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SIMPLE GAMES IN ŁUKASIEWICZ CALCULUS AND THEIR CORES

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We propose a generalization of simple coalition games in the context of games with fuzzy coalitions. Mimicking the correspondence of simple games with non-constant monotone formulas of classical logic, we introduce simple Łukasiewicz games using monotone formulas of Łukasiewicz logic, one of the most prominent fuzzy logics. We study the core solution on the class of simple Łukasiewicz games and show that cores of such games are determined by finitely-many linear constraints only. The non-emptiness of core is completely characterized in terms of balanced systems and by the presence of strong veto players.

Keywords: simple game, game with fuzzy coalitions, core, McNaughton function, Łukasiewicz logic

Classification: 91A12, 06D35

1. INTRODUCTION

Simple games are coalitional games [12] that describe voting in committees and legislative bodies. For example, the United Nations Security Council, which consists of 15 members, approves its decisions by a weighted majority system in which any of the 5 permanent members can veto a proposal. Specifically, decisions on all substantive matters require the affirmative votes of at least 9 members of the council. A simple game is determined by a finite player set and by a family of all winning coalitions, which (i) contains the set of all players, (ii) does not contain the empty set, and (iii) is closed with respect to supersets. Equivalently, every simple game can be identified with a non-constant monotone Boolean function.

Games with fuzzy coalitions, which are just vectors in the unit cube, were introduced by Aubin [2] and since then the theory has been developed in a number of papers and books—see e.g. [4, 5, 6]. The main goal of this paper is to investigate the class of games with fuzzy coalitions that results from changing the logical framework for simple games. Namely, we use the infinite-valued Łukasiewicz logic [8] in place of the Boolean logic and work with the associated logical functions (the so-called McNaughton functions). In this way, we define the class of simple Łukasiewicz games (Definition 3.3), which are argued to be suitable many-valued generalizations of simple games. This step has a few methodological consequences: (i) we obtain a faithful 'completion' of simple games over all fuzzy coalitions since the restriction of every McNaughton function to coalitions is a Boolean function, (ii) we replace the yes/no voting interpretation of simple games by the degree-based utility interpretation of simple Łukasiewicz games. Further, we focus on the core solution concept for simple Łukasiewicz games, which was defined for general games with fuzzy coalitions by Aubin [2].

The paper is structured as follows. In Section 2 we recall basic notions and results in coalitional games (simple games, in particular) and games with fuzzy coalitions. The formal analogues of simple games in Łukasiewicz calculus, the so-called simple Łukasiewicz games, are introduced and studied in Section 3, which also includes the necessary background on Łukasiewicz logic. The main result therein is Corollary 3.6 showing that each simple fuzzy game can be described by a non-constant monotone formula of Łukasiewicz logic; this generalizes the well-known fact about classical simple games. The Aubin's core of simple fuzzy games is investigated thoroughly in Section 4, where we give a wealth of examples and prove that the core of a simple Łukasiewicz game is a convex polytope included in the standard unit simplex (Theorem 4.4). Finally, in Section 5, we find two necessary and sufficient conditions for existence of a payoff distribution in the core: first, using the notion of balanced system of fuzzy coalitions (Theorem 5.1) and, second, using the notion of strong veto players (Theorem 5.5).

2. BASIC NOTIONS

We repeat the basic terminology regarding cooperative coalitional games with transferable utilities [12]. Let the player set be $N = \{1, \dots, n\}$ for some $n \in \mathbb{N}$. Each element $i \in N$ is called a *player* and each element A in the set 2^N of all subsets of N is a *coalition.* The coalition of all the players N is said to be the *grand coalition.* A *(coalition) game* is a function $v\colon 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The value $v(A)$ is said to be a *worth* of coalition $A \subseteq N$. Worth $v(A)$ is the total amount of utility resulting from the cooperation among members of A on some economic or social project. In particular, if the range of a game v is the two-valued scale $\{0, 1\}$ only, then each coalition $A \subseteq N$ is either winning ($v(A) = 1$) or loosing ($v(A) = 0$). Moreover, it is natural to assume that a winning coalition cannot become loosing after any player joins it. In this way we obtain the class of simple games cite[Section 2.2.3]PelegSudholter07.

Definition 2.1. A *simple game* is a $\{0, 1\}$-valued game such that $v(N) = 1$ and v is non-decreasing: $v(A) \leq v(B)$, whenever $A \subseteq B$ for each $A, B \subseteq N$.

Each simple game v can be associated with a unique Boolean function f_v. To this end, notice that a coalition $A \subseteq N$ can be viewed as a vector $1_A \in \{0, 1\}^n$ with coordinates

$$(1_A)_i = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

In case that $A = \{i\}$, we write simply 1_i in place of $1_{\{i\}}$. Consequently, put $f_v(1_A) = v(A)$ for each $A \subseteq N$. Clearly, $f_v\colon \{0, 1\}^n \to \{0, 1\}$ is a non-decreasing Boolean function such that $f_v(1_\emptyset) = 0$ and $f_v(1_N) = 1$.

Example 2.2. (Majority voting) Assume that $N = \{1, 2, 3\}$ is the set of players. The majority voting is captured by a simple game w such that

$$w(A) = \begin{cases} 1 & \text{if } |A| \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding Boolean function is

$$f_w(a_1, a_2, a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3), \quad a_1, a_2, a_3 \in \{0, 1\}.$$

The next game is a simplified version of the UNSC voting.

