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NUMERICAL ANALYSIS OF A SEMI-IMPLICIT DDFV SCHEME FOR THE REGULARIZED CURVATURE DRIVEN LEVEL SET EQUATION IN 2D

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Stability and convergence of the linear semi-implicit discrete duality finite volume (DDFV) numerical scheme in 2D for the solution of the regularized curvature driven level set equation is proved. Numerical experiments concerning comparison with exact solution and image filtering problem using proposed scheme are included.

Keywords: mean curvature flow, level set equation, numerical solution, semi-implicit scheme, discrete duality finite volume method, stability, convergence

Classification: 35K65,65M08, 65M12

1. INTRODUCTION

The curvature driven level set equation [16]

$$u_t - |\nabla u| \nabla . \left(\frac{\nabla u}{|\nabla u|} \right) = 0, \tag{1.1}$$

as well as its nontrivial generalizations, is used in the applications as the motion of interfaces (free boundaries) in thermomechanics (solidification, crystal growth) and computational fluid dynamics (free surface flows, multi-phase flows of immiscible fluids, thin films), the smoothing and segmentation of images and the surface reconstructions in the image processing, computer vision and computer graphics (see e.g. [16, 15, 10] and many others see references therein).

As in the previous papers see e.g. [4],[6] the regularization of the original level set equation is used. We will study the following equation

$$u_t - f(|\nabla u|) \nabla . \left(\frac{\nabla u}{f(|\nabla u|)} \right) = 0, \tag{1.2}$$

where the function f and its properties will be described in assumptions below. The unknown function $u(t,x)$ in (1.2), defined in $I \times \Omega$, $\Omega \subset I\!R^2$ is a bounded Lipschitz domain, $I = [0,T]$, $T > 0$ is a time interval, and we will consider the equation with zero Dirichlet boundary conditions and with an initial condition:

$$\begin{aligned} u(t,x) &= 0 \quad \text{on } I \times \partial\Omega, && (1.3) \\ u(0,x) &= u^0(x). && (1.4) \end{aligned}$$

We will study our numerical scheme under the following hypothesis, called hypothesis (H0):

- Ω is a rectangular domain in $I\!R^2$;
- $u^0 \in H_0^1(\Omega)$.

For the regularization function we will use two possibilities. First is well known Evans-Spruck regularization (see [4]) and for our purpose we call it hypothesis
($\overline{\mathrm{H}}1$):

$$f(z) = \sqrt{z^2 + \varepsilon^2},$$

with fixed regularization parameter $\varepsilon > 0$.

For convergence study we need further regularization assumption as in [6]; we will call it hypothesis
(H1):

$$f(z) = \min(\sqrt{z^2 + \varepsilon^2}, b),$$

with fixed regularization parameter $\varepsilon > 0$ and another real fixed parameter b, $\varepsilon < b$.

The derivation of our numerical method for solving equation (1.2) is based on the finite volume methodology (see e.g. [5]). There are also many interesting results in this topic for numerical schemes obtained by finite difference method and finite element method as well, some of them are mentioned in [6] so here we focus only for those that are based on finite volume methods.

The mathematical analysis of finite volume methods for mean curvature flow level set equation is partly proposed in [9, 13, 7], applied to the co-volume scheme initially proposed by Walkington [17] for the Evans-Spruck regularisation of the problem (hypothesis ($\overline{\mathrm{H}}1$)). Walkington's initial scheme is nonlinear and its linear semi-implicit variant is suggested in [9]. Such semi-implicit scheme is proved to be efficient, as keeping all theoretical properties of Walkington's scheme. It is used in solving various practical 2D and 3D (large-scale) image analysis problems [3]. In [9, 13] the L^∞ stability of the solution and the L^1 stability of its gradient are given. Moreover, in [7], the consistency of the scheme is proved using the Barles and Souganidis [2] approach for solving nonlinear PDEs. However, the convergence of the co-volume semi-implicit scheme to the exact solution remains an open problem.

Another approach using finite volume scheme but with the additional points on edges of the finite volume is used in [6] but with further regularization parameter (hypothesis as (H1)). In this case the convergence analysis is presented for proposed scheme.

We construct the so-called discrete duality finite volume scheme based on idea [1] as was presented in [11] and [8]. From construction point of view this scheme is most closer to those in [9]. The main difference is that for DDF scheme we use two meshes, primal and dual and we have new additional unknowns for the dual mesh. Together with further regularization as in (H1) we are able to prove convergence results for DDF scheme for the regularization level set equation.

Remark. (see [6]) Function $\frac{x}{f(x)}$ is strictly monotone and it holds for every $c, d \in R_+$:

$$\left(\frac{c}{f(c)} - \frac{d}{f(d)}\right)(c-d)) \geq \alpha(c-d)^2, \ \alpha > 0. \tag{1.5}$$

Now, following [6], we define the weak solution of our problem.

Definition 1.1. (Weak solution of (1.2)-(1.3)-(1.4)) Under hypotheses (H0), ($\overline{\text{H}}$1) or (H1) we say that u is a weak solution of (1.2)-(1.3)-(1.4) if, for all $T > 0$:

1. $u \in L^2((0,T); H_0^1(\Omega))$ and $u_t \in L^2((0,T) \times \Omega)$ (hence $u \in C^0((0,T); L^2(\Omega))$);
2. $u(\cdot, 0) = u_0$;
3. the following holds

$$\int_0^T \int_\Omega \left(\frac{u_t(t,x)\ v(t,x)}{f(|\nabla u(t,x)|)} + \frac{\nabla u(t,x) \cdot \nabla v(t,x)}{f(|\nabla u(t,x)|)} \right) \mathrm{d}x\mathrm{d}t = 0, \tag{1.6}$$

$$\forall v \in L^2((0,T); H_0^1(\Omega)). \tag{1.7}$$

In the next section we present in a detail our numerical scheme and in the section 3 we prove its stability. In the section 4 the convergence results are shown. Finally section 5 is devoted to the numerical experiments using proposed scheme.

2. SEMI-IMPLICIT DDFV SCHEME

We choose a uniform discrete time step $\tau = \frac{T}{N_T}$ and replace the time derivative in (1.2) by the backward difference. The nonlinear terms of the equation are treated from the previous time step while the linear ones are considered on the current time level, this means semi-implicitness of the time discretization.

Definition 2.1. (Semi-implicit in time discretization) Let τ be a given time step, and u^0 be a given initial level set function. Then, for $n = 1, \ldots, N_T$, $N_T \cdot \tau = T$, we look for a function u^n, solution of the equation

$$\frac{1}{f(|\nabla u^{n-1}|)} \frac{u^n - u^{n-1}}{\tau} = \nabla . \left(\frac{\nabla u^n}{f(|\nabla u^{n-1}|)} \right). \tag{2.1}$$

Let us introduce now the fully discrete semi-implicit scheme. In the image processing applications, a digital image is given on a structure of pixels with rectangular shape in general (dashed lines rectangles in Figure 1). This set of pixels can represent original rectangular finite volume mesh. We denote it by $\mathcal{T}_h$. Since in every discrete time step of the scheme (2.1) we have to evaluate gradient of the level set function at the previous step $|\nabla u^{n-1}|$, we put a diamond shaped regions to edge, as can be seen in Figure 2, onto the computational domain and then take an approximation by finite differences in this region. Such approach allows simple, fast and clear construction of fully-discrete system of equations. Now we describe proposed discretization in details.

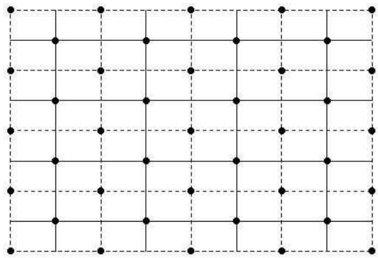

Fig. 1. Original (dashed lines rectangles) and dual (solid lines rectangles) mesh

2.1. Definition of the mesh

We restrict to the uniform mesh in $2D$ space as is it usual in image processing problems. The numerical scheme can be created on the general mesh too with the properties as in [1]. In image processing problems we often deal with the rectangular domains as the union of the set of pixels; so it is natural to use uniform mesh in this case.

Our volume mesh will consist of cells $V_{ij} \in \mathcal{T}_h$, associated with nodes x_{ij}, say $i = 1, \ldots, N_1$, $j = 1, \ldots, N_2$ with the property $\overline{\Omega} = \bigcup_{V_{ij} \in \mathcal{T}_h} V_{ij}$. The numerical solution computed in the representative point x_{ij} we denote by u_{ij}. All finite volumes $V_{ij} \in \mathcal{T}_h$ are squares with the edge of the length h.

Duality mesh, shifted to the north-east direction, consists of cells $\overline{V}_{ij} \in \overline{\mathcal{T}}_h$ associated with nodes $\overline{x}_{ij}$, say $i = 0, \ldots, N_1$, $j = 0, \ldots, N_2$ in such a way that $\overline{x}_{ij}$ is the right top corner for the volume V_{ij} of the original mesh and the boundary finite volumes are degenerated in such a way that $\overline{\Omega} = \bigcup_{\overline{V}_{ij} \in \overline{\mathcal{T}}_h} \overline{V}_{ij}$ (see Figure 1). The numerical solution computed in the representative point $\overline{x}_{ij}$ we denote by $\overline{u}_{ij}$. Again all inner finite volumes $\overline{V}_{ij} \in \overline{\mathcal{T}}_h$ are squares with the edge of the length h.

For each $V_{ij} \in \mathcal{T}$ by N_{ij} we denote the set of all neighbouring (west, east, south, north) finite volumes $V_{i+p,j+q}$, $p, q \in \{-1, 0, 1\}$, $|p| + |q| = 1$. Let $m(V_{ij})$ denote the volume of V_{ij}. In our case $m(V_{ij}) = h^2$. The segment connecting the center of V_{ij} and the center of its neighbour $V_{i+p,j+q} \in N_{ij}$ is denoted by σ_{ij}^{pq} and its length is in our discretization h for all of the finite volumes. The sides of the finite volume V_{ij} are denoted by e_{ij}^{pq} with the length h. For the dual mesh the notation will be the same, but "overlined". For the evaluation of the gradients we use a diamond mesh which is the union of $\mathcal{D}_h$ and $\overline{\mathcal{D}}_h$, where

$$\mathcal{D}_h = \bigcup_{(i,j)=(0,0),\ldots,(N_1,N_2)} D_{ij},$$

where D_{ij} has the vertices $\{x_{ij}, \overline{x}_{i,j-1}, x_{i+1,j}, \overline{x}_{ij},\}$ with degenerated (triangles) diamonds on the boundaries (for $i = 0$, or $i = N_1$) and

$$\overline{\mathcal{D}}_h = \bigcup_{(i,j)=(0,0),\ldots,(N_1,N_2)} \overline{D}_{ij},$$

where $\overline{D}_{ij}$ has the vertices $\{x_{ij}, \overline{x}_{ij}, x_{i,j+1}, \overline{x}_{i-1,j}\}$ with degenerated (triangles) diamonds on the boundaries (for $j = 0$, or $j = N_2$). Under this notation it is clear that $\overline{\Omega} = \mathcal{D}_h \cup \overline{\mathcal{D}}_h$.

