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A SAMPLE-TIME ADJUSTED FEEDBACK FOR ROBUST BOUNDED OUTPUT STABILIZATION

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This paper deals with a bounded control design for a class of nonlinear systems where the mathematical model may be not explicitly given. This class of uncertain nonlinear systems governed by a system of ODE with quasi-Lipschitz right-hand side and containing external perturbations as well. The Attractive Ellipsoid Method (AEM) application permits to describe the class of nonlinear feedbacks (containing a nonlinear projection operator, a linear state estimator and a feedback matrix-gain) guaranteeing a boundedness of all possible trajectories around the origin. To fulfill this property some modification of AEM are introduced: basically, some sort of sample-time corrections of the feedback parameters are required. The optimization of feedback within this class of controllers is associated with the selection of the feedback parameters which provide the trajectory converges within an ellipsoid of a "minimal size". The effectiveness of the suggested approach is illustrated by its application to a flexible arm system).

Keywords: sample-time data, attractive ellipsoid, state estimation, saturated control process, flexible arm system

Classification: 93E12, 62A10

1. INTRODUCTION

The reason that *Robust Control Design* for nonlinear uncertain systems attracts a lot of attention during last two decades, is that it provides a workable instrument to design feedback controllers which are able to operate successfully in the lack of complete information on the plants containing internal uncertainties as well as external perturbations [5, 7, 12].

When only an output (but not states) of a controlled system is available, the most important results are dedicated to the output control designing problem, contain as a sub-part the construction of a state estimator or observer (see, for example, [17, 27, 28] and [29]). In fact, the most of them are nonlinear ones [4, 13, 22, 29]. The, so-called, high-gain observer has some advantages permitting to work with uncertain systems, but, if initial conditions of the real system are far from desired dynamics, such observers turn out to be not effective [1, 14]. The most widely used output controllers, recently applied, contain inside the observers of the Luenberger-like type [29], which were combined recently with robust controllers using the AEM application (see for example [8, 18]).

Some successful theoretical results are related with the, so-called, attractive (stable invariant) sets having successful applications in control engineering, robust analysis and synthesis, control under constraints and disturbance (see, for example, [2, 3, 16]). The main instrument of this approach is the, so-called, *Invariant Ellipsoid Method* (IEM). The IEM provides a convergence of any possible controlled trajectories to a neighbor of the origin for a whole class of nonlinear models. If this property is guaranteed for any initial condition then IEM in this case is referred as the *Attractive Ellipsoid Method* (AEM) which can be undertaken in a variety of settings [6, 18, 19, 21]. Furthermore, the AEM is able to generate a switched control signal with a time-varying feedback based on current system information obtained on-line .

Here we study the workability of AEM when control actions (generated by a designed feedback) are bounded by their physical nature (see, for example, [25], [26] and [24]) and are designed based only on sample-time output data (see [10, 11] and [21]). In general, such constraint are described by a membership of a control action to a priory given bounded convex set (compact) [23]). The convexity property is basically topological depending on the connectedness of considered subsets. Although the Robust Control is applied in many different branches of control theory (like linear and non linear control [7, 21, 27], adaptive control [9, 15], and others) there is a few evidence of the robust control design taking in to account the boundedness (or saturation) of a set of admissible control actions.

Saturation is probably the most commonly encountered nonlinearity in control engineering. By this reason a projector, participating in a designed nonlinear feedback, plays an important role in the description of a saturated control. So in [6] and [19], the invariant ellipsoid for the sliding mode control is constructed by means of linear difference inclusions. Here we consider more general types of nonlinear bounded feedbacks which may be corrected (adjusted) on-line during a control process [18].

1.1. Main contribution

- On our opinion the most important result of this paper is the suggested methodology (the numerical procedure) which permits to apply AEM for designing a nonlinear saturated output control based on only sample data outputs within a class of nonlinear systems without exact knowledge of the dynamic mathematical model and in the presence of external bounded perturbations.

- Here we suggest to associate the notion "Optimal Robust Output Feedback" with a set of feedback parameters which guarantee the "minimal size" of the attractive ellipsoid among all ellipsoids containing all possible trajectories of a controlled nonlinear system.

- In this contribution, the suggested methodology is effectively applied to a vertical underactuated pendulum (commonly known as a flexible arm) of two degrees of freedom.

1.2. Structure of the paper

The outline of this paper is as follows. In Section 2 the structure of uncertain nonlinear system and problem formulation are given. Next section presents the main result dealing with designing of the optimal robust bounded output control based on AEM concept. Numerical aspects dealing with the implementation of the applied methodology are discussed in the section 4. The illustrative example, related to the stabilization of a flexible arm, is presented in section 6. Finally, we present the conclusions.

2. THE CLASS OF UNCERTAIN NONLINEAR SYSTEMS AND PROBLEM FORMULATION

2.1. System description

Hereafter we show how the combination of the traditional AEM and the Projection Concept serves for designing of a *Robust Nonlinear Bounded-Output Feedback*. Consider the sufficiently wide class of perturbed uncertain nonlinear systems, governed by the following system of ODEs:

$$\begin{gathered}\dot{x}(t) = f\left[x(t)\right] + Bu(t) + \zeta(t), \text{ a.e. on } \mathbb{R}_+ \\ y(t) = h\left[x(t)\right] + \zeta_y(t),\ x(0) = x_0 \in \mathbb{R}^n\end{gathered} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector at time $t \geq 0$, $y(t) \in \mathbb{R}^p$ is the output system at time $t \geq 0$, the vector-functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ define the dynamics and output mapping of the system (1) respectively, $B \in \mathbb{R}^{n\times m}$ is the matrix realizing the actuator-mapping, $u(t) \in \mathbb{R}^m$ is the control input at time $t \geq 0$, $\zeta_x(t) \in \mathbb{R}^n$ and $\zeta_y(t) \in \mathbb{R}^p$ are external perturbations.

This nonlinear system, can be represented in a "quasi-linear" format as:

$$\begin{gathered}\dot{x}\,(t) = Ax\,(t) + Bu\,(t) + \xi_x[x(t),t] \\ y(t) = Cx(t) + \xi_y[x(t),t],\ x\,(0) = x_0, \\ A \in \mathbb{R}^{n\times n},\ C \in \mathbb{R}^{p\times n}, \\ \xi_x[x(t),t] := \Delta f[x(t)] + \zeta_x\,(t)\,,\ \Delta f(x) := f\,(x) - Ax, \\ \xi_y[x(t),t] := \Delta h[x(t)] + \zeta_y\,(t)\,,\ \Delta h(x) := h\,(x) - Cx.\end{gathered} \tag{2}$$

Here the vectors $\xi_x[x(t),t]$, and $\xi_y[x(t),t]$ characterize the *uncertain part* (or unmodelled dynamics) of the system (2) which contains both a bounded external perturbations $\zeta_x(t)$ and $\zeta_y(t)$ which assumed to be bounded:

$$\sup_{t\in\mathbb{R}_+} \|\zeta_x(t)\| \leq c_4, \quad \sup_{t\in\mathbb{R}_+} \|\zeta_y(t)\| \leq c_5, \quad c_4,\ c_5 < \infty. \tag{3}$$

The formal definition of the class of the *quasi-Lipschitz* functions $g(x)$ is formulated in the definition below.

Definition 2.1. A vector function $g : \mathbb{R}^n \to \mathbb{R}^k$ is said to be from the class $\mathcal{C}(G, \delta_0, \delta_1)$ of *quasi-Lipschitz* functions if there exist matrix $G \in \mathbb{R}^{k\times n}$ and nonnegative constants δ_0 and δ_1 such that for any $x \in \mathbb{R}^n$ the following inequality holds

$$\|g(x) - Gx\|^2 \le \delta_0 + \delta_1 \|x\|^2. \tag{4}$$

This implies that the growth rates of $g(x)$ as $\|x\| \to \infty$ are not faster than linear (see Figure 1 illustrating the single dimensional case $n = k = 1$ for the class $\mathcal{C}(a, \delta_0, \delta_1)$ when $a > \delta_1 > 0$). Notice that if $G = 0$, $\delta_0 = 0$ and $g(0) = 0$ the inequality (4) characterizes the Lipschitz continuity property of the function $g(x)$ with the Lipschitz constant $L = \sqrt{\delta_1}$.

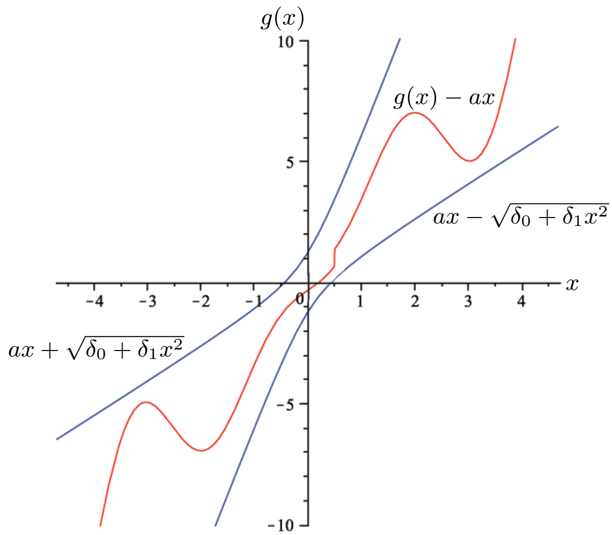

Fig. 1: The class $\mathcal{C}(a, \delta_0, \delta_1)$ of quasi-Lipschitz functions (scalar case).

Then, the mappings f and h may be unknown exactly but are quasi-Lipschitz, that is,

$$f \in \mathcal{C}(A, c_0, c_1), \quad h \in \mathcal{C}(C, c_2, c_3). \tag{5}$$

The parameters A, C, c_0, c_1, c_2, c_3 are supposed to be known a priory.