Example 2.3. Let us consider two groups of players, $N_1 = \{1, 2, 3\}$ and $N_2 = \{4, 5\}$. The simple coalition game over the player set $N = N_1 \cup N_2$ is given by

$$u(A) = \begin{cases} 1 & \text{if } A \supsetneq N_1, \\ 0 & \text{otherwise.} \end{cases}$$

This game is associated with the Boolean function

$$f_u(a_1, a_2, a_3, a_4, a_5) = a_1 \wedge a_2 \wedge a_3 \wedge (a_4 \vee a_5), \quad a_i \in \{0, 1\},\ i = 1, \ldots, 5.$$

It is well-known that each Boolean function corresponds to a formula of classical (Boolean) logic, and vice-versa. We say that a formula of classical logic is *monotone* whenever it is equivalent to a formula built from propositional variables and truth constants **0** and **1** using disjunction and conjunction only.

Theorem 2.4. (Wegener [16, Theorem 4.1]) Let $v\colon 2^N \to \{0, 1\}$ be a non-constant function. Then v is a simple game iff there is a monotone formula φ such that f_v is the Boolean function corresponding to φ.[1]

We use Theorem 2.4 as a guideline in order to single out our class of 'simple' games with fuzzy coalitions in the next section.

The core is one of the basic solution concepts in cooperative game theory [12, Chapter 3]. Let v be a (not necessarily simple) coalition game. The *core* of v is the set of all efficient payoff vectors $x \in \mathbb{R}^n$ upon which no coalition can improve, that is,

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in A} x_i \geq v(A) \text{ for each } A \subseteq N \right\}.$$

The Bondareva–Shapley theorem [12, Theorem 3.1.4] gives a necessary and sufficient condition for the core non-emptiness in terms of the so-called balanced systems, which we discuss in Section 4 in case of simple fuzzy games. If v is a simple game, then we can employ another criterion for the core non-emptiness. This condition is related to the notion of veto player (see Section 5 for explanation of this notion). A player $i \in N$ is said to be a *veto player* provided that $v(N \setminus \{i\}) = 0$. Observe that no veto players are

[1] In fact, we can assume that φ is in the irreducible disjunctive normal form.

present in the game described by Example 2.2 and, in the same time, its core is empty. Such a situation is no coincidence. If the set of veto players $M \subseteq N$ in a simple game v is non-empty, we can uniformly distribute all the payoffs among the veto players only: let

$$x_i = \begin{cases} \frac{1}{|M|} & \text{if } i \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in \mathcal{C}(v)$. This situation applies, for example, to the United Nations Security Council voting game in which any of the five permanent members is a veto player. Similarly, in Example 2.3 each player from set N_1 is a veto player. The following fact follows easily.

Proposition 2.5. A simple game has a non-empty core if and only if there is at least one veto player in the game.

Since the publication of Aubin's seminal paper [2], cooperative scenarios involving fractional membership degrees in coalitions have been studied. In such situation, the subsets of N are no longer proper models for coalitions. Instead, fuzzy coalitions have to be introduced in order to represent the intermediate membership degrees. We assume that a membership degree of player $i \in N$ is determined by a number a_i in the unit interval $I = [0,1]$. A *fuzzy coalition* is a vector $a = (a_1, \dots, a_n) \in I^n$. The n-dimensional cube I^n is thus identified with the set of all fuzzy coalitions. Several definitions of fuzzy games appear in the literature (see e. g. [2, 7]). We adopt the one used by Azrieli and Lehrer [4].

Definition 2.6. Let the set of players be $N = \{1, \dots, n\}$. A *game (with fuzzy coalitions)* is a bounded function $v\colon I^n \to \mathbb{R}$ satisfying $v(1_\emptyset) = 0$.

Most solution concepts (the core, in particular) have been generalized to games with fuzzy coalitions—see [5] for a survey. In our paper we introduce a new subclass of fuzzy games that represent a formal counterpart of Boolean voting systems. This is achieved by considering logical functions corresponding to a logic weaker than the classical two-valued logic. Example 3.7 in Section 3 provides a possible motivation for our approach by describing a game in which players are represented by groups of players in some classical simple game (e.g. game u from Example 2.3).

3. SIMPLE GAMES IN ŁUKASIEWICZ CALCULUS

We provide a survey of Łukasiewicz infinite-valued propositional logic and its associated Lindenbaum algebra (for more details see e.g. [8, Chapter 4]). We restrict ourselves to finitely-many propositional variables $A_1, \dots, A_n$; formulas φ, ψ, ... are then constructed from these variables and the truth-constant $\mathbf{0}$ using the following basic connectives: negation $\neg$ and strong disjunction $\oplus$. The set of all such formulas is denoted by FORM_n.

The standard semantics for connectives of Łukasiewicz logic is given by the corresponding operations of the *standard* MV-*algebra [8],* which is just the real unit interval I endowed with constant falsum $\mathbf{0}$ and the operations of negation $\neg$ and strong disjunction $\oplus$ defined as:

$$\mathbf{0} = 0, \qquad \neg a = 1 - a, \qquad a \oplus b = \min(1, a + b).$$

We also introduce derived connectives (we list them together with their standard semantics; the same symbol is used to denote the connective and the corresponding binary operation on I^2):

implication	$\to$	$\varphi \to \psi$	$= \neg\varphi \oplus \psi$	$\min\{1, 1-a+b\}$
strong conjunction	$\odot$	$\varphi \odot \psi$	$= \neg(\neg\varphi \oplus \neg\psi)$	$\max\{0, a+b-1\}$
lattice disjunction	$\vee$	$\varphi \vee \psi$	$= (\varphi \to \psi) \to \psi$	$\max\{a, b\}$
lattice conjunction	$\wedge$	$\varphi \wedge \psi$	$= \neg(\neg\varphi \vee \neg\psi)$	$\min\{a, b\}$
verum	$\mathbf{1}$	$\mathbf{1}$	$= \neg\mathbf{0}$	1