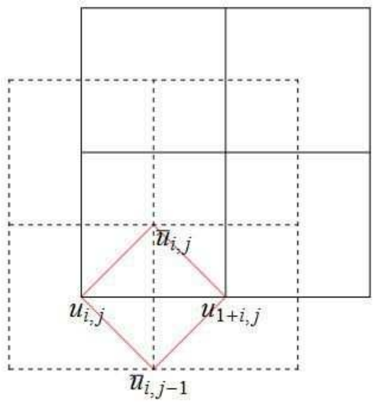

Fig. 2. Values of u in the original mesh and values of $\overline{u}$ in the dual mesh

As it is usual in finite volume methods [5], we integrate (2.1) over every co-volume V_{ij}, $i = 1, \dots, N_1$, $j = 1, \dots, N_2$, and then using the divergence theorem we get an integral formulation of (2.1)

$$\int\limits_{V_{ij}} \frac{1}{f(|\nabla u^{n-1}|)} \frac{u^n - u^{n-1}}{\tau} \, \mathrm{d}x = \sum_{|p|+|q|=1} \int\limits_{e_{ij}^{pq}} \frac{1}{f(|\nabla u^{n-1}|)} \frac{\partial u^n}{\partial \nu_{ij}^{pq}} \, \mathrm{d}s, \tag{2.2}$$

where ν_{ij}^{pq} is a unit outer normal to the edge e_{ij}^{pq} of V_{ij}. For the approximation of the left-hand side of (2.2) we get

$$\int\limits_{V_{ij}} \frac{1}{f(|\nabla u^{n-1}|)} \frac{u^n - u^{n-1}}{\tau} \, \mathrm{d}x \approx \frac{u_{ij}^n - u_{ij}^{n-1}}{AQ_{ij}^{n-1}} \, \frac{h^2}{\tau}, \tag{2.3}$$

where AQ_{ij} is the value of function f of an average modulus of gradient in V_{ij}.

This average will be computed using the values of the gradients on the sides e_{ij}^{pq} of the finite volume, which we have to approximate on the right-hand side of (2.2) as well. On the right-hand side of (2.2), the normal derivative is naturally expressed by the finite difference of neighbouring pixel values divided by the distance between pixel centers. To approximate the modulus of gradients on the pixel sides, we use the approximation of a gradient of diamond mesh. We use the following definitions for $p, q \in \{-1, 0, 1\}$, $|p| + |q| = 1$ and $\alpha(p) = 0$, if $p \geq 0$, and $\alpha(p) = -1$ if $p = -1$.

$$\nabla^{p0} u_{ij}^n = \left(\frac{p(u_{i+p,j}^n - u_{ij}^n)}{h}, \frac{\overline{u}_{i+\alpha(p),j}^n - \overline{u}_{i+\alpha(p),j-1}^n}{h} \right), \tag{2.4}$$

$$\nabla^{0q} u_{ij}^n = \left(\frac{\overline{u}_{i,j-\alpha(q)}^n - \overline{u}_{i-1,j-\alpha(q)}^n}{h}, \frac{q(u_{i,j+q}^n - u_{ij}^n)}{h} \right). \tag{2.5}$$

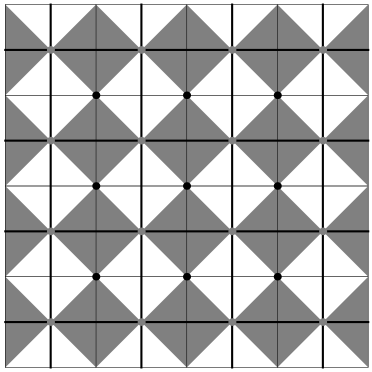

Fig. 3. Diamonds D_{ij} (gray) and diamonds $\overline{D}_{ij}$ (white) with primal-dual mesh and unknowns u (gray points) and $\overline{u}$ (black points)

The formulas (2.4)-(2.5) can be understood as an approximation of the gradient in D_{ij} and $\overline{D}_{ij}$, respectively. Both sets of diamonds one can see in Figure 3.

Now we define in accordance with a hypothesis ($\overline{\text{H}}1$):

$$Q_{ij}^{pq;n-1} = \sqrt{|\nabla^{pq} u_{ij}^{n-1}|^2 + \varepsilon^2} = f(|\nabla^{pq} u_{ij}^{n-1}|). \tag{2.6}$$

For the further analysis we need hypothesis (H1) and then we denote

$$Q_{ij}^{pq;n-1} = \min(\sqrt{|\nabla^{pq} u_{ij}^{n-1}|^2 + \varepsilon^2},\ b) = f(|\nabla^{pq} u_{ij}^{n-1}|). \tag{2.7}$$

We will use the same notation because the first case will be used only for Stability 1. This is a regularized norm of the gradient on the sides of the finite volume V_{ij} (this is the same as in diamond mesh) computed by the solution known from the previous time step $n-1$. The regularized averaged gradient inside the finite volume V_{ij} can be expressed in several ways, for example as an arithmetic or harmonic mean. If we take an arithmetic mean we can denote it in the following way:

$$AQ_{ij}^{n-1} = \frac{1}{4} \sum_{|p|+|q|=1} Q_{ij}^{pq;n-1}. \tag{2.8}$$

Note that for the dual mesh we have:

$$\begin{aligned} Q_{ij}^{1,0;n-1} &= \overline{Q}_{ij}^{0,-1;n-1}, \quad Q_{ij}^{-1,0;n-1} = \overline{Q}_{i-1j}^{0,-1;n-1}, \\ Q_{ij}^{0,1;n-1} &= \overline{Q}_{ij}^{-1,0;n-1}, \quad Q_{ij}^{0,-1;n-1} = \overline{Q}_{ij-1}^{-1,0;n-1}. \end{aligned} \tag{2.9}$$

Combining all the above considerations we end up with the following approximation (the same for the dual mesh)

$$\sum_{|p|+|q|=1} \int_{e_{ij}^{pq}} \frac{1}{f(|\nabla u^{n-1}|)} \frac{\partial u^n}{\partial \nu_{ij}^{pq}} \, \mathrm{d}s \approx \sum_{|p|+|q|=1} \frac{u_{i+p,j+q}^n - u_{ij}^n}{Q_{ij}^{pq;n-1}}. \tag{2.10}$$

Note that the length of each side of the finite volume is h as well as the distance between the points where the numerical solution is computed. Now we can define the discrete initial values in the following way:

$$\begin{aligned} u_{ij}^0 &= \frac{1}{m(V_{ij})} \int\limits_{V_{ij}} u^0(x)\ \mathrm{d}x \quad \forall V_{ij} \in \mathcal{T}_h, \\ \overline{u}_{ij}^0 &= \frac{1}{m(\overline{V}_{ij})} \int\limits_{\overline{V}_{ij}} u^0(x)\ \mathrm{d}x \quad \forall \overline{V}_{ij} \in \overline{\mathcal{T}_h}. \end{aligned} \tag{2.11}$$

If we put together the right-hand sides of (2.3) and (2.10) and consider zero Dirichlet boundary conditions, we can write the following linear system of equations which has to be solved at every discrete time step n, $n = 1, \dots, N_T$.

Definition 2.2. (Fully-discrete semi-implicit DDFV scheme) Let u_{ij}^0, $\overline{u}_{ij}^0$, $i = 1, \dots, N_1$, $j = 1, \dots, N_2$ be given discrete initial values for the original and the dual meshes, respectively as in (2.11). Then, for $n = 1, \dots, N_T$ we look for u_{ij}^n, $\overline{u}_{ij}^n$, $i = 1, \dots, N_1$, $j = 1, \dots, N_2$, satisfying

$$\frac{u_{ij}^n}{AQ_{ij}^{n-1}} \frac{h^2}{\tau} + \sum_{|p|+|q|=1} \frac{(u_{ij}^n - u_{i+p,j+q}^n)}{Q_{ij}^{pq;n-1}} = \frac{u_{ij}^{n-1}}{AQ_{ij}^{n-1}} \frac{h^2}{\tau}, \tag{2.12}$$

$$\frac{\overline{u}_{ij}^n}{\overline{AQ}_{ij}^{n-1}} \frac{h^2}{\tau} + \sum_{|p|+|q|=1} \frac{(\overline{u}_{ij}^n - \overline{u}_{i+p,j+q}^n)}{\overline{Q}_{ij}^{pq;n-1}} = \frac{\overline{u}_{ij}^{n-1}}{\overline{AQ}_{ij}^{n-1}} \frac{h^2}{\tau}. \tag{2.13}$$

We define discrete functions in the following way:

$$u_h(x) = \frac{1}{2} \left(u_{h,V}(x) + \overline{u}_{h,\overline{V}}(x) \right), \tag{2.14}$$

where $u_{h,V}(x) = u_{ij}$, for $x \in V_{ij}$ and $\overline{u}_{h,\overline{V}}(x) = \overline{u}_{ij}$, for $x \in \overline{V}_{ij}$ are piecewise constant functions.

Now, after definition of a gradient and its regularization on the side, we can define constant gradients on D_{ij} and $\overline{D}_{ij}$ in the form

$$\begin{aligned} \nabla u_{ij} &= \left(\frac{u_{i+1,j} - u_{ij}}{h}, \frac{\overline{u}_{ij} - \overline{u}_{i,j-1}}{h} \right) \text{ on } D_{ij}, \\ \nabla \overline{u}_{ij} &= \left(\frac{\overline{u}_{ij} - \overline{u}_{i-1,j}}{h}, \frac{u_{i,j+1} - u_{ij}}{h} \right) \text{ on } \overline{D}_{ij} \end{aligned} \tag{2.15}$$

and

$$\nabla u_h(x) = \begin{cases} \nabla u_{ij} & \text{for } x \in D_{ij}, \\ \nabla \overline{u}_{ij}^n & \text{for } x \in \overline{D}_{ij}. \end{cases} \tag{2.16}$$

Definition 2.3. For the space discretization described in this section we denote by (h) the space discretization of Ω. As in [1], we define L_h, the discrete function space with function in the form (2.14) endowed with the inner product

$$[[u_h, v_h]] \;=\; \frac{1}{2}\left(\sum_{V_{ij}\in\mathcal{T}_h} u_{ij}\; v_{ij}\; h^2 + \sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h} \overline{u}_{ij}\;\overline{v}_{ij}\; h^2\right)$$

as a discrete L_2 inner product. Now by H_h we define the discrete function space, with the functions defined in (2.14) with the gradient defined in (2.16). For the L^2 gradient norm we use the usual norm, but for piecewise constant functions (on D_{ij}, $\overline{D}_{ij}$, respectively).