The constant matrices A and C as well as the constants c_k $(k = 0, 3)$ are supposed to be a priory known. The memberships $f \in \mathcal{C}(A, c_1, c_2)$ and $h \in \mathcal{C}(C, c_3, c_4)$ exactly mean that the growth rates of these functions (as $\|x\| \to \infty$) are not faster than linear. In (5) the matrices A and C characterizes the, so-called, a "*nominal linear plant*" contained within the $\mathcal{C}$ class, the scalars c_k, $k = \overline{0, 3}$ are nonnegative constants defining a permitted deviation of any nonlinearity from this class with respect to a nominal linear plant. Under the additional information that $f(0) = 0$ for any function $f \in \mathcal{C}(A, c_0, c_1)$ one can take $c_0 = 0$, and we shall deal with the class of Lipschitz functions commonly considered within the Modern Control Theory. Under the conditions (3)–(5) we may conclude that

$$\|\xi_x[x(t),t]\|^2 \leq d_0 + d_1 \|x(t)\|^2 ,\ d_0 = 2(c_0 + c_4),\ d_1 = 2c_2,$$
$$\|\xi_y[x(t),t]\|^2 \leq d_2 + d_3 \|x(t)\|^2 ,\ d_2 = 2(c_1 + c_5),\ d_3 = 2c_3. \quad (6)$$

2.2. Basic assumptions

We suppose hereafter that

A1. The nonlinearity $f(x)$ in (1) belongs to the class $\mathcal{C}(A, c_0, c_1)$ (5) (see, Figure 1). Certainly, the knowledge of the matrix A (characterizing the "nominal linear plant") as well as two scalar parameters gives very "approximative" information about the nonlinear function f. Nevertheless, the approximative values of these class-parameters can be estimated *a* priory based on the following consideration:

— $A \simeq \nabla_x f(x)|_{x=0}$ if the vector field $f(x)$ is differential (and hence, $c_1 = 0$) in the origin;

— the parameter c_1 defines a possible upper bound of the velocity norm in the origin, i. e.,

$$\|f(x)\|_{x=x_0=0} \simeq \|\dot{x}(0)\| \leq c_1;$$

— the parameter c_2 characterizes the maximum possible linear increment of the difference, i. e.,

$$\sup_{x\in\mathbb{R}^n} \|f(x) - Ax\| / \|x\| \leq c_2.$$

The same interpretation can be done for the parameters of the class $\mathcal{C}(C, c_3, c_4)$.

A2. Based on the upper estimate (6) below we accept that

$$\xi^\mathsf{T}\xi = \begin{pmatrix} \xi_x[x(t),t] \\ \xi_y[x(t),t] \end{pmatrix}^\mathsf{T} \begin{pmatrix} \xi_x[x(t),t] \\ \xi_y[x(t),t] \end{pmatrix}$$
$$= \xi_x^\mathsf{T}[x(t),t]\,\xi_x[x(t),t] + \xi_y^\mathsf{T}[x(t),t]\,\xi_y[x(t),t] \quad (7)$$
$$\leq b_0 + b_1 \|x(t)\|^2 ,\ b_0 = d_0 + d_2,\ b_1 = d_1 + d_3.$$

A3. The set of all admissible control actions $\mathcal{U}$ is a convex closed bounded complete set (compact):

$$u^+ = \operatorname{diam} \mathcal{U} := \min_{p\in\mathcal{U}} \|s - p\| < \infty.$$

If so, then for any $s \in \mathbb{R}^m$ there exists an unique $p_0 \in \mathcal{U}$, called the projection of s to the set $\mathcal{U}$, such that $\|s - p\| \geq \|s - p_0\|$ for any $p \in U$ (see Figure 2 and [23]). In other words:

$$\|s - p_0\| = \min_{p\in\mathcal{U}} \{\|s - p\| | s \in \mathbb{R}^m\}.$$

The control action

$$u(t) \subset \mathcal{U} \in \mathbb{R}^m \quad (8)$$

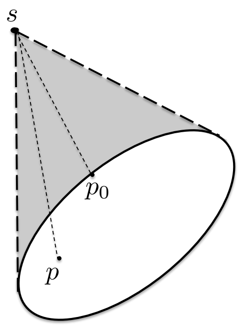

Fig. 2: Projection concept (scalar case).

is obtained as a result of the application of the nonlinear operator $\pi_{\mathcal{U}}(\cdot)$ acting as

$$\begin{aligned} u(t) &:= \pi_{\mathcal{U}}(K_{t_i}\hat{x}(t)) \\ \pi_{\mathcal{U}}(s) &:= \{\bar{u} \in \mathcal{U} \mid \|\bar{u} - s\| \le \|u - s\| \;\; \forall s \in \mathbb{R}^m, u \in \mathcal{U}\} \end{aligned} \tag{9}$$

where $K_{t_i} \in \mathbb{R}^{m \times n}$ is a gain-matrix which also should be designed on-line, so that K_{t_i} remains to be constant within the semi-open a priory given intervals $(t_{i-1}, t_i]$, for all $i = 1, 2, \ldots$, but subjected by tuning in sample-times t_i.

In (9) $\hat{x}(t) \in \mathbb{R}^n$ is an estimate of the state $x(t)$.

A4. The state estimates $\hat{x}(t)$ are generated by the observer (for some fixed $\hat{x}(0) = \hat{x}_0$)

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L_{t_i}[y(x) - C\hat{x}(t)] \tag{10}$$

where $L_{t_i} \in \mathbb{R}^{n \times p}$ is a time invariant gain-matrix to be designed, keeping a constant value on each interval $(t_{i-1}, t_i]$.

2.3. Extended dynamic form

For the observer (10), in view of (9), we have

$$\begin{aligned} \dot{\hat{x}}(t) &= [A + BK_{t_i}(t)]\hat{x}(t) + L_{t_i}[y(t) - C\hat{x}(t)] + B\Delta\pi[K_{t_i}\hat{x}(t)] \\ \Delta\pi[K_{t_i}\hat{x}(t)] &:= \pi_{\mathcal{U}}[K_{t_i}\hat{x}(t)] - K_{t_i}\hat{x}(t) \end{aligned} \tag{11}$$

and define the state estimation error

$$e(t) = x(t) - \hat{x}(t) \tag{12}$$

which implies

$$\dot{e}(t) = [A - L_{t_i}C]e(t) + \xi_x[x(t), t] - L_{t_i}\xi_y[x(t), t], \; e(0) = e_0. \tag{13}$$

Combining (11) and (13) for the extended vector $z^{\intercal}(t) = [\hat{x}^{\intercal}(t),\ e^{\intercal}(t)]$, we derive

$$\dot{z}(t) = \mathcal{A}_{t_i}\left(K_{t_i}, L_{t_i}\right) z(t) + \mathcal{F}_{t_i}\left(L_{t_i}\right) \xi\left[x\left(t\right), t\right] + \mathcal{B}\upsilon_{t_i}, \quad z(0) = z_0, \tag{14}$$

where

$$\mathcal{A}_{t_i}\left(K_{t_i}, L_{t_i}\right) := \begin{bmatrix} A+BK_{t_i} & L_{t_i}C \\ 0_{n\times n} & A-L_{t_i}C \end{bmatrix} \in \mathbb{R}^{2n\times 2n},\ \mathcal{F}_{t_i}\left(L_{t_i}\right) := \begin{bmatrix} 0_{n\times n} & L_{t_i} \\ I_{n\times n} & -L_{t_i} \end{bmatrix} \in \mathbb{R}^{2n\times(n+p)},$$

$$\xi[x(t), t] := \begin{bmatrix} \xi_x[x(t),t] \\ \xi_y[x(t),t] \end{bmatrix} \in \mathbb{R}^{n+p},\ \mathcal{B} := \begin{bmatrix} B & 0_{n\times m} \\ 0_{n\times m} & 0_{n\times m} \end{bmatrix} \in \mathbb{R}^{2n\times(n+m)},\ \upsilon_{t_i} := \begin{bmatrix} \Delta\pi[K_{t_i}\hat{x}(t)] \\ 0_m \end{bmatrix} \in \mathbb{R}^{2m}.$$

2.4. Problem formulation

Notice that, in the presence of the unmodelled dynamics ($\xi\left[x\left(t\right), t\right] \neq 0$) it is impossible to stabilize the given dynamics exactly providing asymptotic origin convergence. The boundedness of the trajectories can be guaranteed only (certainly, if it is possible within the admissible feedbacks (9)). Since any bounded trajectories can be imposed into some convex set (here we will select an ellipsoid), the "best designing" what one can do is to minimize the "size" of this ellipsoid varying the gains matrix K_{t_i} and L_{t_i} using on-line information $\{\hat{x}\left(t\right), u\left(t\right)\}_{t\geq 0}$.

Definition 2.2. We say that a trajectory $\{x(t)\}_{t\geq 0}$ belongs asymptotically to the *attractive ellipsoid*

$$\mathcal{E}\left(\ell, \bar{P}\right) = \left\{x \in \mathbb{R}^n : x^{\intercal}\bar{P}x \leq 1,\ \bar{P} = \bar{P}^{\intercal} > 0\right\}$$

with the center at the point $x = \ell$ and the corresponding matrix $\bar{P}$, if

$$\limsup_{t\to\infty}\ [x(t) - \ell]^{\intercal}\,\bar{P}\,[x(t) - \ell] \leq 1.$$

This means that asymptotically all trajectories of a considered system arrive to the ellipsoid-set $\mathcal{E}\left(\ell, \bar{P}\right)$ refereed below to as an *attractive ellipsoid*.

Now we are ready to formulate the problem which we are going to solve.

