Let $A \in L^{L\times L}$ and $e_L \in L$.

1. If $A \le U_{sM}^{e_L}$, then $[A)_s^{e_L} = U_{sW}^{e_L} \vee A_u$.
2. If $U_{sW}^{e_L} \le A$, then $(A]_s^{e_L} = U_{sM}^{e_L} \wedge A_l$.
3. If $A \le {U_{sM}^{e_L}}^*$ and A is non-decreasing in its first variable and right infinitely $\vee$-distributive, then $[A)_{s\vee}^{e_L} = U_{sW}^{e_L} \vee A$.
4. If ${U_{sW}^{e_L}}^* \le A$ and A is non-decreasing in its first variable and right infinitely $\wedge$-distributive, then $(A]_{s\wedge}^{e_L} = U_{sM}^{e_L} \wedge A$.

P r o o f. We only prove the statements (1) and (3) hold.

1. Let $U = U_{sW}^{e_L} \vee A_u$. Clearly, $U \ge A$ and $U_{sW}^{e_L} \le U \le U_{sM}^{e_L}$. Thus, $U(e_L, x) = x$ for all $x \in L$. By Theorem 3.4(3) and the monotonicity of $U_{sW}^{e_L}$, we see that U is non-decreasing in its each variable. So, $U \in \mathcal{U}_s^{e_L}(L)$. If $A \le U_1$ and $U_1 \in \mathcal{U}_s^{e_L}(L)$, then $U_1 = (U_1)_u \ge A_u$ and $U_1 \ge U_{sW}^{e_L} \vee A_u = U$. Therefore, $[A)_s^{e_L} = U_{sW}^{e_L} \vee A_u$.

3. Let $U^* = U_{sW}^{e_L} \vee A$. If A is non-decreasing in its first variable and right infinitely $\vee$-distributive, then A is non-decreasing in its each variable and so $A_u = A$. Noting that $U_{sW}^{e_L}$ and A are all right infinitely $\vee$-distributive, we can see that U^* is also right infinitely $\vee$-distributive. By the proof of statement (1), we have that $[A)_{s\vee}^{e_L} = U_{sW}^{e_L} \vee A$. □

In Theorem 3.6(3), $A(x, 0) = 0$ for any $x \in L$ when A is right infinitely $\vee$-distributive. Thus, $A \le {U_{sM}^{e_L}}^*$ can be replaced by $A \le U_{sM}^{e_L}$.

Similarly, ${U_{sW}^{e_L}}^* \le A$ can be replaced by $U_{sW}^{e_L} \le A$ in Theorem 3.6(4).

Analogous to Theorem 3.6, we have the following theorem.

Theorem 3.7. Let $A \in L^{L\times L}$ and $e_R \in L$.

1. If $A \le U_{sM}^{e_R}$, then $[A)_s^{e_R} = U_{sW}^{e_R} \vee A_u$.
2. If $U_{sW}^{e_R} \le A$, then $(A]_s^{e_R} = U_{sM}^{e_R} \wedge A_l$.
3. If $A \le U_{sM}^{e_R}$ and A is non-decreasing in its second variable and left infinitely $\vee$-distributive, then $[A)_{\vee s}^{e_R} = U_{sW}^{e_R} \vee A$.
4. If $U_{sW}^{e_R} \le A$ and A is non-decreasing in its second variable and left infinitely $\wedge$-distributive, then $(A]_{\wedge s}^{e_R} = U_{sM}^{e_R} \wedge A$.

The following example shows that analogous to the above theorems may not hold for calculating the right (left) infinitely $\wedge$-distributive upper approximation left (right) semi-uninorm and the right (left) infinitely $\vee$-distributive lower approximation left (right) semi-uninorm of a binary operation.

Example 3.8. Let $L = \{0, a, b, 1\}$ be a lattice, where $0 < a < 1$, $0 < b < 1$, $a \vee b = 1$ and $a \wedge b = 0$. Define two binary operations A and B on L as follows:

A	0	a	b	1
0	0	0	0	0
a	a	1	a	1
b	0	0	0	0
1	a	1	a	1

B	0	a	b	1
0	0	b	0	b
a	1	1	1	1
b	0	b	0	b
1	1	1	1	1

Clearly, $A \leq U^{0_L}_{sM}$, $U^{1_L}_{sW} \leq B$, A is non-decreasing in its first variable and right infinitely $\wedge$-distributive, and B is non-decreasing in its first variable and right infinitely $\vee$-distributive. Let $U_1 = {U^{0_L}_{sW}}^* \vee A$ and $U_2 = {U^{1_L}_{sM}}^* \wedge B$. Then

U_1	0	a	b	1
0	0	a	b	1
a	a	1	1	1
b	0	a	b	1
1	a	1	1	1

U_2	0	a	b	1
0	0	0	0	b
a	0	a	b	1
b	0	0	0	b
1	0	a	b	1

It is easy to see that U_1 is not right infinitely $\wedge$-distributive and U_2 is not right infinitely $\vee$-distributive. This shows that U_1 is not the right infinitely $\wedge$-distributive upper approximation left semi-uninorm of A and U_2 is not the right infinitely $\vee$-distributive lower approximation left semi-uninorm of B.