Definition 2.4. For time and space discretization described in this section we denote by (τ, h) the time-space discretization of $(0, T)\times\Omega$. Then the function $u_{\tau,h}$ is in time and space piecewise constant function defined as follows:

$$u_{\tau,h}(t,x) \;=\; \frac{1}{2}\left(u_{\tau,h,V}(t,x) + u_{\tau,h,\overline{V}}(t,x)\right), \tag{2.17}$$

where

$$\begin{aligned} u_{\tau,h,V}(t,x) &= u^n_{ij} \text{ for } x\in V_{ij},\; t\in((n-1)\tau, n\tau] \\ u_{\tau,h,\overline{V}}(t,x) &= \overline{u}^n_{ij} \text{ for } x\in \overline{V}_{ij},\; t\in((n-1)\tau, n\tau]. \end{aligned}$$

3. PROPERTIES OF THE SCHEME

3.1. Stability 1

In this subsection we present the stability properties for proposed scheme only under the hypothesis $(\overline{H}1)$.

Lemma 3.1. Let the hypotheses (H0) and $(\overline{H}1)$ hold. Then there exist unique solutions

$$u^n_h = (u^n_{11},\ldots,u^n_{N_1N_2}),\;\; \overline{u}^n_h = (\overline{u}^n_{00},\ldots,\overline{u}^n_{N_1N_2})$$

of the scheme (2.12), (2.13) for any value of the regularization parameter $\varepsilon > 0$ and for any time step $n = 1,\ldots,N_T$. Moreover, for the fully discrete numerical solution $u_{\tau,h}$ the following estimates hold

$$\|u_{\tau,h}\|_{L_\infty(I\times\Omega)} \le \|u^0_h\|_{L_\infty(\overline{\Omega})}. \tag{3.1}$$

Lemma 3.2. Let the hypotheses (H0) and $(\overline{H}1)$ hold. Then for the fully discrete scheme (2.12), (2.13) the following stability result holds for any time step $m = 1,\ldots,N_T$:

$$\sum_{n=1}^{m}\sum_{V_{ij}\in\mathcal{T}_h}\frac{\left(u^n_{ij}-u^{n-1}_{ij}\right)^2}{AQ^{n-1}_{ij}}\frac{h^2}{\tau} + \sum_{n=1}^{m}\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h}\frac{\left(\overline{u}^n_{ij}-\overline{u}^{n-1}_{ij}\right)^2}{\overline{AQ}^{n-1}_{ij}}\;\frac{h^2}{\tau} + \tag{3.2}$$

$$\sum_{V_{ij}\in\mathcal{T}_h}\;\sum_{(p,q)\in\{(0,1),(1,0)\}}\sqrt{\nabla^{pq}u^n_{ij}+\varepsilon^2}\; h^2 \le C,$$

where C is a generic constant and depends only on data of the problem, not on h or τ.

Remark. Proofs for both lemmas can be found in [8].

3.2. Stability 2

In this subsection we want to show another stability properties of our approximation of the solution. For this purpose we need further regularization assumption. We have introduced it in hypothesis (H1). As we have described in the section of space-time discretization we recall that on D_{ij} and $\overline{D}_{ij}$ we have defined constant gradients as in (2.16).

The regularization of the absolute value of a gradient on D_{ij} and $\overline{D}_{ij}$ in the sense (2.7)

$$\begin{aligned} Q_{ij}^n &= \min(\sqrt{|\nabla u_{ij}^n|^2 + \varepsilon^2},\ b) = f(|\nabla u_{ij}^n|) \text{ on } D_{ij} \\ \overline{Q}_{ij}^n &= \min(\sqrt{|\nabla \overline{u}_{ij}^n|^2 + \varepsilon^2},\ b) = f(|\nabla \overline{u}_{ij}^n|) \text{ on } \overline{D}_{ij}. \end{aligned} \tag{3.3}$$

As in [6], let F be the function defined by

$$\forall s \in R_+,\ F(s) = \int_0^s \frac{z}{f(z)}\, \mathrm{d}z \in \left[\frac{s^2}{2b}, \frac{s^2}{2\varepsilon}\right]. \tag{3.4}$$

Lemma 3.3. Let the hypotheses (H0) and (H1) hold. Then for the solution of discrete scheme (2.12), (2.13) the following stability results hold:

$$\begin{aligned} &\sum_{n=1}^{m} \sum_{V_{ij}\in\mathcal{T}} \frac{\left(u_{ij}^n - u_{ij}^{n-1}\right)^2}{AQ_{ij}^{n-1}} \frac{h^2}{\tau} + \sum_{n=1}^{m} \sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}} \frac{\left(\overline{u}_{ij}^n - \overline{u}_{ij}^{n-1}\right)^2}{\overline{AQ}_{ij}^{n-1}} \frac{h^2}{\tau} + \\ &\frac{1}{2b}\sum_{n=1}^{m}\left(\sum_{D_{ij}\in\mathcal{D}_h} (|\nabla u_{ij}^n| - |\nabla u_{ij}^{n-1}|)^2\ h^2 + \sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h} (|\nabla \overline{u}_{ij}^n| - |\nabla \overline{u}_{ij}^{n-1}|)^2\ h^2\right) + \\ &\sum_{D_{ij}\in\mathcal{D}_h} F(|\nabla u_{ij}^m|)\ h^2 + \sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h} F(|\nabla \overline{u}_{ij}^m|)\ h^2 \ \le\ C \end{aligned} \tag{3.5}$$

where C is a generic constant and depends only on data of the problem, not on h or τ.

P r o o f. The main idea is similar to that in [6] but in our case we work with two different sets of equations. First we multiply the fully discrete scheme (2.12) and (2.13) with the term $u_{ij}^n - u_{ij}^{n-1}$ and $\overline{u}_{ij}^n - \overline{u}_{ij}^{n-1}$, respectively and sum over all of the volumes V_{ij} and $\overline{V}_{ij}$. Then we put the equations together and we immediately have

$$\sum_{V_{ij}\in\mathcal{T}_h} \frac{\left(u_{ij}^n - u_{ij}^{n-1}\right)^2}{AQ_{ij}^{n-1}} \frac{h^2}{\tau} + \sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h} \frac{\left(\overline{u}_{ij}^n - \overline{u}_{ij}^{n-1}\right)^2}{\overline{AQ}_{ij}^{n-1}} \frac{h^2}{\tau} +$$

$$\sum_{V_{ij}\in\mathcal{T}_h}\sum_{|p|+|q|=1}\frac{(u_{ij}^n-u_{i+p,j+q}^n)(u_{ij}^n-u_{ij}^{n-1})}{Q_{ij}^{pq;n-1}}+$$

$$\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h}\sum_{|p|+|q|=1}\frac{(\overline{u}_{ij}^n-\overline{u}_{i+p,j+q}^n)(\overline{u}_{ij}^n-\overline{u}_{ij}^{n-1})}{\overline{Q}_{ij}^{pq;n-1}} \;=\; 0.$$

We can rearrange last two terms on the left-hand side using the discrete duality formula (2.9) which gives the constant gradients on D_{ij} and $\overline{D}_{ij}$ respectively. We get:

$$\sum_{V_{ij}\in\mathcal{T}_h}\frac{\left(u_{ij}^n-u_{ij}^{n-1}\right)^2}{AQ_{ij}^{n-1}}\;\frac{h^2}{\tau}+\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h}\frac{\left(\overline{u}_{ij}^n-\overline{u}_{ij}^{n-1}\right)^2}{\overline{AQ}_{ij}^{n-1}}\;\frac{h^2}{\tau}+$$

$$\sum_{D_{ij}\in\mathcal{D}_h}\frac{(u_{ij}^n-u_{i+1,j}^n)^2+(\overline{u}_{ij}^n-\overline{u}_{i,j-1}^n)^2}{Q_{ij}^{n-1}}-$$

$$\sum_{D_{ij}\in\mathcal{D}_h}\frac{(u_{ij}^n-u_{i+1,j}^n)(u_{ij}^{n-1}-u_{i+1,j}^{n-1})+(\overline{u}_{ij}^n-\overline{u}_{i,j-1}^n)(\overline{u}_{ij}^{n-1}-\overline{u}_{i,j-1}^{n-1})}{Q_{ij}^{n-1}}+$$

$$\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h}\frac{(u_{ij}^n-u_{i,j+1}^n)^2+(\overline{u}_{ij}^n-\overline{u}_{i-1,j}^n)^2}{Q_{ij}^{n-1}}-$$

$$\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h}\frac{(u_{ij}^n-u_{i,j+1}^n)(u_{ij}^{n-1}-u_{i,j+1}^{n-1})+(\overline{u}_{ij}^n-\overline{u}_{i-1,j}^n)(\overline{u}_{ij}^{n-1}-\overline{u}_{i-1,j}^{n-1})}{Q_{ij}^{n-1}} \;=\; 0,$$

which can be rewritten into the form

$$\sum_{V_{ij}\in\mathcal{T}_h}\frac{\left(u_{ij}^n-u_{ij}^{n-1}\right)^2}{AQ_{ij}^{n-1}}\;\frac{h^2}{\tau}+\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h}\frac{\left(\overline{u}_{ij}^n-\overline{u}_{ij}^{n-1}\right)^2}{\overline{AQ}_{ij}^{n-1}}\;\frac{h^2}{\tau}+$$

$$\sum_{D_{ij}\in\mathcal{D}_h}\frac{|\nabla u_{ij}^n|^2-\nabla u_{ij}^n.\nabla u_{ij}^{n-1}}{Q_{ij}^{n-1}}h^2+\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h}\frac{|\nabla\overline{u}_{ij}^n|^2-\nabla\overline{u}_{ij}^n.\nabla\overline{u}_{ij}^{n-1}}{\overline{Q}_{ij}^{n-1}}h^2 \;=\; 0.$$

From the definition of the functions F and f in (3.4) we have

$$F(|\nabla^{pq}u_{ij}^n|)-F(|\nabla^{pq}u_{ij}^{n-1}|) \;=\; \int\limits_{|\nabla^{pq}u_{ij}^{n-1}|}^{|\nabla^{pq}u_{ij}^n|}\frac{z\;\mathrm{d}z}{f(z)}.$$