Problem 2.3. Based on the available information $\{y(t), \hat{x}\left(t\right), u\left(t\right)\}_{t\geq 0}$ find the sequences $\{K_{t_i}\}_{i=1,2,\ldots}$ and $\{L_{t_i}\}_{i=1,2,\ldots}$ of the gain matrices K_{t_i}, L_{t_i}, which for any plant with uncertainties from the given class $\mathcal{C}$ (5) may guarantee the existence of an attractive ellipsoid of a minimal possible "size" (traditionally, the size $\mathcal{E}\left(0, \bar{P}\right)$ is associated with the trace of the ellipsoidal matrix $\bar{P}$):

$$\operatorname{tr}\left\{\bar{P}_{t_i}\right\} \to \sup_{K_{t_i},\ L_{t_i}\ (i=1,2,\ldots)}. \tag{15}$$

The sequences $\{K_{t_i}, L_{t_i}\}_{i=1,2,\ldots}$ of the gain matrices in (9) and in (10), realizing (15), guarantees the, so-called, *zone-stability* under bounded control signals (8) for any uncertain system (14).

We do not mean optimization related to its volume because a volume of an ellipsoid (or, equivalently, its determinant) in fact is a bad function for the characterization of its "size". Since

$$\det \bar{P} = \prod_{j=1}^{N} \lambda_j(\bar{P}) \text{ and } \sqrt{\lambda_j(\bar{P})} = \frac{1}{\rho_j(\bar{P})}$$

where $\lambda_j(\bar{P})$ are the eigenvalues of the ellipsoidal matrix $\bar{P}$, and $\rho_j(\bar{P})$ are the distance from center to each semi-axises of the ellipsoid, then the maximization of $\det(\bar{P})$ is equivalent to minimization of

$$\prod_{j=1}^{N} \rho_j(\bar{P}) = \text{vol}(\bar{P})$$

that is, the minimization of its volume may provoke a bad situation: the product may be very small, but one of semi-diameter may be very large. But considering $\text{tr}\left\{\bar{P}^{-1}\right\}$ which one wants to minimize (or maximize $\text{tr}\left\{\bar{P}\right\}$), we have

$$\text{tr}\left\{\bar{P}^{-1}\right\} = \sum_{i=1}^{n} \lambda_j^{-1}(\bar{P}) = \sum_{i=1}^{n} \rho_j^2(\bar{P}) \geq \rho_{\max}^2(\bar{P}) \tag{16}$$

which means that minimization of $\text{tr}\left\{\bar{P}^{-1}\right\}$ leads to minimization of the maximal semi-diameter of an ellipsoid.

3. ROBUST BOUNDED OUTPUT CONTROL SYNTHESIS

In this section we present the main contribution of this paper, related to the design of an bounded output controller which provides a robust performance for system (14) under perturbations and the unknown dynamic, based on AEM concept. Firstly, let us select the feedback controller as projectional control (9). Notice that for the system (14) the gain matrix K_{t_i} can be found for each time $t \in (t_{i-1}, t_i]$. But in this case the available information to find K_{t_i} are given by previous data information, that is, in the time t_i we use information up to t_{i-1}, i. e., we use t_{i-1} data information. Thus now the problem we are interested in is to find the gain matrices K_{t_i} and L_{t_i} based on the data information within the time interval $(t_{i-1}, t_i]$.

First, let us formulate an auxiliary result used below in the proof of the main result.

3.1. Storage function

Proposition 3.1. (On the time-interval storage function) If the collection

$$P_{t_i} \in \mathbb{R}^{2n\times 2n},\ Q_{t_i} \in \mathbb{R}^{n\times n},\ K_{t_i} \in \mathbb{R}^{m\times n},\ L_{t_i} \in \mathbb{R}^{n\times p},\ \varepsilon_{1,i},\ \varepsilon_{2,i},\ \alpha_i$$

satisfies the following matrix constraints

$$W_i := \begin{bmatrix} \begin{matrix} P_{t_i}\hat{\mathcal{A}}_{\alpha_i}(K_{t_i},L_{t_i})+ \\ +\hat{\mathcal{A}}_{\alpha_i}^{\intercal}(K_{t_i},L_{t_i})P_{t_i}+R_{t_i}(Q_{t_i}) \end{matrix} & P_{t_i}\mathcal{F}_{t_i}(L_{t_i}) & P_{t_i}\mathcal{B} \\ \mathcal{F}_{t_i}^{\intercal}(L_{t_i})P_{t_i} & -\varepsilon_{1,i}I_{(n+p)\times(n+p)} & 0_{(n+p)\times 2m} \\ \mathcal{B}^{\intercal}P_{t_i} & 0_{2m\times(n+p)} & -\varepsilon_{2,i}I_{2m\times 2m} \end{bmatrix} < 0, \tag{17}$$

$$\varepsilon_{1,i} > 0,\ \varepsilon_{2,i} > 0,\quad 0 < Q_{t_i} = Q_{t_i}^{\intercal},$$
$$K_{t_i}^{\intercal} K_{t_i} \leq Q_{t_i},$$
$$0 < P_{t_i} = P_{t_i}^{\intercal} = \operatorname{diag}\left[P_{A,t_i},\ P_{A,t_i}\right],$$
$$\hat{\mathcal{A}}_{\alpha_i}\left(K_{t_i},, L_{t_i}\right) := \mathcal{A}_{t_i}\left(K_{t_i},, L_{t_i}\right) + \frac{\alpha_i}{2} I_{2n},$$
$$R_{t_i}\left(Q_{t_i}\right) := \varepsilon_{1,i} d_1 I_{2n\times 2n} + 2\varepsilon_{2,i} \operatorname{diag}(Q_{t_i}, 0_{n\times n}),$$

then for the energetic ("*storage*") function

$$V_i[z(t)] := z^{\intercal}(t) P_{t_i} z(t), \tag{18}$$

defined on $(t_{i-1}, t]$, satisfies the following differential inequality

$$\dot{V}_i[z(t)] \leq -\alpha_i V_i[z(t)] + \beta_i, \tag{19}$$

$$\beta_i := \varepsilon_{1,i} d_0 + 2\varepsilon_{2,i} \left(u^{+}\right)^2. \tag{20}$$

Proposition 3.1 is proven in the appendix A. It is well-known that the concept of an energetic function was rigorously formalized by means of the Lyapunov stability theory as well as the notion of a positive invariant set. Here we just note that if there exists a set of solutions $(P_{A,t_i}, Q_{t_i}, K_{t_i}, L_{t_i}, \varepsilon_{1,i}, \varepsilon_{2,i}, \alpha_i)$ within the time interval $(t_{i-1}, t_i]$, such that (17) holds, the storage function (18) is not obligatory monotonically non increasing, that is, $V_i(z)$ is not a Lyapunov function for the considered system at least for this time-interval. Below we suggest the construction of a *Lyapunov-Like* function whose derivative on the trajectories of the considered controlled system is strictly negative outside of an ellipsoid. So, below we present a Lyapunov zone-convergence analysis.

3.2. Zone-convergence analysis

Let us consider the function

$$\begin{gathered} G(t) := \sum_{i=1}^{\infty} \chi_i(t)\, \mathcal{G}_i(t) \\ \chi_i(t) := \begin{cases} 1 & \text{if} \quad t \in (t_{i-1}, t_i] \\ 0 & \text{if} \quad t \notin (t_{i-1}, t_i] \end{cases},\quad \sum_{i=1}^{\infty} \chi_i(t) = 1, \end{gathered} \tag{21}$$

where

$$\begin{gathered} \mathcal{G}_i(t) = \left(\left[\sqrt{V_i[z(t)]} - \sqrt{\frac{\beta_i}{\alpha_i}} \right]_+ \right)^2,\ t \in (t_{i-1}, t_i], \\ [\gamma]_+ := \begin{cases} \gamma & \text{if} \quad \gamma \geq 0 \\ 0 & \text{if} \quad \gamma < 0. \end{cases} \end{gathered} \tag{22}$$

Notice that the function $[\gamma]_+$ is not differentiable in the point $\gamma = 0$, but the function $\left([\cdot]_+\right)^2$ is differential everywhere. In (22) the process $z(t)$ is defined by (14), therefore the function $G(t)$ is defined on all possible trajectories of (14).

Proposition 3.2. (on a zone convergence) If

1) the collection $(P_{A,t_i}, Q_{t_i}, K_{t_i}, L_{t_i}, \varepsilon_{1,i}, \varepsilon_{2,i}, \alpha_i)$ satisfies the set of matrix inequalities in Proposition 3.1 within each the time-interval $(t_{i-1}, t_i]$, $i = 1, 2, \ldots$;

2) the following additional dynamic constraint is fulfilled at each stage $i = 1, 2, \ldots$:

$$\begin{aligned} \mathcal{G}_{i-1}(t_i) &= \left(\left[\sqrt{z^{\mathsf{T}}(t_i)\, P_{i-1} z(t_i)} - \sqrt{\frac{\beta_{i-1}}{\alpha_{i-1}}}\right]_+\right)^2 \\ &\geq \left(\left[\sqrt{z^{\mathsf{T}}(t_i)\, P_i z(t_i)} - \sqrt{\frac{\beta_i}{\alpha_i}}\right]_+\right)^2 = \mathcal{G}_i(t_i) \end{aligned} \tag{23}$$

then the function $G(t)$ (21) is the Lyapunov function for the dynamic system (14), namely,

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t) \leq 0 \tag{24}$$

for all $t \geq 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t) < 0 \text{ if } \sqrt{V_{i(t)}[z(t)]} > \mu_{i(t)} \tag{25}$$

providing the "attractivity property"

$$\left[\sqrt{V_{i(t)}[z(t)]} - \mu_{i(t)}\right]_+ \to 0 \text{ as } t \to \infty,$$

$$\mu_{i(t)} := \sqrt{\beta_{i(t)}/\alpha_{i(t)}},\ i(t) := \{i : t \in (t_{i-1}, t_i]\}.$$

The Proposition 3.2 is proven in the appendix B.