The operations $\odot$ and $\oplus$ are also known as *Łukasiewicz t-norm* and *Łukasiewicz t-conorm,* respectively (see [9]). A *valuation* is a mapping $\nu\colon \mathsf{FORM}_n \to I$ such that for each $\varphi, \psi \in \mathsf{FORM}_n$:

$$\nu(\mathbf{0}) = 0, \qquad \nu(\neg\varphi) = 1 - \nu(\varphi), \qquad \nu(\varphi \oplus \psi) = \min\{1, \nu(\varphi) + \nu(\psi)\}.$$

Formulas $\varphi, \psi \in \mathsf{FORM}_n$ are called *equivalent* when $\nu(\varphi) = \nu(\psi)$ for every valuation ν. The *equivalence class* of φ is denoted by $[\varphi]$. Another example of an MV-algebra is the *Lindenbaum algebra* $\mathcal{M}_n$ of Łukasiewicz logic over n propositional variables, which is the set of all equivalence classes $[\varphi]$ endowed with the operations:[2]

$$\mathbf{0} = [\mathbf{0}], \qquad \neg[\varphi] = [\neg\varphi], \qquad [\varphi] \oplus [\psi] = [\varphi \oplus \psi].$$

Since every valuation ν is completely determined by its restriction to the propositional variables $\nu \mapsto (\nu(A_1), \ldots, \nu(A_n)) \in I^n$, every 'possible world' ν is matched with a unique point $x_\nu \in I^n$. Conversely, for any $x \in I^n$, let ν_x be the valuation uniquely defined by $(\nu_x(A_1), \ldots, \nu_x(A_n)) = x$. Therefore, for each formula φ, its equivalence class $[\varphi]$ can be viewed as a function $[\varphi]\colon I^n \to I$ defined as $[\varphi](x) = \nu_x(\varphi)$ for every $x \in I^n$. Accordingly, $\mathcal{M}_n$ can be rendered as an algebra of functions with operations $\neg$ and $\oplus$ defined pointwise and with $\mathbf{0}$ being the constant zero function.

It is easy to see by induction on the complexity of formula φ that each such function $[\varphi]$ is continuous and piecewise linear with pieces having integer coefficients. The celebrated McNaughton theorem [10] states that those conditions are also sufficient for a function $v\colon I^n \to I$ to belong to $\mathcal{M}_n$.

McNaughton theorem. Let $v\colon I^n \to I$. Then $v \in \mathcal{M}_n$ (i.e., $v = [\varphi]$ for some $\varphi \in \mathsf{FORM}_n$), if and only if, v is continuous and piecewise linear with each linear piece having integer coefficients.

Each function in $\mathcal{M}_n$ is called an *(n-variable) McNaughton function.* Before we proceed further, we need to recall some facts about McNaughton functions, especially their well-known decomposition by min-max combinations of linear polynomials [1, 14]. For any real function $f\colon I^n \to \mathbb{R}$ we define $f^\sharp(x) = \min\{1, \max\{0, f(x)\}\}$. In that follows, by a *polytope* we mean a convex hull of finitely-many points in $\mathbb{R}^n$.

[2] The Lindenbaum algebra $\mathcal{M}_n$ is usually presented as the free n-generated MV-algebra, our way is equivalent due to the so-called standard completeness theorem, see [8, Proposition 4.5.5].

Proposition 3.1. (Aguzzoli [1, Theorem 1.4.4]) For each McNaughton function v, there are unique sets

- Poly^v of linear polynomials $p_1^v, \dots, p_k^v \colon I^n \to \mathbb{R}$ with integer coefficients and
- $\mathcal{P}^v$ of n-dimensional polytopes $P_1^v, \dots, P_m^v \subseteq I^n$

such that $\bigcup_{i=1}^m P_i^v = I^n$, for each $i \le m$ there is $j_i \le k$ such that for each $x \in P_i^v$ holds $v(x) = p_{j_i}^v(x)$, and

$$v = \bigwedge_{i=1}^{m} \bigvee_{j=1}^{j_i} (p_j^v)^\sharp .$$

Remark 3.2. This proposition is formulated in a simplified form sufficient for our needs, for a precise elaboration the reader is advised to consult [1]. Let us now just hint how $\mathcal{P}^v$ is determined by Poly^v. Given a permutation π of the set $\{1, \dots, k\}$, define $P_\pi \subseteq I^n$ by

$$P_\pi = \left\{ a \in I^n \;\middle|\; p_{\pi(1)}(a) \le \dots \le p_{\pi(k)}(a) \right\}.$$

Each P_π is a (possibly empty) polytope since P_π is the intersection of cube I^n with finitely-many halfspaces. If Π is the set of all permutations π of $\{1, \dots, k\}$ making polytope P_π n-dimensional, then $\mathcal{P}^v = \{ P_\pi \mid \pi \in \Pi \}$.

We can say that $p_1^v, \dots, p_k^v$ are the linear pieces of v and $P_1^v, \dots, P_m^v$ are the corresponding 'domains of linearity'. We omit the superscript ^{v} when clear from the context.