As in [6], we use the properties of the functions F and f:

$$\forall c,d\in R_+,\;\;\int\limits_c^d\frac{z\;\mathrm{d}z}{f(z)}+\frac{(d-c)^2}{2f(c)}\leq\frac{d}{f(c)}(d-c), \tag{3.6}$$

which results in

$$F(|\nabla u_{ij}^n|) - F(|\nabla u_{ij}^{n-1}|) + \frac{1}{2b}(|\nabla u_{ij}^n| - |\nabla u_{ij}^{n-1}|)^2 \le \frac{|\nabla u_{ij}^n|}{f(|\nabla u_{ij}^{n-1}|)}(|\nabla u_{ij}^n| - |\nabla u_{ij}^{n-1}|) =$$

$$\frac{|\nabla u_{ij}^n|}{Q_{ij}^{n-1}}(|\nabla u_{ij}^n| - |\nabla u_{ij}^{n-1}|) \le \frac{|\nabla u_{ij}^n|^2 - \nabla u_{ij}^n \cdot \nabla u_{ij}^{n-1}}{Q_{ij}^{n-1}}.$$

The same estimation we can obtain for the ”overlined” terms, too. Using these facts and the hypothesis (H1), we have

$$\sum_{V_{ij}\in\mathcal{T}_h} \frac{\left(u_{ij}^n - u_{ij}^{n-1}\right)^2}{AQ_{ij}^{n-1}} \frac{h^2}{\tau} + \sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h} \frac{\left(\overline{u}_{ij}^n - \overline{u}_{ij}^{n-1}\right)^2}{\overline{AQ}_{ij}^{n-1}} \frac{h^2}{\tau} +$$

$$\frac{1}{2b}\sum_{D_{ij}\in\mathcal{D}_h} \left(|\nabla u_{ij}^n| - |\nabla u_{ij}^{n-1}|\right)^2 h^2 + \frac{1}{2b}\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h} \left(|\nabla \overline{u}_{ij}^n| - |\nabla \overline{u}_{ij}^{n-1}|\right)^2 h^2 +$$

$$\sum_{D_{ij}\in\mathcal{D}_h} \left(F(|\nabla u_{ij}^n|) - F(|\nabla u_{ij}^{n-1}|)\right) h^2 + \sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h} \left(F(|\nabla \overline{u}_{ij}^n|) - F(|\nabla \overline{u}_{ij}^{n-1}|)\right) h^2 \le 0.$$

Summing the last inequality over $n = 1, \dots, m$ we obtain

$$\sum_{n=1}^{m}\sum_{V_{ij}\in\mathcal{T}_h} \frac{\left(u_{ij}^n - u_{ij}^{n-1}\right)^2}{AQ_{ij}^{n-1}} \frac{h^2}{\tau} + \sum_{n=1}^{m}\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h} \frac{\left(\overline{u}_{ij}^n - \overline{u}_{ij}^{n-1}\right)^2}{\overline{AQ}_{ij}^{n-1}} \frac{h^2}{\tau} +$$

$$\frac{1}{2b}\sum_{n=1}^{m}\left(\sum_{D_{ij}\in\mathcal{D}_h} \left(|\nabla u_{ij}^n| - |\nabla u_{ij}^{n-1}|\right)^2 h^2 + \sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h} \left(|\nabla \overline{u}_{ij}^n| - |\nabla \overline{u}_{ij}^{n-1}|\right)^2 h^2\right) +$$

$$\sum_{D_{ij}\in\mathcal{D}_h} F(|\nabla u_{ij}^m|)h^2 + \sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h} F(|\nabla \overline{u}_{ij}^m|)h^2 \ \le \ \sum_{D_{ij}\in\mathcal{D}_h} F(|\nabla u_{ij}^0|)h^2 + \sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h} F(|\nabla \overline{u}_{ij}^0|)h^2$$

which, using the properties of the initial condition, gives the final result. □

4. CONVERGENCE

First we define, as in [1] and [6], the approximation of the function, its discrete gradient and other useful approximations. The discrete solution we have already defined in (2.17).

$$\begin{aligned} \delta u_{\tau,h}(t,x) &= \frac{1}{2}\left(\delta u_{\tau,h,V}(t,x) + \delta u_{\tau,h,\overline{V}}(t,x)\right) \qquad (4.1)\\ \delta u_{\tau,h,V}(t,x) &= \frac{u_{ij}^n - u_{ij}^{n-1}}{\tau} \ \text{for } x \in V_{ij},\ t \in ((n-1)\tau, n\tau] \\ \delta u_{\tau,h,\overline{V}}(t,x) &= \frac{\overline{u}_{ij}^n - \overline{u}_{ij}^{n-1}}{\tau} \ \text{for } x \in \overline{V}_{ij},\ t \in ((n-1)\tau, n\tau] \end{aligned}$$

$$\begin{aligned} AQ_{\tau,h}(t,x) &= \frac{1}{2}\Big(AQ_{\tau,h,V}(t,x) + AQ_{\tau,h,\overline{V}}(t,x)\Big) \\ AQ_{\tau,h,V}(t,x) &= AQ^n_{ij} \text{ for } x \in V_{ij},\ t \in ((n-1)\tau, n\tau] \\ AQ_{\tau,h,\overline{V}}(t,x) &= \overline{AQ}^n_{ij} \text{ for } x \in \overline{V}_{ij},\ t \in ((n-1)\tau, n\tau] \end{aligned} \tag{4.2}$$

$$\begin{aligned} \overline{AQ}_{\tau,h}(t,x) &= \frac{1}{2}\Big(\overline{AQ}_{\tau,h,V}(t,x) + \overline{AQ}_{\tau,h,\overline{V}}(t,x)\Big) \\ \overline{AQ}_{\tau,h,V}(t,x) &= AQ^{n-1}_{ij} \quad \text{for } x \in V_{ij},\ t \in ((n-1)\tau, n\tau] \\ \overline{AQ}_{\tau,h,\overline{V}}(t,x) &= \overline{AQ}^{n-1}_{ij} \quad \text{for } x \in \overline{V}_{ij},\ t \in ((n-1)\tau, n\tau] \end{aligned} \tag{4.3}$$

$$\nabla u_{\tau,h}(t,x) = \begin{cases} \nabla u^n_{ij} & \text{for } x \in D_{ij},\ t \in ((n-1)\tau, n\tau] \\ \nabla \overline{u}^n_{ij} & \text{for } x \in \overline{D}_{ij},\ t \in ((n-1)\tau, n\tau] \end{cases} \tag{4.4}$$

$$\overline{\nabla} u_{\tau,h}(t,x) = \begin{cases} \nabla u^{n-1}_{ij} & \text{for } x \in D_{ij},\ t \in ((n-1)\tau, n\tau] \\ \nabla \overline{u}^{n-1}_{ij} & \text{for } x \in \overline{D}_{ij},\ t \in ((n-1)\tau, n\tau] \end{cases} \tag{4.5}$$

$$\begin{aligned} w_{\tau,h}(t,x) &= \frac{1}{2}\Big(w_{\tau,h,V}(t,x) + w_{\tau,h,\overline{V}}(t,x)\Big) \\ w_{\tau,h,V}(t,x) &= w^n_{ij} = \frac{u^n_{ij} - u^{n-1}_{ij}}{\tau AQ^{n-1}_{ij}} \quad \text{for } x \in V_{ij},\ t \in ((n-1)\tau, n\tau] \\ w_{\tau,h,\overline{V}}(t,x) &= \overline{w}^n_{ij} = \frac{\overline{u}^n_{ij} - \overline{u}^{n-1}_{ij}}{\tau \overline{AQ}^{n-1}_{ij}} \quad \text{for } x \in \overline{V}_{ij},\ t \in ((n-1)\tau, n\tau] \end{aligned} \tag{4.6}$$

$$H_{\tau,h}(t,x) = \frac{\nabla u_{\tau,h}(t,x)}{f(|\overline{\nabla} u_{\tau,h}(t,x)|)} \tag{4.7}$$

Now we can rewrite our scheme (2.12), (2.13) into the form

$$\begin{aligned} \sum_{|p|+|q|=1} \frac{(u^n_{ij} - u^n_{i+p,j+q})}{Q^{pq;n-1}_{ij}} &= -w^n_{ij} h^2 \\ \sum_{|p|+|q|=1} \frac{(\overline{u}^n_{ij} - \overline{u}^n_{i+p,j+q})}{\overline{Q}^{pq;n-1}_{ij}} &= -\overline{w}^n_{ij} h^2. \end{aligned} \tag{4.8}$$

Under these definitions and from the stability results we can conclude:

Theorem 4.1. Let the hypotheses (H0) and (H1) hold. Then for the solution $u_{\tau,h}$ of the discrete scheme (2.12), (2.13) defined in (2.17) and for the gradient $\nabla u_{\tau,h}$ defined in (4.4) the following stability results hold for arbitrary $m = 1, \ldots, N_T$:

$$\int_0^{m\tau}\int_\Omega \frac{\delta u_{\tau,h}^2}{\overline{AQ}_{\tau,h}}\,\mathrm{d}x\mathrm{d}t + \frac{1}{\tau}\int_0^{m\tau}\int_\Omega \left(|\nabla u_{\tau,h}| - |\overline{\nabla} u_{\tau,h}|\right)^2\,\mathrm{d}x + \int_\Omega F(|\nabla u_{\tau,h}(m\tau,x)|)\,\mathrm{d}x \;\le\; C. \tag{4.9}$$

Moreover, due to the hypothesis (H1), the term $\overline{AQ}_{\tau,h}$ is bounded and we have

$$\int_0^{m\tau}\int_\Omega \delta u_{\tau,h}^2\,\mathrm{d}x\mathrm{d}t \;\le\; C, \tag{4.10}$$

where C is a generic constant and depends only on data of the problem, not on h or τ.

Let us denote by (H) the following hypothesis:

- Hypotheses (H0), (H1) are fulfilled;
- The sequence $(\tau_m, h_m)_{m\in\mathbf{N}}$ denotes a sequence of space-time discretization of $(0,T)\times\Omega$ in the sense of Definition 2.4 such that h_m and $\tau_m > 0$ tend to 0 as $m \longrightarrow \infty$;
- For all $m \in \mathbf{N}$, the family $\{u_{ij}^n,\ i = 1,\ldots,N_1,\ j = 1,\ldots,N_2,\ \overline{u}_{ij}^n,\ i = 0,\ldots,N_1,\ j = 0,\ldots,N_2, n \in \mathbf{N}\}$ is such that (2.12), (2.13) hold and the function u_{τ_m,h_m} is defined by (2.17).

To obtain convergence of an approximate solution we need the following lemma.