Corollary 3.3. If in Proposition 3.2 the numerical sequence $\{\mu_{i(t)}\}$ and the matrix sequence $\{P_i^{-1}\}$ monotonically decreases, that is,

$$\mu_{i(t')} \geq \mu_{i(t'')} \text{ for } t' > t'',\ P_{i-1}^{-1} \geq P_i^{-1}, \tag{26}$$

then, by the Weierstrass theorem, both have their limits

$$\lim_{t\to\infty} \mu_{i(t)} = \tilde{\mu},\ \lim_{i\to\infty} P_i^{-1} = \tilde{P}^{-1},$$

which means that the ellipsoid $\mathcal{E}\left(0, \frac{1}{\tilde{\mu}}\tilde{P}\right)$ is *attractive* for all possible trajectories generated by (14) fulfilling the inequality

$$\limsup_{t\to\infty} z^{\mathsf{T}}(t) \left(\frac{1}{\tilde{\mu}}\tilde{P}\right) z(t) \leq 1. \tag{27}$$

The details of the proof of this Corollary are also given in Appendix.

3.3. On the attractive ellipsoid of a "minimal size"

As it has been mentioned above (16), one can arrange the selection of the parameters

$$(P_{A,t_i}, Q_{t_i}, K_{t_i}, L_{t_i}, \varepsilon_{1,i}, \varepsilon_{2,i}, \alpha_i)$$

(which satisfy the conditions of Proposition 3.2 at each time-interval $(t_{i-1}, t_i]$ and fulfills additionally the monotonicity condition (26)) in such a way that the attractive allipsoid $\mathcal{E}\left(0, \frac{1}{\tilde{\mu}}\tilde{P}\right)$ would be of a "minimal size". This process corresponds to the solution of the following optimization problem at each time-interval $(t_{i-1}, t_i]$:

$$\operatorname{tr}\left\{\frac{\alpha_{i(t)}}{\beta_{i(t)}}P_i^{-1}\right\} \to \inf_{P_{A,t_i}, Q_{t_i}, K_{t_i}, L_{t_i}, \varepsilon_{1,i}, \varepsilon_{2,i}, \alpha_i} \tag{28}$$
$$\text{subject to the constraints (17), (20) and (26).}$$

Denote the solution of this optimization problem (28) at each step $i = 1, 2, \ldots$ as

$$P^*_{A,t_i}, Q^*_{t_i}, K^*_{t_i}, L^*_{t_i}, \varepsilon^*_{1,i}, \varepsilon^*_{2,i}, \alpha^*_i.$$

By the monotonicity conditions (26) we may conclude that there exist the limits

$$\lim_{i\to\infty} P^*_{A,t_i} := \tilde{P}^*_A, \ \lim_{t\to\infty} \mu^*_{i(t)} := \tilde{\mu}^*,$$

so that we can consider the ellipsoid $\mathcal{E}\left(0, \frac{1}{\tilde{\mu}^*}\tilde{P}^*\right)$ with $\tilde{P}^* = \begin{bmatrix} \tilde{P}^*_A & 0 \\ 0 & \tilde{P}^*_A \end{bmatrix}$ is the asymptotically attractive ellipsoid of a "minimal size" in the extended space of the variable $z_t^\intercal = \left(\hat{x}_t^\intercal, \quad x_t - \hat{x}_t^\intercal\right)$. To estimate the size of the "optimal attractive ellipsoid" in the state space of the system (1) notice that

$$x_t = Hz_t, \ H := \begin{bmatrix} I_{n\times n} & I_{n\times n} \end{bmatrix}$$

and therefore

$$x_t^\intercal P_x x_t = z_t^\intercal H^\intercal P_x H z_t.$$

From the other side, as it follows from (27), the attractive ellipsoid in the extended $z$space is $\dfrac{1}{\tilde{\mu}^*}\tilde{P}^*$, so there should be fulfilled the equality $\dfrac{1}{\tilde{\mu}^*}\tilde{P}^* = H^\intercal P_x H$. Unfortunately, in general case this identity can not be fulfilled since the size of the matrix P_x, which we are interested in, is less than the size of the matrix $\tilde{P}^*$. Therefore, we suggest to estimate the "minimal ellipsoid" in the x-space as the solution P_x^* of the following optimization problem

$$\left\|\frac{1}{\tilde{\mu}^*}\tilde{P}^* - H^\intercal P_x H\right\|^2 \to \min_{P_x \geq 0} \tag{29}$$

where the norm in the Hilbert space of finite-dimensional matrices is defined as

$$\langle A, B\rangle := \operatorname{tr}\left\{AB^\intercal\right\}, \ \ \|A\|^2 := \langle A, A\rangle = \operatorname{tr}\left\{AA^\intercal\right\}.$$

The solution P_x^* of the optimization problem (29) is as follows[1] :

$$P_x^* = \frac{1}{\tilde{\mu}^*} \left(H^\intercal\right)^+ \tilde{P}^* H^+$$

(H^+ is the pseudo-inverse matrix to H defined in the Moore–Penrose sense [20]). Since in our case

$$H^+ = \begin{bmatrix} I_{n\times n} & I_{n\times n} \end{bmatrix}^+ = \frac{1}{2}\begin{bmatrix} I_{n\times n} \\ I_{n\times n} \end{bmatrix}$$

we obtain

$$P_x^* = \frac{1}{4\tilde{\mu}^*}\begin{bmatrix} I_{n\times n} & I_{n\times n} \end{bmatrix}\begin{bmatrix} \tilde{P}_A^* & 0 \\ 0 & \tilde{P}_A^* \end{bmatrix}\begin{bmatrix} I_{n\times n} \\ I_{n\times n} \end{bmatrix} = \frac{1}{2\tilde{\mu}^*}\tilde{P}_A^*.$$

Hence, the attractive ellipsoid of the “minimal size” in the state space, which guarantees the property $\limsup\limits_{t\to\infty} x_t^\intercal P_x^* x_t \le 1$, is defined by the ellipsoid matrix

$$P_x^* = \frac{1}{2\tilde{\mu}^*}\tilde{P}_A^*. \tag{30}$$

So, the attractive ellipsoid in x-space is twice more than the corresponding minimal ellipsoid in z-space.

4. NUMERICAL ASPECTS

4.1. Transformation of BMI's constraints in to LMI's

The optimization problem (28) is a nonlinear optimization problem, subject (with fixed $\alpha_i, \varepsilon_{1,i}, \varepsilon_{2,i}, \lambda_i$) to the bilinear matrix inequality (BMI) (17). This bilinear (under fixed scalars) optimization problem can be transformed to a linear one (containing only LMI‘s constraints) using the transformation of variables given in the following Proposition.

---

[1]The solution X^* of the optimization problem

$$\|A - XB\|^2 \to \min_X \tag{a}$$

satisfies the identity

$$\begin{gathered}\frac{\partial}{\partial X}\|A - XB\|^2 = \frac{\partial}{\partial X}\operatorname{tr}\left\{(A - XB)\left(A^\intercal - B^\intercal X^\intercal\right)\right\} \\ = -2\left(AB^\intercal - X^+ BB^\intercal\right) = 0\end{gathered}$$

or equivalently $X^* BB^\intercal = AB^\intercal$, which solution (in the case $BB^\intercal > 0$) is

$$X^* = AB^+ + Y\left(I - BB^+\right),\ B^+ := B^\intercal\left(BB^\intercal\right)^{-1}$$

where Y is any matrix of the corresponding size. So, one has

$$\begin{gathered}\|X^*\|^2 = \left\|AB^+\right\|^2 + \left\|Y\left(I - BB^+\right)\right\|^2 \\ +2\operatorname{tr}\left\{AB^+\left(I - \left(\left(BB^\intercal\right)^{-1}B\right)B^\intercal\right)Y^\intercal\right\} \\ = \left\|AB^+\right\|^2 + \left\|Y\left(I - BB^+\right)\right\|^2 \ge \left\|AB^+\right\|^2.\end{gathered}$$

This means that the solution of the optimization problem (a), which has the minimal norm, is $X^* = AB^+$.

Proposition 4.1. The solution $(P_{A,t_i}, Q_{t_i}, K_{t_i}, L_{t_i})$ of the optimization problem (28) under fixed scalar parameters α_i, $\varepsilon_{1,i}$, $\varepsilon_{2,i}$ and λ_i is "*isomorphic*" to the set of variables

$$X_{t_i} := \operatorname{diag}(X_{11}^{t_i}, X_{22}^{t_i}),\ Y_1^{t_i} := P_{A,t_i} B K_{t_i},\ Y_2^{t_i} := P_{A,t_i} L_{t_i}$$

and uniquely related with previous one as

$$X_{11}^{t_i} := P_{t_i}^{B,11} > 0,\ X_{22}^{t_i} := P_{t_i}^{B,22} > 0,$$
$$Y_1^{t_i} := \begin{bmatrix} 0_{(n-m)\times n} \\ Y_i^{1,2} \end{bmatrix},\ Y_i^{1,2} := P_{t_i}^{B,22} K_{t_i},\ Y_{i,2} := P_{t_i}^{A} L_{t_i}$$

satisfying the following *LMI's*

$$\begin{bmatrix} \bar{W}_{11} & \bar{W}_{12} & \bar{W}_{13} \\ \bar{W}_{12}^{\intercal} & -\varepsilon_1 I_{(n+p)\times(n+p)} & 0_{(n+p)\times 2m} \\ \bar{W}_{13}^{\intercal} & 0_{2m\times(n+p)} & -\varepsilon_2 I_{2m\times 2m} \end{bmatrix} < 0 \tag{31}$$

$$\begin{bmatrix} (u^+)^2 & x_{t_{i-1}}^{\intercal} Q_{t_i} \\ Q_{t_i} x_{t_{i-1}} & Q_{t_i} \end{bmatrix} > 0,\ t \in [t_{i-1}, t_i) \tag{32}$$

with the following elements

$$\bar{W}_{11} = \begin{bmatrix} \begin{matrix} X_{t_i} A_\alpha + A_\alpha^{\intercal} X_{t_i} + Y_1^{t_i} + \left(Y_1^{t_i}\right)^{\intercal} \\ + d_1 \varepsilon_{1,i} I_{n\times n} + \varepsilon_{2,i} Q_{t_i} \end{matrix} & Y_2^{t_i} C \\ C^{\intercal} \left(Y_2^{t_i}\right)^{\intercal} & \begin{matrix} X_{t_i} A_\alpha + A_\alpha^{\intercal} X_{t_i} - Y_2^{t_i} C - C^{\intercal} \left(Y_2^{t_i}\right)^{\intercal} \\ + d_1 \varepsilon_{1,i} I_{n\times n} \end{matrix} \end{bmatrix},$$

$$\bar{W}_{12} = \begin{bmatrix} 0_{n\times n} & Y_2^{t_i} \\ X_{t_i} & -Y_2^{t_i} \end{bmatrix}, \quad \bar{W}_{13} = \begin{bmatrix} X_{t_i} B & 0_{n\times m} \\ 0_{n\times m} & 0_{n\times m} \end{bmatrix}.$$