The McNaughton theorem shows that McNaughton functions stand to Łukasiewicz logic as Boolean functions stand to classical two-valued logic: they are functions 'expressible' by the formulas of the logic. Therefore we can think of them the key ingredients for developing many-valued analogues of simple games in coalitional game theory. Analogously to simple games, which can be identified with non-decreasing Boolean functions, we define simple Łukasiewicz games as non-decreasing McNaughton functions. As our intuition here is based on a particular fuzzy logic we opt for a more specific name than just 'simple fuzzy games'.

Definition 3.3. Let the player set be $N = \{1, \dots, n\}$. A *simple Łukasiewicz game* is a non-decreasing McNaughton function $v \colon I^n \to I$ such that $v(1_\emptyset) = 0$ and $v(1_N) = 1$. By SLG_n we denote the set of all simple Łukasiewicz games over I^n.

Every simple Łukasiewicz game is indeed a game with fuzzy coalitions in the sense of Definition 2.6. Note that the conditions $v(1_\emptyset) = 0$ and $v(1_N) = 1$ could be replaced by demanding that v is non-constant.

The rest of this section is dedicated to generalizing Theorem 2.4, which will justify the notion of simple Łukasiewicz game. We say that $\varphi \in \mathsf{FORM}_n$ is *monotone* if φ is equivalent to a formula ψ over the propositional language containing only the symbols $A_1, \dots, A_n, \oplus, \odot, \vee, \wedge, \mathbf{0}, \mathbf{1}$. We start with the proof of an important lemma.

Lemma 3.4. Let $p \colon I^n \to \mathbb{R}$ be a non-decreasing linear polynomial with integer coefficients. Then there exists a monotone formula φ of Łukasiewicz logic such that the McNaughton function $p^\sharp = [\varphi]$.

Proof. We know that

$$p(a_1,\dots,a_n)=\sum_{j=1}^{n}\alpha_j a_j+\beta,\quad \text{for every } (a_1,\dots,a_n)\in I^n,$$

and that $\alpha_j \geq 0$ for each $j \leq n$ (as p is non-decreasing). We proceed by induction on $\alpha=\sum_{j=1}^{n}\alpha_j$. First suppose that $\alpha=0$, i.e., $p(a_1,\dots,a_n)=\beta$. Then clearly:

$$\varphi=\begin{cases}\mathbf{1} & \text{if } \beta>0,\\ \mathbf{0} & \text{otherwise.}\end{cases}$$

Induction step: let $\max(\alpha_1,\dots,\alpha_n)=\alpha_j\geq 1$. Put $q=p-a_j$ and, by induction hypothesis, we know that there are monotone formulas $\chi,\delta\in\mathsf{FORM}_n$ such that $[\chi]=q^\sharp$ and $[\delta]=(q+1)^\sharp$. We can routinely check that the equation

$$(q(a)+a_j)^\sharp=(q(a)^\sharp\oplus a_j)\odot(q(a)+1)^\sharp$$

is satisfied for every $a=(a_1,\dots,a_n)\in I^n$ (see e.g. [8, Lemma 3.1.9]). Therefore we can get the sought formula as $\varphi=(\chi\oplus A_j)\odot\delta$. □

Theorem 3.5. A function $v\colon I^n\to I$ is a monotone McNaughton function iff there is a monotone formula of Łukasiewicz logic φ such that $v=[\varphi]$.

Proof. Left-to-right direction: let us consider $p\in \mathit{Poly}^v$. Because v is non-decreasing, so is p and thus we can employ Lemma 3.4 to recover a monotone formula φ_p such that $[\varphi_p]=p^\sharp$. To complete the proof we use Proposition 3.1 to entail that the sought formula can be defined as:

$$\varphi=\bigwedge_{i=1}^{m}\bigvee_{j=1}^{j_i}\varphi_{p_j}.$$

The converse direction is an easy consequence of monotonicity of all the operations involved. □

Corollary 3.6. Let $v\colon I^n\to I$ be a non-constant function. Then the game v is a simple Łukasiewicz game iff there is a monotone formula of Łukasiewicz logic φ such that $v=[\varphi]$.

The game u from Example 2.3 describes the UNSC-style voting with two distinguished player sets. The model can be further improved by assessing how results of voting in the two groups contribute to the overall result. This naturally results in a simple Łukasiewicz game.

Example 3.7. Let the player set be $N=\{1,2\}$, where each $i=1,2$ corresponds to the respective set of players N_i from Example 2.3. Each degree of membership $a_i\in I$ captures the ratio of affirmative votes within the group. Hence we have the following membership degrees: $a_1\in\left\{0,\frac{1}{3},\frac{2}{3},1\right\}$ and $a_2\in\left\{0,\frac{1}{2},1\right\}$. Put

$$v_0(a_1,a_2)=\begin{cases}0 & \text{if } a_2=0,\\ a_1 & \text{if } a_2\in\left\{\frac{1}{2},1\right\}.\end{cases}$$

Values of $v_0(a_1, a_2)$ are thus the ratios of affirmative votes within the group of veto players, provided that $a_2 \neq 0$. If v is a linear interpolation of v_0 on the square I^2 subdivided into two triangles meeting at the segment with endpoints $(0, \frac{1}{2})$ and $(1, 0)$, then we obtain a simple Łukasiewicz game $v(a_1, a_2) = a_1 \wedge (a_1 + 2a_2 - 1)^\sharp$. By Corollary 3.6, this game corresponds to a formula $A_1 \wedge (A_1 \odot (A_2 \oplus A_2))$. Although the linear interpolation of v_0 may seem artificial, observe that we have arrived at a simple closed-form formula for v. Moreover, we will show in next section that already function v_0 contains sufficient information about payoff distributions.