Lemma 4.2. (Strong convergence of the approximate gradient of φ) Let hypothesis (H) be fulfilled. For all $\varphi \in C_c^\infty((0,T)\times\Omega)$, we denote by $r_{ij}^n = \varphi(n\tau, x_{ij})$ and $\overline{r}_{ij}^n = \varphi(n\tau, \overline{x}_{ij})$. We introduce the approximations

$$\nabla_{ij}^n\varphi = \left(\frac{r_{i+1,j}^n - r_{ij}^n}{h}, \frac{\overline{r}_{ij}^n - \overline{r}_{i,j-1}^n}{h}\right) \tag{4.11}$$

and

$$\overline{\nabla}_{ij}^n\varphi = \left(\frac{\overline{r}_{ij}^n - \overline{r}_{i-1,j}^n}{h}, \frac{r_{i,j+1}^n - r_{ij}^n}{h}\right), \tag{4.12}$$

$$\nabla_{\tau,h}\varphi(t,x) = \begin{cases} \nabla_{ij}^n\varphi \text{ for } x \in D_{ij},\ t \in ((n-1)\tau, n\tau] \\ \overline{\nabla}_{ij}^n\varphi \text{ for } x \in \overline{D}_{ij},\ t \in ((n-1)\tau, n\tau]. \end{cases} \tag{4.13}$$

Then $\nabla_{\tau,h}\varphi$ strongly converges in $L^\infty((0,T)\times\Omega)$ to $\nabla\varphi$ as $h \longrightarrow 0$ and $\tau \longrightarrow 0$.

Proof. For such a regular space discretization the outward normal to the vertical edges is $\mathbf{n}_1 = (1,0)$ and to the horizontal edges $\mathbf{n}_2 = (0,1)$. The function φ is smooth enough, so using Taylor expansion for $x \in D_{ij}$ we immediately have

$$\begin{aligned} \nabla\varphi(t,x)\cdot\mathbf{n}_1 &= \frac{\varphi(t,x_{i+1,j}) - \varphi(t,x_{ij})}{h} + C_{ij}(t)h, \\ \nabla\varphi(t,x)\cdot\mathbf{n}_2 &= \frac{\varphi(t,\overline{x}_{ij}) - \varphi(t,\overline{x}_{i,j-1})}{h} + \overline{C}_{ij}(t)h, \end{aligned}$$

where $C_{ij}(t)$, $\overline{C}_{ij}(t)$ are bounded independently of the discretization and similarly for $x \in \overline{D}_{ij}$. The convergence result is then the consequence of these estimations. □

Theorem 4.3. (Convergence properties) Let hypothesis (H) be fulfilled. Then there exists a subsequence of $(\tau_m, h_m)_{m\in\mathbf{N}}$, again denoted by $(\tau_m, h_m)_{m\in\mathbf{N}}$ and there exists a function $\overline{u} \in L^\infty((0,T); H_0^1(\Omega)) \cap C^0((0,T); L^2(\Omega))$, such that

$$\begin{aligned} &\overline{u}_t \in L^2((0,T)\times\Omega), \\ &u(0,.) = u^0, \\ &u_{\tau_m,h_m} \longrightarrow \overline{u} \in L^\infty((0,T); L_2(\Omega)), \\ &u_{\tau_m,h_m} \rightharpoonup \overline{u} \in L^\infty((0,T); H_0^1(\Omega)) \end{aligned}$$

and there exist functions $\overline{H} \in L^2((0,T)\times\Omega)^2$, $\overline{w} \in L^2((0,T)\times\Omega)$ such that $H_{\tau_m,h_m} \rightharpoonup \overline{H}$ weakly in $L^2((0,T)\times\Omega)^2$ (see (4.7)), and such that $w_{\tau_m,h_m} \rightharpoonup \overline{w}$ and $\delta u_{\tau_m,h_m} \rightharpoonup \overline{u}_t$ weakly in $L^2((0,T)\times\Omega)$ as $m \to \infty$. Moreover,

$$|\nabla u_{\tau_m,h_m}| - |\overline{\nabla u}_{\tau_m,h_m}| \to 0 \text{ in } L^2((0,T)\times\Omega) \text{ (see (4.4), (4.5))}$$

and the following relation holds:

$$\lim_{m\to\infty} \int_0^T \int_\Omega \frac{|\nabla u_{\tau_m,h_m}(t,x)|^2}{f(|\nabla u_{\tau_m,h_m}(t,x)|)}\, \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \overline{H}(t,x)\cdot\nabla\overline{u}(t,x)\, \mathrm{d}x\mathrm{d}t. \tag{4.14}$$

Proof. From the definition of F (3.4), the estimation (4.9) and hypothesis (H), we have

$$F(s) \geq s^2/2b, \quad \int_\Omega |\nabla u_{\tau_m,h_m}(n\tau,x)|^2\, \mathrm{d}x \leq C \;\forall n = 1,2,\ldots,N_T.$$

Hence we can apply the results of [1] (Lemma 3.6) and [6] (Theorem 6.1), which is a generalization of Ascoli's theorem and shows the convergence

$$u_{\tau_m,h_m}(t,\cdot) \to \overline{u} \in L^\infty((0,T); L^2(\Omega)).$$

Thanks to (2.11), we have $\overline{u}(\cdot,0) = u_0$. We also get, thanks to [1] (Lemma 3.6), that $\overline{u} \in L^\infty((0,T); H_0^1(\Omega))$ and that

$$\nabla u_{\tau_m,h_m} \rightharpoonup \nabla\overline{u}$$

weakly in $L^2((0,T)\times\Omega)^2$.

From (4.10) and hypothesis (H) we get that w_{τ_m,h_m} is bounded in $L^2((0,T)\times\Omega)$ for all $m\in\mathbf{N}$. Therefore there exists a function $\overline{w}\in L^2((0,T)\times\Omega)$ such that, up to a subsequence of the preceding one, $w_{\tau_m,h_m}\rightharpoonup\overline{w}$ weakly in $L^2((0,T)\times\Omega)$. Similarly, from (4.10) we have $\delta_{\tau_m,h_m}\rightharpoonup\overline{u}_t$ weakly in $L^2((0,T)\times\Omega)$, which shows that $\overline{u}_t\in L^2((0,T)\times\Omega)$ (see [6]).

Similarly, from (4.9) and hypothesis (H), $H_{\tau_m,h_m}\rightharpoonup\overline{H}$ weakly in $L^2((0,T)\times\Omega)^2$, up to a subsequence of the preceding one.

Let us now focus on the difference between $|\nabla u_{\tau_m,h_m}|$ and $|\overline{\nabla}u_{\tau_m,h_m}|$. Using (4.9), we get the existence of $C>0$ independent of m such that

$$\||\nabla u_{\tau_m,h_m}|-|\overline{\nabla}u_{\tau_m,h_m}|\|^2_{L^2((0,T)\times\Omega)}\le C\tau_m,$$

which provides

$$\lim_{m\to\infty}\||\nabla u_{\tau_m,h_m}|-|\overline{\nabla}u_{\tau,h_m}|\|_{L^2((0,T)\times\Omega)}=0. \tag{4.15}$$

Now the idea is similar to that in [6]. For simplicity we now for a moment omit the indices m for h and τ.

Let $\varphi\in C_c^\infty((0,T)\times\Omega)$ be given. We again denote by $r_{ij}^n=\varphi(n\tau,x_{ij})$ and $\overline{r}_{ij}^n=\varphi(n\tau,\overline{x}_{ij})$. Multiplying first equation of (4.8) by τr_{ij}^n, and the second equation of (4.8) by $\tau\overline{r}_{ij}^n$, summing over all of the finite volumes and all n, we get after small rearrangement

$$-\sum_{n=1}^{N_T}\tau\left(\sum_{D_{ij}\in\mathcal{D}_h}\frac{(u_{ij}^n-u_{i+1,j}^n)(r_{ij}^n-r_{i+1,j}^n)}{Q_{ij}^{n-1}}+\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h}\frac{(u_{ij}^n-u_{i,j+1}^n)(r_{ij}^n-r_{i,j+1}^n)}{\overline{Q}_{ij}^{n-1}}\right)$$
$$=\sum_{n=1}^{N_T}\tau\sum_{V_{ij}\in\mathcal{T}_h}w_{ij}^n r_{ij}^n h^2,$$
$$-\sum_{n=1}^{N_T}\tau\left(\sum_{D_{ij}\in\mathcal{D}_h}\frac{(\overline{u}_{ij}^n-\overline{u}_{i,j-1}^n)(\overline{r}_{ij}^n-\overline{r}_{i,j-1}^n)}{Q_{ij}^{n-1}}+\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h}\frac{(\overline{u}_{ij}^n-\overline{u}_{i-1,j}^n)(\overline{r}_{ij}^n-r_{i-1,j}^n)}{\overline{Q}_{ij}^{n-1}}\right)$$
$$=\sum_{n=1}^{N_T}\tau\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h}\overline{w}_{ij}^n h^2\overline{r}_{ij}^n. \tag{4.16}$$

Now, putting two equations together, we obtain

$T_1=-T_2$ with

$$T_1=\sum_{n=1}^{N_T}\tau\left(\sum_{D_{ij}\in\mathcal{D}_h}\frac{(u_{ij}^n-u_{i+1,j}^n)(r_{ij}^n-r_{i+1,j}^n)+(\overline{u}_{ij}^n-\overline{u}_{i,j-1}^n)(\overline{r}_{ij}^n-\overline{r}_{i,j-1}^n)}{Q_{ij}^{n-1}}\right)$$
$$+\sum_{n=1}^{N_T}\tau\left(\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h}\frac{(u_{ij}^n-u_{i,j+1}^n)(r_{ij}^n-r_{i,j+1}^n)+(\overline{u}_{ij}^n-\overline{u}_{i-1,j}^n)(\overline{r}_{ij}^n-\overline{r}_{i-1,j}^n)}{\overline{Q}_{ij}^{n-1}}\right)$$

and

$$T_2=\sum_{n=1}^{N_T}\tau\sum_{V_{ij}\in\mathcal{T}_h}w_{ij}^n r_{ij}^n h^2+\sum_{n=1}^{N_T}\tau\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h}\overline{w}_{ij}^n h^2\overline{r}_{ij}^n.$$

Using the approximation $\nabla_{\tau,h}\varphi$ of $\nabla\varphi$ as in (4.13), we can write that

$$T_1 = 2\int_0^T\int_\Omega H_{\tau,h}\cdot\nabla_{\tau,h}\varphi\ \mathrm{d}x\mathrm{d}t.$$