The solution $P_{t_i}^*$, $K_{t_i}^*$, and $L_{t_i}^*$ are obtained using the, so-called, "*regular form*" [27]. The matrix $P_{A,t_i} := X_{11}^{t_i}$ can be found as follows:

$$P_{A,t_i} := G^{\intercal} P_{B,t_i} G, \quad B := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$G := \begin{bmatrix} I_{(n-m)\times(n-m)} & -B_1 B_2^{-1} \\ 0_{m\times(n-m)} & B_2^{-1} \end{bmatrix}, \quad P_{B,t_i} := \begin{bmatrix} P_{B,1}^{t_i} & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & P_{B,2}^{t_i} \end{bmatrix}$$

where $B_1 \in \mathbb{R}^{(n-m)\times m}$, $B_2 \in \mathbb{R}^{m\times m}$ and $\det(B_2) \neq 0$. Finally, the solution of the problem (28) becomes to be as

$$L_{t_i}^* := (\mathcal{X}_i^*)^{-1} Y_{i,2}^*, \quad K_{t_i}^* := \frac{\mathcal{X}_i^{B,22*}}{\det\left(B_2^{-1}\right)} \left[B_1 B_2^{-1},\ B_2^{-1}\right] Y_i^{1,2*}.$$

Remark 4.2. Notice that the optimization problem (28), as it is formulated in the Proposition above, contains the additional constraint (32) which is introduced here to update the initial value of the matrix Q_{t_i} which restricts the admissible set of gain matrices K_{t_i} as $K_{t_i}^{\intercal} K_{t_i} \leq Q_{t_i}$ at each time-interval $(t_{i-1}, t_i]$.

4.2. Computational aspects

The problem (28) can be solved numerically with the MATLAB Toolbox SeDuMi and Yalmip. The calculation of K_{t_i}, P_{t_i}, $i = 1, 2, \ldots$. an be obtained recursively using the following procedure:

1. First, fixing some initial values of scalar parameters $\alpha_i = \alpha_i^0$, $\varepsilon_{1,i} = \varepsilon_{1,i}^0$, $\varepsilon_{2,i} = \varepsilon_{2,i}^0$, $\lambda_i = \lambda_i^0$ and we apply the MATLAB Toolbox SeDuMi to solve the corresponding constraint optimization problem (28). As the result, we obtain the matrices P_{t_0}, Q_{t_0}, K_{t_0}, and L_{t_0}.

2. Under found matrices P_{t_0}, Q_{t_0}, K_{t_0}, and L_{t_0} we suggest to increase the parameters α_i and λ_i taking
$$\alpha_i^1 = \alpha_i^0 + \Delta\alpha_i,\ 0 < \Delta\alpha_i \ll 1,$$
$$\lambda_i^1 = \lambda_i^0 + \Delta\lambda_i,\ 0 < \Delta\lambda i \ll 1,$$
and to decrease $\varepsilon_{1,i}$ and $\varepsilon_{2,i}$, making
$$\varepsilon_{1,i}^1 = \varepsilon_{1,i}^0 - \Delta\varepsilon_{1,i} > 0,\ 0 < \Delta\varepsilon_{1,i} \ll 1,$$
$$\varepsilon_{2,i}^1 = \varepsilon_{2,i}^0 - \Delta\varepsilon_{2,i} > 0,\ 0 < \Delta\varepsilon_{2,i} \ll 1.$$

3. When the SeduMi Toolbox "informs" that the current LMI's (31) has no solution, we stop the procedure. The last admissible parameters are declared as optimal ones: $(P_{A,t_i}^*, Q_{t_i}^*, K_{t_i}^*, L_{t_i}^*, \varepsilon_{1,i}^*, \varepsilon_{2,i}^*, \lambda_i^*)$ for time interval $t \in (t_{i-1}, t_i]$.

4. Apply the switched controller (9) in (10) for the current time interval $(t_{i-1}, t_i]$.

5. Increase $i = i + 1$, and return to "*step 1*", verifying the following conditions:

   a) If the condition (26) holds, we may conclude that the set of solutions P_{A,t_i}, Q_{t_i}, K_{t_i}, L_{t_i}, $\varepsilon_{1,i}, \varepsilon_{2,i}, \lambda_i$ are the final solution and declared as the sub-optimal solution set $(P_{A,t_i}^*, Q_{t_i}^*, K_{t_i}^*, L_{t_i}^*, \varepsilon_{1,i}^*, \varepsilon_{2,i}^*, \lambda_i^*)$ for each time interval $t \in (t_{i-1}, t_i]$.

   b) If (26) does not hold, return back to "*step 1*" with $\alpha_i^0 = \alpha_i^*$, $\varepsilon_{1,i}^0 = \varepsilon_{1,i}^*$, $\varepsilon_{2,i}^0 = \varepsilon_{2,i}^*$, and $\lambda i^0 = \lambda_i^*$.

6. Since $\bar{P}_{A,t_i} \to \bar{P}_A$, the minimal size of ellipsoidal matrix $\bar{P}_A$ is declared as $\bar{P}_{A,t_i}$ for large enough "i".

5. ILLUSTRATIVE EXAMPLE

In this section we consider the flexible arm robot (two-degree of freedom flexible pendulum) depicted in Figure 3. The flexibility of link is a result of lightening a robot arm (for example in space applications). The study of link flexibility is also enforced for some kind of heavy manipulators, such as large-scale systems. If the spring constant of the flexible arm is 0, this system result to the extreme case of the underectuated two-link robot (the Pendubot system). The control to be designed is intended to stabilize the pendulum in the vertical position using the shoulder-torque in the first (lowest) joint.

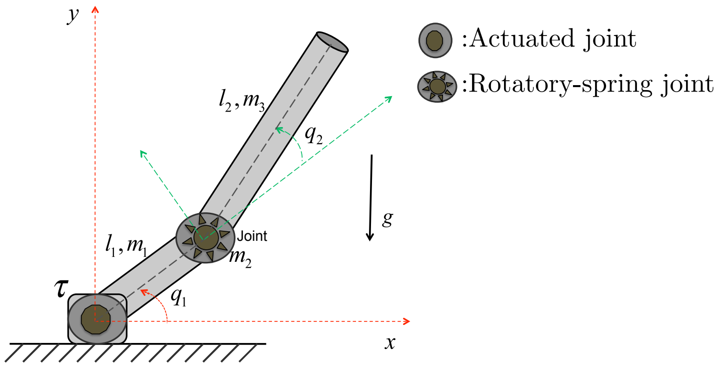

Fig. 3: Two-degree of freedom flexible pendulum.

5.1. Dynamic model

The mathematical model of the considered systems can be presented as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = T \tag{33}$$

where the position coordinates $q \in \mathbb{R}^M$ with associated velocities $\dot{q}$ and accelerations $\ddot{q}$ are controlled by the vector $T \in \mathbb{R}^M$ of driving forces. The generalized moment of inertia $D(q) \in \mathbb{R}^{M\times M}$ is a symmetric and positive definite matrix, the Coriolis (centripetal) forces are $C(q,\dot{q})\dot{q} \in \mathbb{R}^M$, and the gravitational forces are denoted by $G(q) \in \mathbb{R}^M$. All vary along the trajectories, $M = 2$ is the degree of freedom. We can represent (33) in the standard Cauchy affine (with respect to the control) form

$$\dot{\bar{x}} = f(\bar{x}) + g(\bar{x})u + \zeta(t), \;\; f(x) := \left[\, f_1(\bar{x}) \; f_2(\bar{x}) \; f_3(\bar{x}) \; f_4(\bar{x}) \,\right]^{\intercal}, \;\; g(\bar{x}) := \left[\, 0 \; 0 \; g_1(\bar{x}) \; g_2(\bar{x}) \,\right]^{\intercal}.$$

The problem to be solved is to stabilize this system in the upper equilibrium point of the desired unstable position[2]. The functions $f(x)$ and $g(x)$ have the following structure:

$$\begin{bmatrix} f_1(\bar{x}) \\ f_2(\bar{x}) \end{bmatrix} = \begin{bmatrix} \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}, \quad \begin{bmatrix} f_3(\bar{x}) \\ f_4(\bar{x}) \end{bmatrix} = D^{-1}(\bar{x})\left[-C(\bar{x})\left[\begin{smallmatrix} \bar{x}_3 \\ \bar{x}_4 \end{smallmatrix}\right] - G(\bar{x}) - F(x)\right],$$

$$g_1(\bar{x}) = \frac{\theta_2}{\det(D(\bar{x}))}, \quad g_2(\bar{x}) = -\frac{\theta_2+\theta_3\cos\bar{x}_2}{\det(D(\bar{x}))}, \quad \det(D(\bar{x})) = \theta_1\theta_2 + \theta_3^2\cos^2\bar{x}_2$$

$$\begin{cases} d_{11} = \theta_1 + \theta_2 + \theta_3\cos(\bar{x}_2), & d_{12} = d_{21} = \theta_2 + \theta_3\cos(\bar{x}_2) \\ d_{22} = \theta_2, & c_{11} = -\theta_3\sin(\bar{x}_2)\bar{x}_3 \\ c_{12} = -\theta_3\sin(\bar{x}_2)(\bar{x}_3+\bar{x}_4), & c_{21} = -\theta_3\sin(\bar{x}_2)\bar{x}_3 \\ c_{22} = 0, & G_1(x) = \theta_4\cos(\bar{x}_1) + G_2(x) \\ G_2(x) = \theta_5\cos(\bar{x}_1+\bar{x}_2) & F_1(x) = 0 \\ F_2(x) = F_r\bar{x}_2 & \end{cases}.$$

[2]For the flexible pendulum systems we use the following variable coordinate change $\bar{x}_{1:} = q_1$, $\bar{x}_2 := q_2$, $\bar{x}_3 := \dot{q}_1$, and $\bar{x}_4 := \dot{q}_2$. The top position is $\bar{x}_{eq} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = (\pi/2, 0, 0, 0)$.