4. THE RELATIONS BETWEEN THE UPPER APPROXIMATION LEFT (RIGHT) SEMI-UNINORMS OF A GIVEN BINARY OPERATION AND LOWER APPROXIMATION LEFT (RIGHT) SEMI-UNINORMS OF ITS DUAL OPERATION

In section 3, we give out the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation. In this section, we investigate the relations between the upper approximation left (right) semi-uninorm of a given binary operation and the lower approximation left (right) semi-uninorm of its dual operation.

We firstly review some basic concepts and properties which will be used in this section.

Definition 4.1. (Ma and Wu [16]) A mapping $N : L \to L$ is called a negation if

(N1) $N(0) = 1$ and $N(1) = 0$,

(N2) $x \leq y,\ x, y \in L \Rightarrow N(y) \leq N(x)$.

A negation N is called strong if it is an involution, i. e., $N(N(x)) = x$ for any $x \in L$.

Theorem 4.2. (Wang and Yu [23]) Let $x_j \in L$ $(j \in J)$. If N is a strong negation on L, then

$$N(\bigvee_{j\in J} x_j) = \bigwedge_{j\in J} N(x_j),\ N(\bigwedge_{j\in J} x_j) = \bigvee_{j\in J} N(x_j).$$

Definition 4.3. (De Baets [1]) Consider a strong negation N on L. The N-dual operation of a binary operation A on L is the binary operation A_N on L defined by

$$A_N(x,y) = N^{-1}\big(A(N(x),N(y))\big) \ \ \forall x,y \in L.$$

Note that $(A_N)_{N^{-1}} = (A_N)_N = A$ for any binary operation A on L.

The following theorem about N-dual is easily verified.

Theorem 4.4. Let A, B be two binary operations and N a strong negation on L. Then the following statements hold:

1. $(A \wedge B)_N = A_N \vee B_N$ and $(A \vee B)_N = A_N \wedge B_N$.
2. If A is left (right) infinitely $\vee$-distributive, then A_N is left (right) infinitely $\wedge$-distributive.
3. If A is left (right) infinitely $\wedge$-distributive, then A_N is left (right) infinitely $\vee$-distributive.
4. If A is increasing (decreasing) in its ith variable, then A_N is increasing (decreasing) in its ith variable ($i = 1$, 2).
5. The N-dual operation of a left (right) semi-uninorm with a left (right) neutral element e_L (e_R) is a left (right) semi-uninorm with a left (right) neutral element $N(e_L)$ ($N(e_R)$).
6. $(U_{sW}^{e_L})_N = U_{sM}^{N(e_L)}$, $(U_{sM}^{e_L})_N = U_{sW}^{N(e_L)}$, $(U_{sW}^{e_R})_N = U_{sM}^{N(e_R)}$ and $(U_{sM}^{e_R})_N = U_{sW}^{N(e_R)}$.

Theorem 4.5. If A is a binary operation and N a strong negation on L, then $(A_N)_u = (A_l)_N$ and $(A_N)_l = (A_u)_N$.

Proof. By Definition 4.3 and Theorem 4.2, we can see that

$$\begin{aligned}
&(A_N)_u(x,y) = \bigvee\{A_N(u,v) \mid u \le x, v \le y\} \\
= \ & \bigvee\{N^{-1}(A(N(u),N(v))) \mid u \le x, v \le y\} \\
= \ & N^{-1}\big(\bigwedge\{A(N(u),N(v)) \mid u \le x, v \le y\}\big) \\
= \ & N^{-1}\big(\bigwedge\{A(u',v') \mid u' \ge N(x), v' \ge N(y)\}\big) \\
= \ & N^{-1}\big(A_l(N(x),N(y))\big) = (A_l)_N(x,y) \ \ \forall x,y \in L.
\end{aligned}$$

Moreover, we have that $(A_u)_N = \big(((A_N)_N)_u\big)_N = \big(((A_N)_l)_N\big)_N = (A_N)_l$. □

Below, we investigate the relations between the upper approximation left (right) semi-uninorms of a given binary operation and lower approximation left (right) semi-uninorms of its dual operation.