Now we again return back to the subsequence (τ_m, h_m) and denote the expression by T_{1m} and T_{2m} using the definitions (4.6) and (4.7):

$$T_{1m} = 2\int_0^T\int_\Omega H_{\tau_m,h_m}\cdot\nabla_{\tau_m,h_m}\varphi\ \mathrm{d}x\mathrm{d}t,$$

$$T_{2m} = 2\int_0^T\int_\Omega w_{\tau_m,h_m}\varphi_{\tau_m,h_m}\ \mathrm{d}x\mathrm{d}t.$$

Hence, by weak/strong convergence from above and Lemma 4.2,

$$\lim_{m\to\infty} T_{1m} = 2\int_0^T\int_\Omega \overline{H}\cdot\nabla\varphi\ \mathrm{d}x\mathrm{d}t.$$

We have on the other hand

$$\lim_{m\to\infty} T_{2m} = 2\int_0^T\int_\Omega \overline{w}\varphi\ \mathrm{d}x\mathrm{d}t.$$

Hence

$$\int_0^T\int_\Omega \overline{H}\cdot\nabla\varphi\ \mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega \overline{w}\varphi\ \mathrm{d}x\mathrm{d}t.$$

Since the above equality holds for all $\varphi \in C_c^\infty((0,T)\times\Omega)$, it also holds by the density for all $v \in L^2((0,T); H_0^1(\Omega))$. Hence we get

$$\int_0^T\int_\Omega \overline{H}\cdot\nabla v\ \mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega \overline{w}v\ \mathrm{d}x\mathrm{d}t. \tag{4.17}$$

and also

$$\int_0^T\int_\Omega \overline{H}\cdot\nabla\overline{u}\ \mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega \overline{w}\ \overline{u}\ \mathrm{d}x\mathrm{d}t. \tag{4.18}$$

Now we multiply (4.8) by τu_{ij}^n, and $\tau\overline{u}_{ij}^n$, respectively, sum over all of the finite volumes and all n. Again, after similar rearrangement, we get $T_3 = -T_4$, where

$$T_3=\sum_{n=1}^{N_T}\tau\sum_{D_{ij}\in\mathcal{D}_h}\frac{(u_{ij}^n-u_{i+1,j}^n)^2+(\overline{u}_{ij}^n-\overline{u}_{i,j-1}^n)^2}{Q_{ij}^{n-1}}+$$

$$\sum_{n=1}^{N_T}\tau\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h}\frac{(u_{ij}^n-u_{i,j+1}^n)^2+(\overline{u}_{i,j}^n-\overline{u}_{i-1,j}^n)^2}{\overline{Q}_{ij}^{n-1}}=2\int_0^T\int_\Omega\frac{|\nabla u_{\tau,h}|^2}{f(|\overline{\nabla}u_{\tau,h}|)}\,\mathrm{d}x\mathrm{d}t$$

and

$$T_4=\sum_{n=1}^{N_T}\tau\sum_{V_{ij}\in\mathcal{T}_h}w_{ij}^n u_{ij}^n h^2+\sum_{n=1}^{N_T}\tau\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}_h}\overline{w}_{ij}^n h^2\overline{u}_{ij}^n.$$

We use similar notation as in the previous case $T_{3m}=-T_{4m}$:

$$T_{3m}=2\int_0^T\int_\Omega\frac{|\nabla u_{\tau_m,h_m}|^2}{f(|\overline{\nabla}u_{\tau_m,h_m}|)}\,\mathrm{d}x\mathrm{d}t,\tag{4.19}$$

$$T_{4m}=2\int_0^T\int_\Omega\overline{w}_{\tau_m,h_m}\;\overline{u}_{\tau_m,h_m}\,\mathrm{d}x\mathrm{d}t.$$

We have, by weak/strong convergence from

$$\lim_{m\to\infty}T_{4m}=2\int_0^T\int_\Omega\overline{w}\;\overline{u}\,\mathrm{d}x\mathrm{d}t,$$

which leads, using (4.18), to

$$\lim_{m\to\infty}T_{3m}=-2\int_0^T\int_\Omega\overline{w}\;\overline{u}\,\mathrm{d}x\mathrm{d}t=2\int_0^T\int_\Omega\overline{H}\cdot\nabla\overline{u}\,\mathrm{d}x\mathrm{d}t.\tag{4.20}$$

We now define

$$T_{5m}=2\int_0^T\int_\Omega\frac{|\nabla u_{\tau_m,h_m}(t,x)|^2}{f(|\nabla u_{\tau_m,h_m}(t,x)|)}\,\mathrm{d}x\mathrm{d}t.\tag{4.21}$$

Let us now prove that T_{3m} and T_{5m} have the same limit. Writing

$$T_{3m}-T_{5m}=\int_0^T\int_\Omega\frac{|\nabla u_{\tau_m,h_m}(t,x)|^2}{f(|\overline{\nabla}u_{\tau_m,h_m}(t,x)|)}\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega\frac{|\nabla u_{\tau_m,h_m}(t,x)|^2}{f(|\nabla u_{\tau,h_m}(t,x)|)}\,\mathrm{d}x\mathrm{d}t=$$

$$\sum_{n=1}^{N_T}\tau\int_\Omega\frac{\nabla u_{h_m}(n\tau,x)^2}{f(|\nabla u_{h_m}((n-1)\tau,x)|)}-\frac{\nabla u_{h_m}(n\tau,x)^2}{f(|\nabla u_{h_m}(n\tau,x)|)}\,\mathrm{d}x\mathrm{d}t=$$

$$\tau \int_\Omega \frac{\nabla u_{h_m}(0,x)^2}{f(|\nabla u_{h_m}(0,x)|)}\,\mathrm{d}x\mathrm{d}t - \tau \int_\Omega \frac{\nabla u_{h_m}(T,x)^2}{f(|\nabla u_{h_m}(T,x)|)}\,\mathrm{d}x\mathrm{d}t +$$

$$\sum_{n=1}^{N_T} \tau \int_\Omega \frac{\nabla u_{\tau_m,h_m}(n\tau,x)^2 - \nabla u_{\tau_m,h_m}((n-1)\tau,x)^2}{f(|\nabla u_{\tau_m,h_m}((n-1)\tau,x)|)}\,\mathrm{d}x\mathrm{d}t.$$

Now, using the property (4.15), the hypothesis (H) and the relation (3.5), we get

$$\lim_{m\to\infty} (T_{3m} - T_{5m}) = 0.$$

Hence we also get that

$$\lim_{m\to\infty} T_{5m} = -2\int_0^T \int_\Omega \overline{w}\;\overline{u}\,\mathrm{d}x\mathrm{d}t = 2\int_0^T \int_\Omega \overline{H}\cdot\nabla\overline{u}\,\mathrm{d}x\mathrm{d}t, \tag{4.22}$$

which completes the proof of (4.14). □

Lemma 4.4. Let u_h and v_h be the arbitrary functions of the discrete function space H_h. We denote by

$$T_6 = \sum_{D_{ij}\in\mathcal{D}_h} \left(\frac{(\nabla u_{ij})^2}{f(|\nabla u_{ij}|)} - \frac{\nabla u_{ij}\cdot\nabla v_{ij}}{f(|\nabla u_{ij}|)} + \frac{(\nabla v_{ij})^2}{f(|\nabla v_{ij}|)} - \frac{\nabla u_{ij}\cdot\nabla v_{ij}}{f(|\nabla v_{ij}|)}\right)\frac{h^2}{2} +$$

$$\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h} \left(\frac{(\nabla\overline{u}_{ij})^2}{f(|\nabla\overline{u}_{ij}|)} - \frac{\nabla\overline{u}_{ij}\cdot\nabla\overline{v}_{i,j}}{f(|\nabla\overline{u}_{ij}|)} + \frac{(\nabla\overline{v}_{ij})^2}{f(|\nabla\overline{v}_{ij}|)} - \frac{\nabla\overline{u}_{ij}\cdot\nabla\overline{v}_{i,j}}{f(|\nabla\overline{v}_{ij}}\right)\frac{h^2}{2}. \tag{4.23}$$

Then it holds

$$T_6 \geq 0. \tag{4.24}$$

P r o o f. Using Cauchy-Schwartz inequality, we immediately have

$$T_6 \geq \sum_{D_{ij}\in\mathcal{D}_h} \left(\frac{|\nabla u_{ij}|}{f(|\nabla u_{ij}|)} - \frac{|\nabla v_{ij}|}{f(|\nabla v_{ij}|)}\right)(|\nabla u_{ij}| - |\nabla v_{ij}|)\,\frac{h^2}{2} +$$

$$\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h} \left(\frac{|\nabla\overline{u}_{ij}|}{f(|\nabla\overline{u}_{ij}|)} - \frac{|\nabla\overline{v}_{ij}|}{f(|\nabla\overline{v}_{ij}|)}\right)(|\nabla\overline{u}_{ij}| - |\nabla\overline{v}_{ij}|)\,\frac{h^2}{2} \geq 0$$

due to the hypothesis (H1) and the remark below (monotonicity of the function f). □

Lemma 4.5. Let Hypothesis (H) be fulfilled. We assume that the sequence $(\tau_m, h_m)_{m\in\mathbf{N}}$ denotes an extracted sub-sequence, the existence of which is provided by Theorem 4.3. Let $\varphi \in C_c^\infty((0,T)\times\Omega)$ be given and the notation from the Lemma 4.2 is used. We denote

$$T_6^n = \sum_{D_{ij}\in\mathcal{D}_h} \left(\frac{(\nabla u_{ij}^n)^2}{f(|\nabla u_{ij}^n|)} - \frac{\nabla u_{ij}^n\cdot\nabla_{ij}^n\varphi}{f(|\nabla u_{ij}^n|)} + \frac{(\nabla_{ij}^n\varphi)^2}{f(|\nabla_{ij}^n\varphi|)} - \frac{\nabla u_{ij}^n\cdot\nabla_{ij}^n\varphi}{f(|\nabla_{ij}^n\varphi|)}\right)\frac{h^2}{2}+$$

$$\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_h}\left(\frac{(\nabla\overline{u}_{ij}^n)^2}{f(|\nabla\overline{u}_{ij}^n|)}-\frac{\nabla\overline{u}_{ij}^n\cdot\overline{\nabla}_{i,j}^n\varphi}{f(|\nabla\overline{u}_{ij}^n|)}+\frac{(\overline{\nabla}_{ij}^n\varphi)^2}{f(|\overline{\nabla}_{ij}^n\varphi|)}-\frac{\nabla\overline{u}_{ij}^n\cdot\overline{\nabla}_{i,j}^n\varphi}{f(|\overline{\nabla}_{ij}^n\varphi|)}\right)\frac{h^2}{2},$$

and

$$T_m=\sum_{n=0}^{N_T}\tau T_6^n. \tag{4.25}$$

Then this gives

$$T_m=\int_0^T\int_\Omega\left(\frac{\nabla u_{\tau_m,h_m}}{f(|\nabla u_{\tau_m,h_m}|)}-\frac{\nabla_{\tau_m,h_m}\varphi}{f(|\nabla_{\tau_m,h_m}\varphi|)}\right)\cdot(\nabla u_{\tau_m,h_m}-\nabla_{\tau_m,h_m}\varphi)\ \mathrm{d}x\mathrm{d}t \tag{4.26}$$

and the following holds

$$\lim_{m\to\infty}T_m=\int_0^T\int_\Omega(\overline{H}-\frac{\nabla\varphi}{f(|\nabla\varphi|)})(\nabla\overline{u}-\nabla\varphi)\ \mathrm{d}x\mathrm{d}t \tag{4.27}$$

and

$$\int_0^T\int_\Omega\overline{H}\cdot\nabla v\ \mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega\frac{\nabla\overline{u}}{f(|\nabla\overline{u}|)}\cdot\nabla v\ \mathrm{d}x\mathrm{d}t,\ \forall v\in L^2((0,T);H_0^1(\Omega)). \tag{4.28}$$