Notice that this example has the underactuated property i. e., $T = \left[\begin{smallmatrix}\tau\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] u$. The parameters of this system are given by:

$$\begin{array}{lll} \theta_1 = \mathbb{I}_{zz,1} + l_1^2\left[M_2 + M_3\right], & \theta_2 = \mathbb{I}_{zz,2}, & \theta_3 = M_2 l_1 \frac{l_2}{2}, \\ \theta_4 = \frac{3}{2} g M_1 l_1 + g M_3, & \theta_5 = g M_2 \frac{l_2}{2}. & \end{array}$$

There $M_1 = 0.0832\,\mathrm{kg}$, $M_3 = 0.12899\,\mathrm{kg}$ are the mass of ith barr, $M_2 = 0.1659 kg$ is the mass of joint, $l_1 = 0.275\,\mathrm{m}$, $l_2 = 0.467m$ is the length of ith barr, $\mathbb{I}_{zz,1} = 0.0005\,\mathrm{kg{\cdot}m^2}$, $\mathbb{I}_{zz,2} = 0.00045\,\mathrm{kg{\cdot}m^2}$ is the inertia of one of the central moments of ith barr, $i = 1, 2$, the acceleration of the gravity constant $g = 9.81$ m$\cdot$ sec^{-2}, and the spring constant of flexible arm $F_r := 3.56$.

The external perturbations $\zeta_x(t)$ are generated by sensor noise and $\zeta_y(t)$ by communication noise. Defining the deviation vector as $x(t) = \bar{x}(t) - x_{eq}$ and introducing artificial perturbations

$$\zeta_x(t) = \left[0.592 \sin(\omega t),\ 0.52 \sin(\omega t),\ 0.252 \cos(\omega t),\ 0.195 \cos(\omega t)\right]^{\mathsf{T}}$$
$$\zeta_y(t) = \left[0.02 \sin(\omega t).\ 0.004 \cos(\omega t)\right]^{\mathsf{T}}$$

with $\omega = 60$ rad., and taking into account that in a neighbor of the equilibrium point $x_1 \simeq 0$ and $x_2 \simeq 0$, and we may conclude that

$$g(x) \simeq B = \left[\begin{smallmatrix}0 & 0 & B_{31} & B_{41}\end{smallmatrix}\right]^{\mathsf{T}},\ B_{31} = \frac{\theta_2}{\theta_1\theta_2 - 2\theta_3^2},\ B_{41} = -\frac{\theta_2 + \theta_3}{\theta_1\theta_2 - 2\theta_3^2}. \tag{34}$$

By property of inertia matrix $D(\bar{x})$ and the physical construction, all denominators in (34) are non singular.

5.2. Numerical Results

Applying the suggested technique for the flexible link system and using the initial conditions

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 51 & -52 & 0 & 0 \\ -81 & -145 & 0 & 0 \end{bmatrix}.$$

First, we fix the positive scalar parameters and solve our problem with respect to the matrix variables which satisfy LMI- constraints. If the toolbox says that the LMI constrain is not feasible, it is suggested to select 10%-smaller parameter α_0 and 10%-bigger the parameter $\varepsilon_{1,0}$, $\varepsilon_{2,0}$ and etc. Such parameters, providing the feasibility of the considered LMI, obligatory exist since, by the accepted assumptions, the pair (A, B) is controllable and the pair (C, A) is observable. After 25 recurrent steps of the numerical procedure for $\lambda^* = 1$ we got:

$$\alpha_1^* = 0.8,\ \varepsilon_{1,1}^* = 0.23,\ \varepsilon_{2,1}^* = 0.23,$$

$$K_{t_1}^* = 10^3 \begin{bmatrix} 14.8717 \\ 24.4791 \\ 3.6508 \\ 3.1516 \end{bmatrix}^{\mathsf{T}},\ L_{t_0}^* = \begin{bmatrix} 82.5301 & -27.1871 \\ -27.1869 & 44.9827 \\ -147.1981 & -138.3028 \\ 0.0542 & 0.3672 \end{bmatrix},$$

$$\left(P_{t_1}^A\right)^* = 10^2 \begin{bmatrix} 65.9240 & 91.7881 & 16.8821 & 8.0148 \\ 91.7881 & 142.8598 & 25.0746 & 11.9062 \\ 16.8821 & 25.0746 & 4.8337 & 2.2943 \\ 8.0148 & 11.9062 & 2.2943 & 1.0902 \end{bmatrix}.$$

In order to illustrate the numerical results, we chose the initial conditions x_0 as follows:

$$\hat{x}_0 = \begin{bmatrix} 0.52 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}, \; x_0 = \begin{bmatrix} 0.7 \\ 0.2 \\ 0.3 \\ 0.6 \end{bmatrix}.$$

Notice that such initial conditions correspond to the internal part of the constructed attractive ellipsoid. The condition (26) holds during the 4 iteration ($i = 4$), and the current algorithm stopped at the iteration $i = 8$, obtaining the following result:

$$\alpha_8^* = 0.1, \; \varepsilon_{1,8}^* = 0.23, \; \varepsilon_{2,8}^* = 0.23,$$

$$K_{t_8}^* = 10^3 \begin{bmatrix} 66.0606 \\ 108.5089 \\ 16.2737 \\ 14.0701 \end{bmatrix}^{\mathsf{T}}, \quad L_{t_8}^* = \begin{bmatrix} 57.5370 & -14.5432 \\ -14.5426 & 36.1554 \\ -139.5972 & -128.8544 \\ 0.1045 & 0.3347 \end{bmatrix},$$

$$\left(P_{t_8}^A\right)^* = 10^2 \begin{bmatrix} 92.8922 & 118.2234 & 22.6702 & 10.7640 \\ 118.2234 & 170.6464 & 30.9422 & 14.6938 \\ 22.6702 & 30.9422 & 6.1191 & 2.9048 \\ 10.7640 & 14.6938 & 2.9048 & 1.3802 \end{bmatrix}.$$

5.3. Simulation results

The illustrative plots are divided in two figures: Figure 4 presents the trajectory $x_1(t)$ and $x_4(t)$ corresponding to the position and velocity of the first link, and Figure 5 represents the position and velocity corresponding to the second link of the system. The figures 6 and 7 depict how the trajectories goes to minimal invariant ellipsoid. Also the Figures 6 − 7 present how the ellipsoid changes in the time intervals, and how they converge to an ellipsoid of a minimal size.

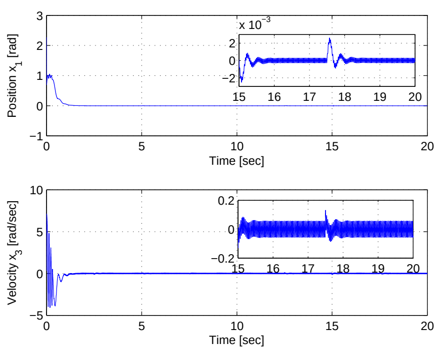

Fig. 4: First link position and velocity.

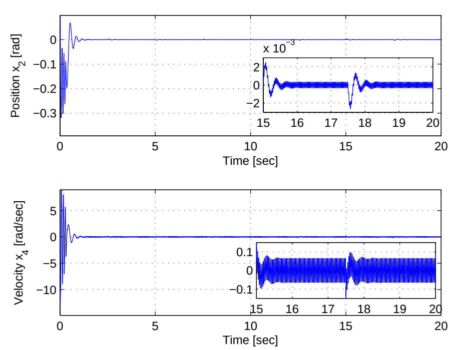

Fig. 5: Second link position and velocity.

Finally, Figure 8 presents the evolution of the bounded control law (9) over all time interval. In this figure one can see how the control action is saturated by the upper control estimate u^+.

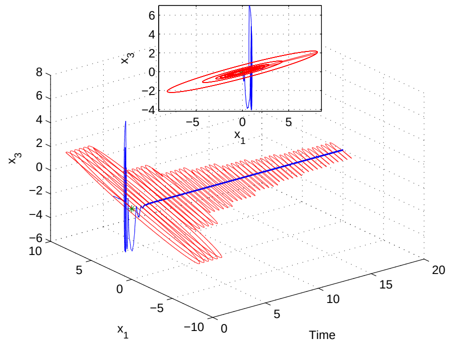

Fig. 6: Invariant ellipsoids $\mathcal{E}_{t_i}$ corresponding to trajectories $x_1(t)$ Vs $x_2(t)$.