4. THE CORES OF SIMPLE ŁUKASIEWICZ GAMES

The core of a game v with fuzzy coalitions and the player set $N = \{1, \dots, n\}$ is a set of payoff vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, each of which is *efficient* (the players can redistribute the total worth $v(1_N)$ of the grand coalition N among themselves) and *coalitionally rational* (no fuzzy coalition a can contest payoff x due to obtaining strictly less than worth $v(a)$ resulting from a's own activity). For every $a, b \in \mathbb{R}^n$, let $\langle a, b \rangle$ denote the standard scalar product in $\mathbb{R}^n$, that is, $\langle a, b \rangle = \sum_{i=1}^n a_i b_i$.

Definition 4.1. The *core* of a game with fuzzy coalitions v is the set

$$\mathcal{C}(v) = \{ x \in \mathbb{R}^n \mid \langle 1_N, x \rangle = v(1_N) \text{ and } \langle a, x \rangle \geq v(a) \text{ for each } a \in I^n \}.$$

This is a generalization of the classical core (see Section 2) introduced by Aubin [2], which was subsequently investigated in a number of papers (see [4, 5], for example). In contrast to classical coalition games, the core of a game with fuzzy coalitions is the intersection of infinitely-many halfspaces with the affine hyperplane. For each $a \in I^n$ we define:

$$\mathcal{C}_a(v) = \begin{cases} \{ x \in \mathbb{R}^n \mid \langle 1_N, x \rangle = v(1_N) \} & \text{if } a = 1_N, \\ \{ x \in \mathbb{R}^n \mid \langle a, x \rangle \geq v(a) \} & \text{otherwise,} \end{cases}$$

and observe that we can write

$$\mathcal{C}(v) = \bigcap_{a \in I^n} \mathcal{C}_a(v).$$

In general, the structure of the core for games with fuzzy coalitions can be fairly complicated. Azrieli and Lehrer provide a complete characterization (based on the notion of a balanced system introduced in the sequel) of games with fuzzy coalitions whose core is non-empty [4, Theorem 1(i)]. Besides, Branzei et al. proved that the core of convex fuzzy games is fully given by Boolean coalitions only [5, Theorem 8.38]. We call a game with fuzzy coalitions v *convex*, whenever the condition

$$v(a \oplus d) - v(a) \leq v(b \oplus d) - v(b)$$

holds true for all fuzzy coalitions $a, b, d \in I^n$ such that $a \leq b$ and $b \odot d = 1_\emptyset$. It is worth emphasizing that, in general, convexity of a game v does not imply and is not implied by convexity of v as an n-variable real function [4].

Theorem 4.2. (Branzei et al. [5]) If v is a convex game with fuzzy coalitions, then

$$\mathcal{C}(v) = \bigcap_{A \subseteq N} \mathcal{C}_{1_A}(v) \neq \emptyset.$$

The previous result enables us to reduce the core of convex simple Łukasiewicz games to the intersection of finitely-many linear constraints. We obtain, however, a more general result for the whole class of simple Łukasiewicz games $\mathsf{SŁG}_n$.

We start with Example 4.3 suggesting the core structure of all games in $\mathsf{SŁG}_n$. In that follows, $\mathcal{N}(v)$ denotes the set of all *nodes* of a piecewise linear function $v \in \mathsf{SŁG}_n$, that is, the finite set of all fuzzy coalitions $a \in I^n$ such that a is a vertex of some polytope in $\mathcal{P}^v$, where $\mathcal{P}^v$ denotes the set of all linearity domains of v (see Proposition 3.1). Notice that $\mathcal{N}(v) \supseteq \{0,1\}^n$.

Example 4.3. Assume that the player set is $N = \{1, 2\}$. Let $\mathrm{co}\{a^1, \ldots, a^k\}$ denote the convex hull of vectors $a^1, \ldots, a^k \in \mathbb{R}^n$. Put

$$P = \mathrm{co}\left\{(1, \tfrac{9}{13}), (\tfrac{9}{13}, 1), (1, \tfrac{7}{10}), (\tfrac{7}{10}, 1)\right\},$$
$$Q = \mathrm{co}\left\{(1, \tfrac{7}{10}), (\tfrac{7}{10}, 1), 1_N\right\}.$$

Define a function $v\colon I^n \to \mathbb{R}$ by

$$v(a) = \begin{cases} 13a_1 + 13a_2 - 22 & \text{if } a \in P, \\ 3a_1 + 3a_2 - 5 & \text{if } a \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

Then $v \in \mathsf{SŁG}_n$ and its node set is

$$\mathcal{N}(v) = \left\{1_\emptyset, 1_1, 1_2, (1, \tfrac{9}{13}), (\tfrac{9}{13}, 1), (1, \tfrac{7}{10}), (\tfrac{7}{10}, 1), 1_N\right\}.$$

We can directly verify that its core coincides with the line segment

$$\mathrm{co}\left\{(\tfrac{1}{10}, \tfrac{9}{10}), (\tfrac{9}{10}, \tfrac{1}{10})\right\} = \bigcap_{a \in \mathcal{N}(v)} \mathcal{C}_a(v).$$

The previous example is just an instance of the general phenomenon: the core of every simple Łukasiewicz game v is fully determined by the values of v at all nodes $a \in \mathcal{N}(v)$. Remarkably, the core is thus given by the intersection of finitely-many linear constraints only.