Theorem 4.6. Let A, N and e_L be a binary operation, strong negation and fixed element on L, respectively. Then the following statements hold:

1. If $A \leq U_{sM}^{e_L}$, then $[A)_s^{e_L} = ((A_N]_s^{N(e_L)})_N$.
2. If $U_{sW}^{e_L} \leq A$, then $(A]_s^{e_L} = ([A_N)_s^{N(e_L)})_N$.
3. If $A \leq U_{sM}^{e_L}$ and A is non-decreasing in its first variable and right infinitely $\vee$-distributive, then $[A)_{s\vee}^{e_L} = ((A_N]_{s\wedge}^{N(e_L)})_N$.
4. If $U_{sW}^{e_L} \leq A$ and A is non-decreasing in its first variable and right infinitely $\wedge$-distributive, then $(A]_{s\wedge}^{e_L} = ([A_N)_{s\vee}^{N(e_L)})_N$.

Proof. We only prove the statements (1) and (3) hold.

1. If $A \leq U_{sM}^{e_L}$, then $[A)_s^{e_L} = U_{sW}^{e_L} \vee A_u$ by Theorem 3.6 and $A_N \geq (U_{sM}^{e_L})_N = U_{sW}^{N(e_L)}$ by Theorem 4.4. Thus, $(A_N]_s^{N(e_L)} = U_{sM}^{N(e_L)} \wedge (A_N)_l$ by Theorem 3.6. Moreover, by virtue of Theorems 3.6, 4.4 and 4.5, we see that

$$\begin{aligned} & \big((A_N]_s^{N(e_L)}\big)_N = \big(U_{sM}^{N(e_L)} \wedge (A_N)_l\big)_N = \big(U_{sM}^{N(e_L)} \wedge (A_u)_N\big)_N \\ = \; & (U_{sM}^{N(e_L)})_N \vee \big((A_u)_N\big)_N = U_{sW}^{e_L} \vee A_u = [A)_s^{e_L}. \end{aligned}$$

3. If $A \leq U_{sM}^{e_L}$ and A is non-decreasing in its first variable and right infinitely $\vee$-distributive, then $A_u = A$ by Theorem 3.4(4), $[A)_{s\vee}^{e_L} = U_{sW}^{e_L} \vee A$ by Theorem 3.6, $A_N \geq (U_{sM}^{e_L})_N = U_{sW}^{N(e_L)}$ and A_N is is non-decreasing in its first variable and right infinitely $\wedge$-distributive by Theorem 4.4. Thus, $(A_N]_{s\wedge}^{N(e_L)} = U_{sM}^{N(e_L)} \wedge A_N$ by Theorem 3.6. Moreover, we see that $[A)_{s\vee}^{e_L} = ((A_N]_{s\wedge}^{N(e_L)})_N$ by the proof of statement (1). □

Analogous to Theorem 4.6, we have the following theorem.

Theorem 4.7. Let A, N and e_R be a binary operation, strong negation and fixed element on L, respectively. Then the following statements hold:

1. If $A \leq U_{sM}^{e_R}$, then $[A)_s^{e_R} = ((A_N]_s^{N(e_R)})_N$.
2. If $U_{sW}^{e_R} \leq A$, then $(A]_s^{e_R} = ([A_N)_s^{N(e_R)})_N$.
3. If $A \leq U_{sM}^{e_R}$ and A is non-decreasing in its second variable and left infinitely $\vee$-distributive, then $[A)_{\vee s}^{e_R} = ((A_N]_{\wedge s}^{N(e_R)})_N$.
4. If $U_{sW}^{e_R} \leq A$ and A is non-decreasing in its second variable and left infinitely $\wedge$-distributive, then $(A]_{\wedge s}^{e_R} = ([A_N)_{\vee s}^{N(e_R)})_N$.

5. CONCLUSIONS AND FUTURE WORKS

Uninorms are important generalizations of triangular norms and conorms, with a neutral element lying anywhere in the unit interval. Noting that the associative binary operators are often used to generate n-ary aggregation operators and the commutativity is not desired for these aggregation operators in a lot of cases, Mas et al. [17, 18] introduced the concepts of left and right uninorms on $[0,1]$ by eliminating the commutativity from the axioms of uninorm, Wang and Fang [25, 26] studied the residual operations and the residual coimplications of left (right) uninorms on a complete lattice, and Liu [15] discussed the concept of semi-uninorms on a complete lattice by removing the associativity and commutativity from the axioms of uninorms. In this paper, motivated by these generalizations, we introduce the concepts of left and right semi-uninorms on a complete lattice, lay bare the formulas for calculating the upper and lower approximation left (right) semi-uninorms of a binary operation, and discuss the relations between the upper approximation left (right) semi-uninorms of a given binary operation and the lower approximation left (right) semi-uninorms of its dual operation.

In a forthcoming paper, we will investigate the relationships among left (right) semi-uninorms, implications and coimplications on a complete lattice.

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