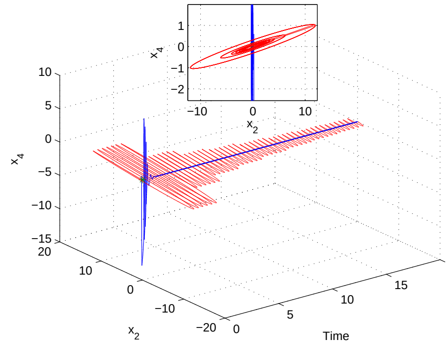

Fig. 7: Invariant ellipsoids $\mathcal{E}_{t_i}$ corresponding to trajectories $x_3(t)$ Vs $x_4(t)$.

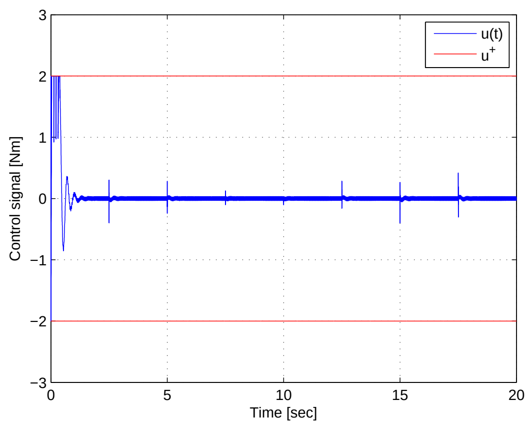

Fig. 8: Control signal.

6. CONCLUSION

In this paper we suggest a new modification of the AEM which permits to use the on-line information, obtained during the process, to adjust the projectional control law. The use of the suggested recurrent procedures together with the resolution of the constraint optimization problem (realizing the minimization of the attractive ellipsoid at each iteration) constitutes the main feature of the Attractive Ellipsoid Method with saturated control law. Numerical example, dealing with a flexible arm pendulum stabilization in unstable steady-state, demonstrates a high effectiveness of the suggested approach.

A. PROOF OF PROPOSITION 3.1

The time derivative of (18) along the system trajectories (14) for the time interval $(t_{i-1}, t_i]$ is

$$\dot V_i[z(t)] = 2z^\intercal(t)P_{t_i}\dot z(t) = 2z^\intercal(t)P_{t_i}\mathcal{A}_{t_i}\left(K_{t_i}, L_{t_i}\right)z(t) + 2z^\intercal(t)P_{t_i}\mathcal{F}_{t_i}\left(L_{t_i}\right)\xi + 2z^\intercal(t)P_{t_i}\mathcal{B}\upsilon_{t_i}$$

$$= \begin{bmatrix} z(t) \\ \xi \\ \upsilon_{t_i} \end{bmatrix}^\intercal \begin{bmatrix} P_{t_i}\mathcal{A}_{t_i}\left(K_{t_i}, L_{t_i}\right) + \mathcal{A}^\intercal_{t_i}\left(K_{t_i}, L_{t_i}\right)P_{t_i} & P_{t_i}\mathcal{F}_{t_i}\left(L_{t_i}\right) & P_{t_i}\mathcal{B} \\ \mathcal{F}^\intercal_{t_i}\left(L_{t_i}\right)P_{t_i} & 0_{(n+p)\times(n+p)} & 0_{(n+p)\times 2m} \\ \mathcal{B}^\intercal P_{t_i} & 0_{2m\times(n+p)} & 0_{2m\times 2m} \end{bmatrix} \begin{bmatrix} z(t) \\ \xi \\ \upsilon_{t_i} \end{bmatrix}.$$

Adding and subtracting the terms $\alpha V_i[z(t)]$, $\varepsilon_{1,i}\xi^\intercal\xi$ and $\varepsilon_{2,i}\upsilon^\intercal_{t_i}\upsilon_{t_i}$ in the right-hands side of the last equality leads to

$$\dot V_i[z(t)] = \begin{bmatrix} z(t) \\ \xi \\ \upsilon_{t_i} \end{bmatrix}^\intercal \begin{bmatrix} P_{t_i}\mathcal{A}_{t_i}\left(K_{t_i},L_{t_i}\right)+\mathcal{A}^\intercal_{t_i}\left(K_{t_i},L_{t_i}\right)P_{t_i} & P_{t_i}\mathcal{F}_{t_i}\left(L_{t_i}\right) & P_{t_i}\mathcal{B} \\ \mathcal{F}^\intercal_{t_i}\left(L_{t_i}\right)P_{t_i} & -\varepsilon_{1,i}I_{(n+p)\times(n+p)} & 0_{(n+p)\times 2m} \\ \mathcal{B}^\intercal P_{t_i} & 0_{2m\times(n+p)} & -\varepsilon_{2,i}I_{2m\times 2m} \end{bmatrix} \begin{bmatrix} z(t) \\ \xi \\ \upsilon_{t_i} \end{bmatrix}$$
$$-\alpha V_i[z(t)] + \varepsilon_{1,i}\xi^\intercal\xi + \varepsilon_{2,i}\upsilon^\intercal_{t_i}\upsilon_{t_i}. \tag{35}$$

Notice that by the identity $x(t) := \hat x(t) + e(t) = I_{2n\times 2n}z(t)$, and in view of the estimate

$$\xi^\intercal\xi \le b_0 + b_1\left\|x\right\|^2$$

(b_0, b_1 are defined in (7)) we conclude that $\xi^\intercal\xi \le b_0 + b_1\|z(t)\|^2$. Hence, in view of (11) the following statement is true

$$\|\Delta\pi[K_{t_i}\hat x(t)]\|^2 \le 2\|\pi_{\mathcal{U}}(K_{t_i}\hat x(t))\|^2 + 2\|K_{t_i}\hat x(t)\|^2 = 2\left(u^+\right)^2 + 2\|K_{t_i}\hat x(t)\|^2.$$

Since $\hat x(t) = Hz(t)$, $H = [I_{n\times n}\ 0_{n\times n}]$, and in view of the conditions of this proposition

$$\|\Delta\pi[K_{t_i}\hat x(t)]\|^2 \le 2\left(u^+\right)^2 + 2z^\intercal(t)H^\intercal Q_{t_i}Hz(t).$$

So, the differential equation (35) results as the following differential inclusion

$$\dot V_i[z(t)] \le$$

$$\begin{bmatrix} z(t) \\ \xi \\ \upsilon_{t_i} \end{bmatrix}^\intercal \begin{bmatrix} P_{t_i}\mathcal{A}_{\alpha,t_i}\left(K_{t_i},L_{t_i}\right)+\mathcal{A}^\intercal_{\alpha,t_i}\left(K_{t_i},L_{t_i}\right)P_{t_i}+R_{t_i} & P_{t_i}\mathcal{F}_{t_i}\left(L_{t_i}\right) & P_{t_i}\mathcal{B} \\ \mathcal{F}^\intercal_{t_i}\left(L_{t_i}\right)P_{t_i} & -\varepsilon_{1,i}I_{(n+p)\times(n+p)} & 0_{(n+p)\times 2m} \\ \mathcal{B}^\intercal P_{t_i} & 0_{2m\times(n+p)} & -\varepsilon_{2,i}I_{2m\times 2m} \end{bmatrix} \begin{bmatrix} z(t) \\ \xi \\ \upsilon_{t_i} \end{bmatrix}$$
$$-\alpha_i V_i[z(t)] + \beta_i,$$

$$\mathcal{A}_{\alpha,t_i}\left(K_{t_i}, L_{t_i}\right) := \mathcal{A}_{t_i}\left(K_{t_i}, L_{t_i}\right) + \frac{\alpha_i}{2}I_{2n\times 2n}, \quad R := \varepsilon_{1,i}d_1I_{2n\times 2n} + 2\varepsilon_{1,i}Q_{t_i},$$

from this inclusion, if

$$W := \begin{bmatrix} P_{t_i}\mathcal{A}_{\alpha,t_i}\left(K_{t_i},L_{t_i}\right)+\mathcal{A}^\intercal_{\alpha,t_i}\left(K_{t_i},L_{t_i}\right)P_{t_i}+R_{t_i} & P_{t_i}\mathcal{F}_{t_i}\left(L_{t_i}\right) & P_{t_i}\mathcal{B} \\ \mathcal{F}^\intercal_{t_i}\left(L_{t_i}\right)P_{t_i} & -\varepsilon_{1,i}I_{(n+p)\times(n+p)} & 0_{(n+p)\times 2m} \\ \mathcal{B}^\intercal P_{t_i} & 0_{2m\times(n+p)} & -\varepsilon_{2,i}I_{2m\times 2m} \end{bmatrix} < 0$$

the inequality is preserved, then

$$\dot V_i[z(t)] \le -\alpha_i V_i[z(t)] + \beta_i, \quad t \in (t_{i-1}, t_i],$$

that completes the proof.