Theorem 4.4. If $v \in \mathsf{SŁG}_n$, then $\mathcal{C}(v) = \bigcap_{a \in \mathcal{N}(v)} \mathcal{C}_a(v)$. Moreover, $\mathcal{C}(v)$ is a (possibly empty) polytope included in the standard $(n-1)$-dimensional simplex

$$\Delta_n = \{\, x \in \mathbb{R}^n \mid \langle 1_N, x \rangle = 1 \text{ and } x_i \geq 0 \text{ for every } i \in N \,\}.$$

P r o o f. As for the first part of the proposition, we only need to show that

$$\mathcal{C}(v) \supseteq \bigcap_{a \in \mathcal{N}(v)} \mathcal{C}_a(v). \tag{1}$$

Assume that $x \in \bigcap_{a \in \mathcal{N}(v)} \mathcal{C}_a(v)$. We will check that no fuzzy coalition $b \in I^n \setminus \mathcal{N}(v)$ can improve upon the choice of payoff vector x, that is, the inequality $\langle b, x \rangle \geq v(b)$ holds true for every $b \in I^n \setminus \mathcal{N}(v)$.

There exists a polytope $P \in \mathcal{P}^v$ such that $b \in P$ and v is linear over P. Let $\mathrm{V}(P)$ be the set of all vertices of P. Minkowski theorem presents b as a convex combination of vertices of P: there exist real numbers $\alpha_c \in I$, one for each vertex $c \in \mathrm{V}(P)$, such that $b = \sum_{c \in \mathrm{V}(P)} \alpha_c c$ and $\sum_{c \in \mathrm{V}(P)} \alpha_c = 1$. Hence

$$\langle b, x \rangle = \sum_{c \in \mathrm{V}(P)} \alpha_c \langle c, x \rangle \geq \sum_{c \in \mathrm{V}(P)} \alpha_c v(c) = v \left(\sum_{c \in \mathrm{V}(P)} \alpha_c c \right) = v(b),$$

which proves (1) and thus $\mathcal{C}(v) = \bigcap_{a \in \mathcal{N}(v)} \mathcal{C}_a(v)$. Next we show that $\mathcal{C}(v)$ is contained in Δ_n; we know that $\mathcal{N}(v) \supseteq \{0,1\}^n$ and thus

$$\mathcal{C}(v) = \bigcap_{a \in \mathcal{N}(v)} \mathcal{C}_a(v) \subseteq \bigcap_{i \in N} \mathcal{C}_{1_i}(v) \cap \mathcal{C}_{1_N}(v) \subseteq \Delta_n.$$

This also implies that $\mathcal{C}(v)$ is a polytope since it is a bounded intersection of finitely-many halfspaces $\mathcal{C}_a(v)$. □

The previous proposition makes it possible to discuss examples of cores of particular simple Łukasiewicz games. For example, the core of $v_\wedge(a) = a_1 \wedge \cdots \wedge a_n$ equals Δ_n and therefore it is the maximal possible core. Indeed, since $\mathcal{N}(v_\wedge) = \{0,1\}^n$ we have

$$\mathcal{C}(v_\wedge) = \bigcap_{A \subseteq N} \mathcal{C}_{1_A}(v_\wedge).$$

However, for $A \neq N$ we have $\mathcal{C}_{1_A}(v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i \in A} x_i \geq 0 \right\}$ and so each such set is redundant in the above intersection, unless A is a singleton. This yields

$$\bigcap_{A \subseteq N} \mathcal{C}_{1_A}(v_\wedge) = \bigcap_{i \in N} \mathcal{C}_{1_i}(v_\wedge) \cap \mathcal{C}_{1_N}(v_\wedge) = \Delta_n.$$

The core can collapse to a point. In case of a 'dictatorial' game given by the ith coordinate projection $v_i(a) = a_i$, we obtain $\mathcal{C}(v_i) = \{1_i\}$, the singleton assigning the whole unit payoff to player i: indeed,

$$\mathcal{C}(v_i) = \bigcap_{a \in \{0,1\}^n} \mathcal{C}_a(v_i) = \mathcal{C}_{1_i}(v_i) \cap \mathcal{C}_{1_N}(v_i) = \{1_i\}.$$

Examples of games with empty cores are easily found: take, for instance, the game defined by $v_\oplus(a) = a_1 \oplus \cdots \oplus a_n$. It results that $\mathcal{C}(v_\oplus) = \emptyset$, since $\mathcal{C}_{1_N}(v_\oplus) \cap \mathcal{C}_{1_i}(v_\oplus) = \{1_i\}$

and so the intersection $\mathcal{C}_{1_N}(v_\oplus) \cap \mathcal{C}_{1_i}(v_\oplus) \cap \mathcal{C}_{1_j}(v_\oplus)$ is empty for $i \neq j$. The similarity of game $v_\oplus$ with majority voting game in Example 2.2 is intuitively appealing. By the same token, we can say that no players with veto power are present in game $v_\oplus$. The role of (generalization of) veto players in simple Łukasiewicz games will be shown in Theorem 5.5.