P r o o f. We can rewrite T_m: $T_m=\frac{1}{2}T_{5m}-T_{7m}-T_{8m}+T_{9m}$, where T_{5m} is defined in (4.21):

$$\frac{1}{2}T_{5m}=\int_0^T\int_\Omega\frac{|\nabla u_{\tau_m,h_m}|^2}{f(|\nabla u_{\tau_m,h_m}|)}\ \mathrm{d}x\mathrm{d}t,$$

and

$$T_{7m}=\int_0^T\int_\Omega\frac{\nabla u_{\tau_m,h_m}\cdot\nabla_{\tau_m,h_m}\varphi}{f(|\nabla u_{\tau_m,h_m}|)}\ \mathrm{d}x\mathrm{d}t,$$

$$T_{8m}=\int_0^T\int_\Omega\frac{\nabla u_{\tau_m,h_m}\cdot\nabla_{\tau_m,h_m}\varphi}{f(|\nabla_{\tau_m,h_m}\varphi|)}\ \mathrm{d}x\mathrm{d}t,$$

$$T_{9m}=\int_0^T\int_\Omega\frac{(\nabla_{\tau_m,h_m}\varphi)^2}{f(|\nabla_{\tau_m,h_m}\varphi|)}\ \mathrm{d}x\mathrm{d}t.$$

We already know from (4.22):

$$\lim_{m\to\infty}\frac{1}{2}T_{5m}=\int_0^T\int_\Omega\overline{w}\ \overline{u}\ \mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega\overline{H}\cdot\nabla\overline{u}\ \mathrm{d}x\mathrm{d}t.$$

From the results of Lemma 4.2 we have $\nabla_{\tau,h}\varphi$ strongly converges in $L^\infty((0,T)\times\Omega)$ to $\nabla\varphi$ as $h \longrightarrow 0$ and $\tau \longrightarrow 0$, which implies together with the results of Theorem 4.3: $H_{\tau_m,h_m} \rightharpoonup \overline{H}$

$$\lim_{m\to\infty} T_{7m} = \lim_{m\to\infty} \int_0^T \int_\Omega H_{\tau_m,h_m}(t,x)\nabla_{\tau_m,h_m}\varphi(t,x)\ \mathrm{d}x\mathrm{d}t =$$

$$\int_0^T \int_\Omega \overline{H}(t,x)\nabla\varphi(t,x)\ \mathrm{d}x\mathrm{d}t.$$

$$\lim_{m\to\infty} T_{8m} = \lim_{m\to\infty} \int_0^T \int_\Omega \frac{\nabla_{\tau_m,h_m}\varphi(t,x)\nabla u_{\tau_m,h_m}(t,x)}{f(\nabla_{\tau_m,h_m}\varphi(t,x))}\ \mathrm{d}x\mathrm{d}t =$$

$$\int_0^T \int_\Omega \frac{\nabla\varphi(t,x)\nabla\overline{u}(t,x)}{f(\nabla\varphi(t,x))}\ \mathrm{d}x\mathrm{d}t.$$

$$\lim_{m\to\infty} T_{9m} = \lim_{m\to\infty} \int_0^T \int_\Omega \frac{(\nabla_{\tau_m,h_m}\varphi(t,x))^2}{f(\nabla_{\tau_m,h_m}\varphi(t,x))}\ \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \frac{(\nabla\varphi(t,x))^2}{f(\nabla\varphi(t,x))}\ \mathrm{d}x\mathrm{d}t.$$

From these results we obtain

$$\lim_{m\to\infty} T_m = \int_0^T \int_\Omega \overline{H}\cdot\nabla\overline{u} - \overline{H}\nabla\varphi - \frac{\nabla\varphi\nabla\overline{u}}{f(\nabla\varphi)} + \frac{(\nabla\varphi)^2}{f(\nabla\varphi)}\ \mathrm{d}x\mathrm{d}t,$$

which gives (4.27). From Lemma 4.4 we know that T_m is non negative, so we have

$$\int_0^T \int_\Omega (\overline{H} - \frac{\nabla\varphi}{f(|\nabla\varphi|)})(\nabla\overline{u} - \nabla\varphi)\ \mathrm{d}x\mathrm{d}t \geq 0. \tag{4.29}$$

By density argument we obtain

$$\int_0^T \int_\Omega (\overline{H} - \frac{\nabla v}{f(|\nabla v|)})(\nabla\overline{u} - \nabla v)\ \mathrm{d}x\mathrm{d}t \geq 0 \quad \forall v \in L^2((0,T);H_0^1(\Omega)).$$

Following the idea of [6], we apply Minty trick with $v = \overline{u} - \lambda\psi,\ \lambda > 0,\ \psi \in C_c^\infty((0,T)\times\Omega)$. After dividing by λ we obtain

$$\int_0^T \int_\Omega (\overline{H} - \frac{\nabla\overline{u} - \lambda\psi}{f(|\nabla\overline{u} - \lambda\psi|)})(\nabla\psi)\ \mathrm{d}x\mathrm{d}t \geq 0.$$

We can let $\lambda \longrightarrow 0$ in the above inequality, using Lebesgue's dominated convergence theorem. We then get

$$\int_0^T \int_\Omega (\overline{H} - \frac{\nabla(u)}{f(|\nabla(u)|)}) \nabla\psi \; \mathrm{d}x\mathrm{d}t \geq 0.$$

Since this holds also for $-\psi$, we get

$$\int_0^T \int_\Omega (\overline{H} - \frac{\nabla(u)}{f(|\nabla(u)|)}) \nabla\psi \; \mathrm{d}x\mathrm{d}t = 0.$$

The above equality can again be extended to all $\psi \in L^2((0,T); H_0^1(\Omega))$, which achieves the proof of (4.28). □

Lemma 4.6. Under the same assumptions as in Lemma 4.5, $|\nabla u_{\tau_m,h_m}|$ converges in $L^2((0,T)\times\Omega)$ to $|\nabla\overline{u}|$ as $m \longrightarrow \infty$ (the notation is the same as in Lemma 4.4).

Proof. Let $\varphi \in C_c^\infty((0,T)\times\Omega)$ be given. We denote by $r_{ij}^n = \varphi(n\tau, x_{ij})$ and $\overline{r}_{ij}^n = \varphi(n\tau, \overline{x}_{ij})$. Let us denote

$$\overline{T}_m = \int_0^T \int_\Omega \left(\frac{|\nabla u_{\tau_m,h_m}|}{f(|\nabla u_{\tau_m,h_m}|)} - \frac{|\nabla_{\tau_m,h_m}\varphi|}{f(|\nabla_{\tau_m,h_m}\varphi|)} \right) (|\nabla u_{\tau_m,h_m}| - |\nabla_{\tau_m,h_m}\varphi|) \; \mathrm{d}x\mathrm{d}t.$$

From (4.26), after using Cauchy-Schwartz inequality we have

$$0 \leq \overline{T}_m \leq T_m.$$

We write $\overline{T}_m = T_{10m} - T_{11m} - T_{12m}$, with

$$T_{10m} = \int_0^T \int_\Omega \left(\frac{|\nabla u_{\tau_m,h_m}|}{f(|\nabla u_{\tau_m,h_m}|)} - \frac{|\nabla\overline{u}|}{f(|\nabla\overline{u}|)} \right) (|\nabla u_{\tau_m,h_m}| - |\nabla\overline{u}|) \; \mathrm{d}x\mathrm{d}t$$

$$T_{11m} = -\int_0^T \int_\Omega \left(\frac{(\nabla\varphi)^2}{f(|\nabla\varphi|)} - \frac{(\nabla\overline{u})^2}{f(|\nabla\overline{u}|)} \right) \; \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \frac{|\nabla u_{\tau_m,h_m}|}{f(|\nabla u_{\tau_m,h_m}|)} (|\nabla\varphi| - |\nabla\overline{u}|) \; \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\nabla u_{h_m\tau_m}| \left(\frac{|\nabla\varphi|}{f(|\nabla\varphi|)} - \frac{|\nabla\overline{u}|}{f(|\nabla\overline{u}|)} \right)$$

$$T_{12m} = \int_0^T \int_\Omega \left(\frac{(\nabla\varphi)^2}{f(|\nabla\varphi|)} - \frac{(\nabla_{\tau_m,h_m}\varphi)^2}{f(|\nabla_{\tau_m,h_m}\varphi|)} \right) \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \frac{|\nabla u_{\tau_m,h_m}|}{f(|\nabla u_{\tau_m,h_m}|)} \left(|\nabla\varphi| - |\nabla_{\tau_m,h_m}\varphi|\right) \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega |\nabla u_{\tau_m,h_m}| \left(\frac{|\nabla\varphi|}{f(|\nabla\varphi|)} - \frac{|\nabla_{\tau_m,h_m}\varphi|}{f(|\nabla_{\tau_m,h_m}\varphi|} \right).$$

Now, using Cauchy-Schwarz inequality and estimates from the previous results, we have

$$0 \le T_{10m} \le T_m + C\ \||\nabla\varphi| - |\nabla\overline{u}|\|_{L_2((0,T)\times\Omega)} + C\ \||\nabla\varphi| - |\nabla_{\tau_m,h_m}\varphi|\|_{L_2((0,T)\times\Omega)}.$$

Passing to the limit and using strong convergence of the approximation of a gradient, we obtain, as in [6]

$$0 \le \limsup_{m\longrightarrow\infty} T_{10m} \le 2\int_0^T \int_\Omega \left(\frac{|\nabla\overline{u}|}{f(|\nabla\overline{u}|)} - \frac{|\nabla\varphi|}{f(|\nabla\varphi|)} \right) \left(|\nabla\overline{u}| - |\nabla\varphi|\right) \mathrm{d}x\mathrm{d}t + C\ \||\varphi| - |\nabla\overline{u}|\|_{L_2((0,T)\times\Omega)}.$$