B. PROOF OF PROPOSITION 3.2

Remember that the "generalized" derivative of the Heaviside function

$$\theta(t) := \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

is the Dirac delta-function $\delta(t) = \theta'(t)$ with the property $\int_{t=-\infty}^{\infty} \delta(t) f(t) \, \mathrm{d}t = f(0)$, for any function $f(t)$ right-continuous in the origin. Since $\chi_i(t) = \theta(t - t_{i-1}) - \theta(t - t_i)$ the differentiation of (21) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} G(t) = \sum_{i=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} [\chi_i(t) \mathcal{G}_i(t)]$$

$$= \sum_{i=1}^{\infty} \chi_i(t) \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{G}_i(t) + \sum_{i=1}^{\infty} [\delta(t - t_{i-1}) - \delta(t - t_i)] \mathcal{G}_i(t).$$

This is a singular-perturbed differential equation which in the equivalent integral form can be represented as follows:

$$G(t) - G(0) = \int_{s=0}^{t} \sum_{i=1}^{\infty} [\delta(s - t_{i-1}) - \delta(s - t_i)] \mathcal{G}_i(s) \, \mathrm{d}s + \int_{s=0}^{t} \left[\sum_{i=1}^{\infty} \chi_i(s) \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{G}_i(s) \right] \mathrm{d}s$$

$$= \sum_{i=1}^{\infty} [\mathcal{G}_i(t_{i-1}) - \mathcal{G}_i(t_i)] + \int_{s=0}^{t} \left[\sum_{i=1}^{\infty} \chi_i(s) \left[\sqrt{V_i(s)} - \sqrt{\frac{\beta_i}{\alpha_i}} \right]_+ \frac{\frac{\mathrm{d}}{\mathrm{d}s} V_i(s)}{\sqrt{V_i(s)}} \right] \mathrm{d}s$$

$$\leq \mathcal{G}_0(t_0) + [-\mathcal{G}_0(t_1) + \mathcal{G}_1(t_1)] + [-\mathcal{G}_1(t_2) + \mathcal{G}_2(t_2)] + \ldots + \left[-\mathcal{G}_{i-1}\left(t_{i(t)}\right) + \mathcal{G}_i\left(t_{i(t)}\right) \right] +$$

$$\ldots. + \int_{s=0}^{t} \left[\sum_{i=1}^{\infty} \chi_i(s) \left[\sqrt{V_i(s)} - \sqrt{\frac{\beta_i}{\alpha_i}} \right]_+ \frac{(-\alpha_i V_i(s) + \beta_i)}{\sqrt{V_i(s)}} \right] \mathrm{d}s.$$

Taking into account the "monotonicity condition" (23), we get

$$G(t) - G(0) \leq \mathcal{G}_0(0) + I(t)$$

where

$$I(t) := \int_{s=0}^{t} \left[\sum_{i=1}^{\infty} \chi_i(s) \left[\sqrt{V_i(s)} - \sqrt{\frac{\beta_i}{\alpha_i}} \right]_+ \frac{(-\alpha_i V_i(s) + \beta_i)}{\sqrt{V_i(s)}} \right] \mathrm{d}s = -\int_{s=0}^{t} \sum_{i=1}^{\infty} \alpha_i \chi_i(s) \left[\sqrt{V_i(s)} - \sqrt{\frac{\beta_i}{\alpha_i}} \right]_+ \frac{\left(V_i(s) - \frac{\beta_i}{\alpha_i} \right)}{\sqrt{V_i(s)}} \mathrm{d}s$$

$$= -\int_{s=0}^{t} \sum_{i=1}^{\infty} \alpha_i \chi_i(s) \left[\sqrt{V_i(s)} - \sqrt{\frac{\beta_i}{\alpha_i}} \right]_+ \frac{\left(\sqrt{V_i(s)} - \sqrt{\frac{\beta_i}{\alpha_i}} \right) \left(\sqrt{\frac{\beta_i}{\alpha_i}} + \sqrt{V_i(s)} \right)}{\sqrt{V_i(s)}} \mathrm{d}s$$

$$= -\int_{s=0}^{t} \sum_{i=1}^{\infty} \alpha_i \chi_i(s) \left[\sqrt{V_i(s)} - \sqrt{\frac{\beta_i}{\alpha_i}} \right]_+^2 \frac{\sqrt{\frac{\beta_i}{\alpha_i}} + \sqrt{V_i(s)}}{\sqrt{V_i(s)}} \mathrm{d}s = -\int_{s=0}^{t} \sum_{i=1}^{\infty} \alpha_i \chi_i(t) G_i(t) \frac{\left(\sqrt{V_i(t)} + \sqrt{\frac{\beta_i}{\alpha_i}} \right)}{\sqrt{V_i(t)}} \mathrm{d}s \leq 0. \tag{36}$$

Introduce the, so-called, "dominating process" $\tilde{G}(t)$ satisfying

$$\tilde{G}(t) - \tilde{G}(0) = \mathcal{G}_0(0) + I(t).$$

Obviously that if $G(0) = \tilde{G}(0)$ then $G(t) \leq \tilde{G}(t)$. Differentiation of the last identity implies

$$\frac{d}{dt}\tilde{G}(t) = -\sum_{i=1}^{\infty} \alpha_i \chi_i(t) G_i(t) \frac{\left(\sqrt{V_i(t)} + \sqrt{\frac{\beta_i}{\alpha_i}}\right)}{\sqrt{V_i(t)}} \leq 0.$$

The right hand side of the last expression is strictly negative if $\sqrt{V_{i(t)}[z(t)]} > \mu_{i(t)}$. Moreover, since $\tilde{G}(t)$ is nonnegative monotonically non-increasing function, and hence, by the Weierstrass theorem, $\tilde{G}(t)$ has a limit:

$$\lim_{t\to\infty} \tilde{G}(t) = \tilde{G}^*.$$

From (36) it follows $0 \leq \tilde{G}(t) + |I(t)| = \mathcal{G}_0(0) + \tilde{G}(0) = \text{const}$. Taking $t \to \infty$ we obtain $0 \leq \tilde{G}^* + \limsup_{t\to\infty} |I(t)| < \infty$, implying $\limsup_{t\to\infty} |I(t)| < \infty$. This means that there exists a time-sequence $\{s_k\}_{k=1,2,..}$ such that

$$\sum_{i=1}^{\infty} \alpha_i \chi_i(s_k) \left[\sqrt{V_i(s_k)} - \sqrt{\frac{\beta_i}{\alpha_i}}\right]_+^2 \frac{\left(\sqrt{V_i(s_k)} + \sqrt{\frac{\beta_i}{\alpha_i}}\right)}{\sqrt{V_i(s_k)}} \underset{k\to\infty}{\to} 0$$

and, as the result,

$$\sum_{i=1}^{\infty} \alpha_i \chi_i(s_k) \left[\sqrt{V_i(s_k)} - \sqrt{\frac{\beta_i}{\alpha_i}}\right]_+^2 = G(s_k) \underset{k\to\infty}{\to} 0.$$

Hence, from the continuity of $\tilde{G}(t)$, it follows that $\tilde{G}(s_k) \underset{k\to\infty}{\to} 0$. But the sequence $G(t)$ converges, and hence, all its subsequence have the same limit point providing $G^* = 0$. Proposition is proven.

C. PROOF OF COROLLARY 3.3

Since by the assumption of the Proposition 3.1, $W_i < 0$, we directly obtain (19). Remark that the functions $\dot{V}_i$ satisfies (19) in the time intervals $(t_{i-1}, t_i]$ for all $i := 1, 2, \ldots,$ and, as the result,

$$V_i[z(t_i)] \leq \beta_i/\alpha_i + \{V_i[z(t_{i-1})] - \beta_i/\alpha_i\} e^{-\alpha_i \tau_i} = \frac{\beta_i}{\alpha_i} - \frac{\beta_i}{\alpha_i} e^{-\alpha_i \tau_i} + e^{-\alpha_i \tau_i} V_{i-1} \quad (37)$$

with $\tau_i := t_i - t_{i-1}$, and under the monotonicity condition (23), given in Corollary 3.1, we have $V_i[z(t_i)] \leq V_{i-1}[z(t_{i-1})]$. The use of the Abel's identity (see for example [20] section 12.2.2),

$$\prod_{s=i_0}^{i} \gamma_s + \sum_{s=i_0}^{i} (1 - \gamma_s) \prod_{l=s+1}^{i} \gamma_l = 1, \prod_{s=i_0}^{i<i_0} (\cdot)_s \equiv 1, \sum_{s=i_0}^{i<i_0} (\cdot)_s \equiv 0,$$

valid for any sequence $\{\gamma_s\}$ of real numbers, implies

$$V_i[z(t_i)] \leq \frac{\beta_i}{\alpha_i}\left(1 - e^{-\alpha_i \tau_i}\right) + e^{-\alpha_i \tau_i} V_{i-1} \leq$$

$$\frac{\beta_i}{\alpha_i}\left(1 - e^{-\alpha_i \tau_i}\right) + e^{-\alpha_i \tau_i} \frac{\beta_{i-1}}{\alpha_{i-1}}\left(1 - e^{-\alpha_{i-1} \tau_{i-1}}\right) + e^{-\alpha_i \tau_i - \alpha_{i-1} \tau_{i-1}} V_{i-2} \leq$$

$$\cdots \leq \sum_{s=i_0}^{i} \frac{\beta_s}{\alpha_s}\left(1 - e^{-\alpha_s \tau_s}\right) e^{-\sum\limits_{l=s+1}^{i} \alpha_l \tau_l} + e^{-\sum\limits_{l=i_0}^{i} \alpha_l \tau_l} V_{i_0} \leq$$

$$\left(\max_{i_0 \leq s \leq i} \frac{\beta_s}{\alpha_s}\right) \sum_{s=i_0}^{i} \left(1 - e^{-\alpha_s \tau_s}\right) e^{-\sum\limits_{l=s+1}^{i} \alpha_l \tau_l} + e^{-\sum\limits_{l=i_0}^{i} \alpha_l \tau_l} V_{i_0}$$

$$= \left(\max_{i_0 \leq s \leq i} \frac{\beta_s}{\alpha_s}\right) \left[1 - e^{-\sum\limits_{l=i_0}^{i} \alpha_l \tau_l}\right] + e^{-\sum\limits_{l=i_0}^{i} \alpha_l \tau_l} V_{i_0} \leq \left(\max_{i_0 \leq s \leq i} \frac{\beta_s}{\alpha_s}\right) + e^{-\sum\limits_{l=i_0}^{i} \alpha_l \tau_l} V_{i_0}$$

and $z^{\mathsf{T}}(t_i) P_{t_i} z(t_i) \leq \mu_{i(t)}$ or equivalently, $z^{\mathsf{T}}(t_i) \frac{P_{t_i}}{\mu_{i(t)}} z(t_i) \leq 1$. Taking upper limit on $i \to \infty$ and using the monotonicity property (26) we complete the proof.

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