Remark 4.5. We may view the core $\mathcal{C}(v)$ of each simple Łukasiewicz game v as the set of states on the unit cube I^n [11, 13], where each such state majorize the game. Specifically, for each $x \in \mathcal{C}(v)$, let $s_x \colon I^n \to I$ be the linear function

$$s_x(a) = \langle a, x \rangle .$$

Since $x \in \Delta_n$ by Theorem 4.4, it follows that s_x is a non-decreasing function with $s_x(1_N) = 1$. Moreover, s_x is additive in the sense of Łukasiewicz calculus:

$$s_x(a \oplus b) = s_x(a) + s_x(b), \quad \text{whenever } a, b \in I^n \text{ satisfy } a \odot b = 0.^3$$

The last condition $a \odot b = 0$ is a natural expression for the disjointness of fuzzy coalitions a and b: two fuzzy coalitions are disjoint unless the sum of each player's membership degree in those coalitions exceeds 1. Every payoff vector $x = (x_1, \ldots, x_n) \in \mathcal{C}(v)$, which is imputed to individual players $i \in N$, is distributed by state s_x linearly to each fuzzy coalition $a \in I^n$ according to the levels of participation a_i of individual members of a. This distribution scheme provides the usual interpretation of states and measures in coalitional game theory (cf. [3]).

5. CHARACTERIZATIONS OF GAMES WITH NON-EMPTY CORES

Azrieli and Lehrer proved a necessary and sufficient condition for non-emptiness of the core on the whole class of games with fuzzy coalitions [4, Theorem 1(i)]. We improve this result for simple Łukasiewicz games in two ways. First we use the notion of balanced families of fuzzy coalitions. Let $\mathcal{B}$ be a finite set of fuzzy coalitions in I^n and $(\lambda_a)_{a \in \mathcal{B}}$ be a real vector with $\lambda_a \in I$. We say that a pair $(\mathcal{B}, (\lambda_a)_{a \in \mathcal{B}})$ is a *balanced system* if

$$\sum_{a \in \mathcal{B}} \lambda_a a = 1_N .$$

Theorem 5.1. Let $v \in \mathsf{SLG}_n$. The core $\mathcal{C}(v)$ is non-empty if and only if the inequality

$$\sum_{a \in \mathcal{N}(v)} \lambda_a v(a) \leq 1 \tag{2}$$

is true for every balanced system $(\mathcal{N}(v), (\lambda_a)_{a \in \mathcal{N}(v)})$.

[3] This easily follows from the fact that $a \odot b = 0$ iff $a + b \leq 1$, which gives $a \oplus b = a + b$.

P r o o f. Given $v \in \mathsf{SŁG}_n$, consider the following linear program:

$$\text{minimize } \langle 1_N, x\rangle \qquad \text{subject to } \langle a, x\rangle \geq v(a) \text{ for every } a \in \mathcal{N}(v). \tag{3}$$

Using Theorem 4.4, the value of (3) is $v(1_N) = 1$ iff $\mathcal{C}(v) \neq \emptyset$. Moreover, the program (3) is feasible: for each player $i \in N$, define

$$\hat{a}_i = \min\{\, a_i \mid (a_1, \dots, a_n) \in \mathcal{N}(v) \text{ and } a_i > 0 \,\}, \qquad \hat{x}_i = \tfrac{1}{\hat{a}_i}, \qquad \hat{x} = (\hat{x}_1, \dots, \hat{x}_n).$$

If $a = 1_\emptyset$, then $\langle 1_\emptyset, \hat{x}\rangle = 0 = v(1_\emptyset)$. If $a \in \mathcal{N}(v) \setminus \{1_\emptyset\}$, then

$$\langle a, \hat{x}\rangle = \sum_{i=1}^{n} \frac{a_i}{\hat{a}_i} \geq 1 \geq v(a)$$

and the program (3) is thus feasible.

The dual program of (3) is

$$\text{maximize} \sum_{a\in\mathcal{N}(v)} \lambda_a v(a) \qquad \text{subject to} \sum_{a\in\mathcal{N}(v)} \lambda_a a = 1_N \text{ where } \lambda_a \in I. \tag{4}$$

The dual program is feasible too; indeed we use the fact that $\{0,1\}^n \subseteq \mathcal{N}(v)$ and set:

$$\lambda_a = \begin{cases} 1 & \text{if } a = 1_i \text{ for some } i \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Then the system $(\mathcal{N}(v), (\lambda_a)_{a\in\mathcal{N}(v)})$ is balanced. By the duality theorem of linear programming (see e. g. [15, Theorem 4.3.4]), both programs (3) and (4) have necessarily the same value. This implies that the core of v is non-empty if and only if

$$\sum_{a\in\mathcal{N}(v)} \lambda_a v(a) \leq 1,$$

for every feasible vector $(\lambda_a)_{a\in\mathcal{N}(v)}$ of (4). The desired conclusion follows by observing that the feasible set of (4) coincides with the set of all balanced systems over $\mathcal{N}(v)$. □

The condition (2) may not be easy to check. Therefore, we provide a sufficient condition for the core non-emptiness based on the notion of strong veto player. For every $v \in \mathsf{SŁG}_n$ and $i \in N$, let $v^i(a) \colon I \to I$ be defined by

$$v^i(a) = v(\underbrace{1, \dots, 1}_{i-1}, a, 1, \dots, 1), \quad a \in I.$$

Loosely speaking, function v^i measures a marginal influence of player i with respect to the grand coalition: the more increasing v^i is in the neighborhood of 1, the greater coalition potential player i executes in the grand coalition. Based on behavior of this function, we define a degree $d^i \in I$ in which player $i \in N$ vetoes a proposition: put

$$d^i = \inf\left\{\, a \in I \;\middle|\; v^i(a) = 1 \,\right\}.$$

Definition 5.2. Let $v \in \mathsf{SŁG}_n$. We say that player $i \in N$ is a

- *null player* if $d^i = 0$,
- *weak veto player* if $0 < d^i < 1$,
- *strong veto player* if $d^i = 1$,
- *veto player* if i is not null player.