This holds for any $\varphi \in C_c^\infty((0,T)\times\Omega)$ we take $\varphi \longrightarrow \overline{u}$ in $L_2((0,T);H_0^1(\Omega))$ and then the right-hand side tends to zero and we have

$$\lim_{m\longrightarrow\infty} \int_0^T \int_\Omega \left(\frac{|\nabla u_{\tau_m,h_m}|}{f(|\nabla u_{\tau_m,h_m}|)} - \frac{|\nabla_{\tau_m,h_m}\varphi|}{f(|\nabla_{\tau_m,h_m}\varphi|)} \right) \left(|\nabla u_{\tau_m,h_m}| - |\nabla_{\tau_m,h_m}\varphi|\right) \mathrm{d}x\mathrm{d}t = 0.$$

From the property (1.5) of f, we immediately have the conclusion of the lemma. □

Theorem 4.7. Let Hypothesis (H) be fulfilled. We assume that the sequence $(\tau_m, h_m)_{m\in\mathbf{N}}$ denotes an extracted sub-sequence, the existence of which is provided by Theorem 4.3. Then the function $\overline{u} \in L^\infty((0,T);H_0^1(\Omega))$, such that $u_{\tau_m,h_m} \to \overline{u}$ in $L^\infty((0,T);L^2(\Omega))$ is a weak solution of (1.2)-(1.3)-(1.4) in the sense of Definition 1.1. Moreover,

$$\nabla u_{\tau_m,h_m} \to \nabla\overline{u} \text{ in } L^2((0,T)\times\Omega)^2 \text{ (see (4.4)) and}$$
$$|\nabla u_{\tau_m,h_m}| \to |\nabla\overline{u}|, \quad |\overline{\nabla} u_{\tau_m,h_m}| \to |\nabla\overline{u}| \text{ in } L^2((0,T)\times\Omega).$$

Proof. We remind first that $\overline{w}$ is the weak limit of the w_{τ_m,h_m} defined in (4.6) from Theorem 4.3. Using Lemma 4.5 and (4.18) we obtain

$$\int_0^T \int_\Omega \overline{w}v\ \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \frac{\nabla\overline{u}}{f(|\nabla\overline{u}|)} \cdot \nabla v\ \mathrm{d}x\mathrm{d}t,\ \forall v \in L^2((0,T);H_0^1(\Omega)). \tag{4.30}$$

From Lemma 4.6 we get $\overline{w} = -\frac{\overline{u}_t}{f(|\nabla \overline{u}|)}$ and this results in that $\overline{u}$ is a weak solution of (1.2)-(1.3)-(1.4) in the sense of Definition 1.1. Moreover, from Lemma 4.6 and Theorem 4.3 we have the weak convergence $\nabla u_{\tau_m,h_m} \rightharpoonup \nabla \overline{u}$ in $L^2((0,T)\times\Omega)^2$ and the strong convergence of $|\nabla u_{\tau_m,h_m}| \to |\nabla \overline{u}|$, $|\overline{\nabla} u_{\tau_m,h_m}| \to |\nabla \overline{u}|$ in $L^2((0,T)\times\Omega)$.

Now we prove the strong convergence of a gradient. Let again $\varphi \in C_c^\infty((0,T)\times\Omega)$ be a given arbitrary function. From Cauchy-Schwartz inequality we have

$$\int_0^T\int_\Omega (\nabla u_{\tau_m,h_m} - \nabla \overline{u})^2 \, \mathrm{d}x\mathrm{d}t \le 3(T_{13m} + T_{14m} + T_{15m}),$$

where

$$T_{13m} = \int_0^T\int_\Omega (\nabla u_{\tau_m,h_m} - \nabla_{\tau_m,h_m}\varphi)^2 \, \mathrm{d}x\mathrm{d}t,$$

$$T_{14m} = \int_0^T\int_\Omega (\nabla_{\tau_m,h_m}\varphi - \nabla\varphi)^2 \, \mathrm{d}x\mathrm{d}t,$$

$$T_{15m} = \int_0^T\int_\Omega (\nabla\varphi - \nabla \overline{u})^2 \, \mathrm{d}x\mathrm{d}t.$$

From Lemma 4.2 we immediately have

$$\lim_{m\longrightarrow\infty} T_{14m} = 0.$$

For T_{13m} we have (we denote by $< .,. >$ the scalar product of a piecewise constant functions in $L_2(\Omega)$)

$$T_{13m} = \int_0^T \; < \nabla u_{\tau_m,h_m} - \nabla_{\tau_m,h_m}\varphi, \nabla u_{\tau_m,h_m} - \nabla_{\tau_m,h_m}\varphi > \; \mathrm{d}t =$$

$$\int_0^T\int_\Omega |\nabla u_{\tau_m,h_m}|^2 \, \mathrm{d}x\mathrm{d}t - 2\int_0^T \; < \nabla u_{\tau_m,h_m}, \nabla_{\tau_m,h_m}\varphi > \; \mathrm{d}t + \int_0^T\int_\Omega |\nabla_{\tau_m,h_m}\varphi|^2 \, \mathrm{d}x\mathrm{d}t.$$

From the previous results we get

$$\lim_{m\longrightarrow\infty} T_{13m} = \int_0^T\int_\Omega |\nabla \overline{u}|^2 \, \mathrm{d}x\mathrm{d}t - 2\int_0^T \; < \nabla \overline{u}, \nabla\varphi > \; \mathrm{d}t + \int_0^T\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x\mathrm{d}t =$$

$$\int_0^T\int_\Omega |\nabla \overline{u} - \nabla\varphi|^2 \, \mathrm{d}x\mathrm{d}t.$$

Now, for arbitrary $\varepsilon > 0$, we can choose φ in such a way that

$$\int_0^T \int_\Omega (\nabla\varphi - \nabla\overline{u})^2 \, \mathrm{d}x\mathrm{d}t < \varepsilon.$$

Collecting these results together completes the proof. □

Remark. The strong convergence of a gradient can be also proved by the classical results of a weak convergence in a Hilbert space:

weak convergence + convergence of norms ⇒ strong convergence.

5. NUMERICAL EXAMPLES

We choose for the illustration only two numerical experiments that use the proposed scheme in 2D. Further examples concerning image processing problems and comparing the results with the exact solution and computing the experimental order of convergence one can find in [8]. The 3D scheme and numerical experiments for this case one can find in [11] and [12].

Example 5.1. In this experiment we use the exact solution to study the EOC for DDF scheme. We use the solution presented in [14] to the level set equation of the following form

$$u(x,y,t) = \min\{\frac{x^2+y^2-1}{2} + t, 0\}.$$

We have computed the problem on the square $\Omega = [-1,25,1.25] \times [-1,25,1.25]$ with zero Dirichlet boundary conditions, in time interval $[0,T] = [0,0.3125]$.

The solution contains flat regions and a singular circular curve with gradient jump, so we cannot expect second order accuracy. However as we see from the table, the numerical schemes converge also in this singular case and naturally, EOC is equal (or close to) 1 for the solution error. We have used SOR algorithm for solving linear system with tolerance 1.0e-10. The regularisation parameter was chosen as $\varepsilon = h^2$ and $\tau = h^2$. The results for L_2 errors both for the solution E_2 and gradient EG_2 are given in Table 1.

n	τ	E_2	EG_2	EOC	
10	6.25e-02	5.51855e-02	-	2.58165e-01	-
20	1.5625e-02	3.17089e-02	0.79870	2.16399e-01	0.25817
40	3.90625e-03	1.68439e-02	0.91266	1.75709e-01	0.30005
80	9.76563e-04	8.86524e-03	0.95538	1.41097e-01	0.31651
160	2.44141e-04	4.41592e-03	0.97606	1.12646e-01	0.32488
320	6.10352e-05	2.22861e-03	0.98657	8.97433e-02	0.32792

Tab. 1. Example 1, error reports and EOCs, $\varepsilon = h^2$ and $\tau = h^2$

Example 5.2. We will use the scheme for the filtering of 20 percent salt and pepper noise added to the image with function of the asteroid, we set $N = 250$ and $\tau = h^2$. In Figure 4 are shown the noisy image and the results of filtering after 1, 2 and 6 time steps. The initial value for $\overline{u}_{ij}$-s were set as the median of the four neighbouring values of u.

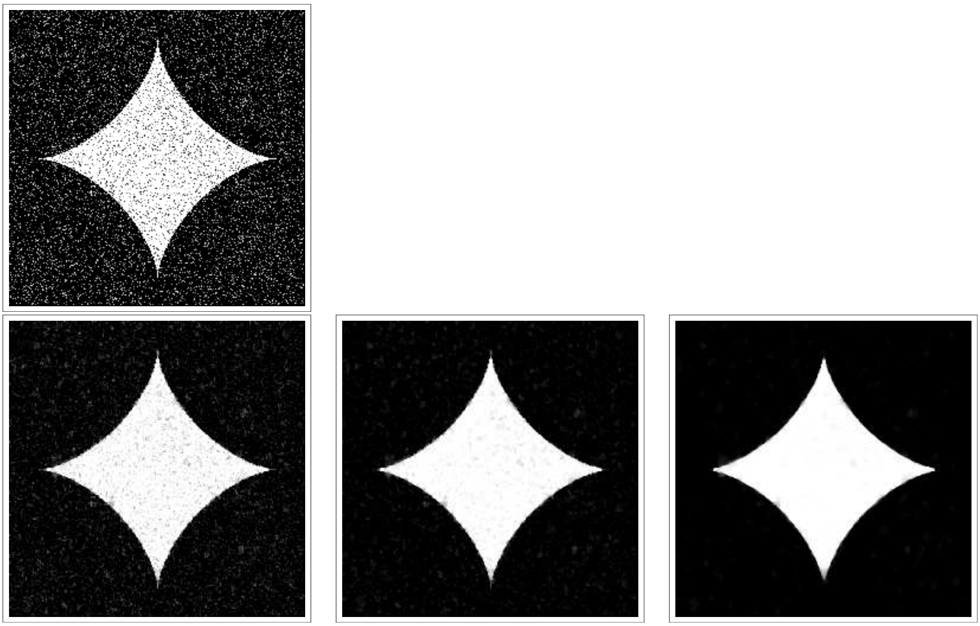

Fig. 4. Initial noisy image with 20 percent salt and pepper noise (top), filtering results after 1 (left), 2 (middle) and 6 (right) time steps

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