Note that the null player can be equivalently defined by condition $v(1_{N\setminus i}) = 1$, which is usual in coalitional game theory. Analogously, the veto player can be defined as a player for which $v(1_{N\setminus i}) = 0$ or, equivalently, $v^i \in \mathsf{SŁG}_1$. Moreover, i is a strong veto player iff v^i is strictly increasing over the line segment $[c, 1]$ for some $c < 1$.

Clearly, any veto player i can effectively turn the (winning) grand coalition 1_N into the loosing coalition $1_{N\setminus i}$, that is, $v(1_{N\setminus i}) = 0$. A finer classification arises, however, in case of fuzzy games than in case of classical coalition games. The strong veto player's actions cause a loss to each fuzzy coalition because $a_i < 1$ implies $v(a) < 1$, whereas the weak veto player's level of membership a_i in a winning fuzzy coalition a can be strictly less than 1. The strong veto players are indispensable in decision-forming since they can profit from any cooperative situation by raising threats to other coalition members.

Example 5.3. Player 1 from Example 3.7 is a strong veto player and player 2 is null. In particular, Theorem 4.4 shows that the core of v is the singleton $\{(1, 0)\}$.

The situation in the example above can be generalized. A *game with strong veto players* is a game $v \in \mathsf{SŁG}_n$ such that there exists a strong veto player in game v. We show that games with strong veto players have non-empty cores. In order to prove this, the following purely technical lemma is needed.

Lemma 5.4. If $p \in \mathsf{SŁG}_n$ and i is a strong veto player, then $p^i(a) \le a$ for every $a \in I$.

Proof. Let us write v instead of p^i. We proceed by contradiction: assume that there is $a \in (0, 1)$ with $v(a) > a$. Due to the continuity of v there has to be interval $[c', c] \subseteq [a, 1]$ such that $v(c') > c'$, $v(c) = c$, and $[c', c]$ is a subset of one domain of linearity of v, i.e., there is a linear polynomial $f(x) = \alpha x + \beta$, where $\alpha, \beta \in \mathbb{Z}$, and $v(x) = f^{\#}(x)$ for each $x \in [c', c]$. Therefore we get

$$\alpha = \frac{c - v(c')}{c - c'}.$$

We distinguish three cases: first, assuming $c < v(c')$ yields $\alpha < 0$, a contradiction with the fact that v is non-decreasing. Second, if $c > v(c')$, then

$$0 < c - v(c') < c - c'$$

and thus we get $0 < \alpha < 1$, a contradiction with $\alpha \in \mathbb{N}$. Finally, assume that $c = v(c')$, then we have $f(x) = \beta$. Again, we distinguish two cases: if $\beta \le 0$, we obtain a $v(c') = 0$ a contradiction. If $\beta \ge 1$, we get $v(c') = 1$, a contradiction with the fact that i is a strong veto player. □

The following result is a generalization of Proposition 2.5.

Theorem 5.5. Any game $v \in \mathsf{SŁG}_n$ is a game with strong veto players iff it has a non-empty core.

P r o o f. Let $M \subseteq N$ be a non-empty set of strong veto players. Put $m = |M|$ and

$$x_i = \begin{cases} \frac{1}{m} & \text{if } i \in M, \\ 0 & \text{otherwise.} \end{cases}$$

We show that $x \in \mathcal{C}(v)$. Clearly $x \in \mathcal{C}_{1_N}(v)$:

$$\langle 1_N, x \rangle = \sum_{i \in M} x_i = m \cdot \tfrac{1}{m} = 1.$$

Second, using Theorem 4.4, it is sufficient to show that $x \in \mathcal{C}_a(v)$ for any fuzzy coalition $a \in \mathcal{N}(v) \setminus \{1_N\}$. Consider the following chain of (in)equalities:

$$v(a) \leq \bigwedge_{j \in M} v^j(a_j) \leq \bigwedge_{j \in M} a_j = \tfrac{1}{m} \sum_{i \in M} \bigwedge_{j \in M} a_j \leq \tfrac{1}{m} \sum_{i \in M} a_i = \langle a, x \rangle .$$

The first one follows from $v(a) \leq v^j(a_j)$ (v is non-decreasing); the second uses Lemma 5.4 and the assumption that player $j \in M$ is a strong veto player, and the remaining ones are trivial.

The converse direction: assume that there are no strong veto players, i. e., for each $i \leq n$ there is $c_i < 1$ such that $v^i(c_i) = 1$. Let us take $c = \max\{c_1, \ldots, c_n\} < 1$ and define fuzzy coalitions a^i as

$$a^i = (\underbrace{1, \ldots, 1}_{i-1}, c, 1, \ldots, 1).$$

We proceed by contradiction: assume there is $x \in \mathcal{C}(v)$. Then $x \in \mathcal{C}_{a^i}(v)$ for each $i \leq n$, which means that:

$$\begin{array}{l} cx_1 + x_2 + \cdots + x_n \geq v(a^1) = v^1(c) = 1 \\ x_1 + cx_2 + \cdots + x_n \geq v(a^2) = v^2(c) = 1 \\ \vdots \\ x_1 + x_2 + \cdots + cx_n \geq v(a^n) = v^n(c) = 1 \end{array}$$

Summing up these inequations we get:

$$(n - 1 + c)x_1 + (n - 1 + c)x_2 + \cdots + (n - 1 + c)x_n = (n - 1 + c) \sum_{i=1}^{n} x_i \geq n.$$

Using the fact $x \in \mathcal{C}_{1_N}(v)$ yields $\sum_{i=1}^{n} x_i = 1$ and thus we obtain $c \geq 1$, a contradiction. □

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